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UNIFORMLY CONTINUOUS SET-VALUED COMPOSITION OPERATORS IN THE SPACE OF TOTAL φ-BIDIMENSIONAL VARIATION IN THE SENSE OF RIESZ

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Abstract. In this paper we prove that if a Nemytskij composition operator, generated by a function of three variables in which the third variable is a function one, maps a suitable large subset of the space of functions of bounded total φ-bidimensional variation in the sense of Riesz, into another such space, and is uniformly continuous, then its generator is an affine function in the function variable. This extends some previous results in the one-dimensional setting.

Keywords: φ-bidimensional variation, uniformly continuous, Nemytskij operator.

Mathematics Subject Classification: 47H30, 39B52.

INTRODUCTION

Let $a = (a_1, a_2)$, $b = (b_1, b_2)$ be points in $\mathbb{R}^2$ such that $a_i < b_i$, $i = 1, 2$. In the sequel, we use the symbol I_a^b to denote the basic rectangle $[a_1, b_1] \times [a_2, b_2]$. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be real normed spaces and let C be a closed and convex set in X. We also denote by $X^{I_a^b}$ the algebra of all functions $f : I_a^b \longrightarrow X$, and by $\mathcal{F}$ the set of all non-decreasing continuous functions $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ such that:

(i) $\varphi(t) = 0$ if and only if $t = 0$, and
(ii) $\lim\limits_{t \longrightarrow \infty} \varphi(t) = \infty$.

If, in addition, $\varphi \in \mathcal{F}$ is a convex map, we say that $\varphi \in \mathcal{N}$ or that φ is an $\mathcal{N}$-function. More details about $\mathcal{N}$-functions can be found in [5].

Let $\mathfrak{X}$ be a subspace of $X^{I_a^b}$. Given a function $h : I_a^b \times X \longrightarrow Y$, the composition (Nemytskij or Superposition, see [2, 14]) operator

$$H : \mathfrak{X} \longrightarrow Y^{I_a^b},$$

generated by h, is defined as

$$(Hf)(t,s) := h(t,s,f(t,s)), \quad (t,s) \in I_a^b.$$

For $\varphi \in \mathcal{F}$, let $(BV_\varphi^R(I_a^b, X), \|\cdot\|_\varphi)$ be the Banach space of all functions $f \in X^{I_a^b}$ which are of bounded total φ-bidimensional variation in the sense of Riesz (see [3] and the next section).

As usual $\mathcal{L}(X,Y)$ denotes the space of all continuous linear operators from a normed space X into a normed space Y.

Assume that H, a Nemytskij composition operator generated by a function of three variables $h : I_a^b \times X \longrightarrow Y$, maps the set of functions $f \in BV_\varphi^R(I_a^b, X)$ such that $f(I_a^b) \subset C$ into $BV_\psi^R(I_a^b, Y)$. In this article, we prove that, if H is uniformly continuous, then its generator is an affine function in the function variable. This extends previous results (see [1, 11]) in the one-dimensional setting.

1. PRELIMINARIES

Let $\xi = \{t_i\}_{i=0}^m$ and $\eta = \{s_j\}_{j=0}^n$ be partitions of $[a_1, b_1]$ and $[a_2, b_2]$, respectively; i.e., $m, n \in \mathbb{N}$,

$$a_1 = t_0 < t_1 < \ldots < t_m = b_1$$

and

$$a_2 = s_0 < s_1 < \ldots < s_n = b_2.$$

For each function $f \in X^{I_a^b}$, let us introduce the following notation:

$$\Delta s_j := s_j - s_{j-1},$$
$$\Delta t_i := t_i - t_{i-1}$$

and

$$\Delta_{10} f(t_i, s_j) := f(t_i, s_j) - f(t_{i-1}, s_j),$$
$$\Delta_{01} f(t_i, s_j) := f(t_i, s_j) - f(t_i, s_{j-1}),$$
$$\Delta_{11} f(t_i, s_j) := f(t_{i-1}, s_{j-1}) - f(t_{i-1}, s_j) - f(t_i, s_{j-1}) + f(t_i, s_j).$$

Definition 1.1 (See [3]). Let $\varphi \in \mathcal{F}$, $(X; \|\cdot\|)$ be a real normed space and $f \in X^{I_a^b}$.

(a) Let $x_2 \in [a_2, b_2]$ be fixed. Consider the function

$$f(\cdot, x_2) \ : \ [a_1, b_1] \times \{x_2\} \longrightarrow X,$$

defined as

$$f(\cdot, x_2)(t) := f(t, x_2), \quad t \in [a_1, b_1].$$

Then the (one-dimensional) φ-variation in the sense of Riesz (see [18]) of the function $f(\cdot, x_2)$, on an subinterval $[x_1, y_1] \subseteq [a_1, b_1]$, is the quantity

$$V_{\varphi,[x_1,y_1]}^R(f(\cdot, x_2)) := \sup_{\Pi_1} \sum_{i=1}^{m} \varphi\left[\frac{\|\Delta_{10} f(t_i, x_2)\|}{|\Delta t_i|}\right] |\Delta t_i|, \tag{1.1}$$

where the supremum is taken over all partitions $\Pi_1 = \{t_i\}_{i=0}^m$ $(m \in \mathbb{N})$ of the interval $[x_1, y_1]$.

(b) A similar argument applies for the variation $V^R_{\varphi,[x_2,y_2]}$, where $x_1 \in [a_1, b_1]$ is fixed and $[x_2, y_2]$ is a subinterval of $[a_2, b_2]$. That is, for the function $f(x_1, \cdot) : \{x_1\} \times [a_2, b_2] \longrightarrow X$, the φ-variation in the sense of Riesz, is the quantity

$$V^R_{\varphi,[x_2,y_2]}(f(x_1,\cdot)) := \sup_{\Pi_2} \sum_{j=1}^{n} \varphi\left[\frac{\|\Delta_{01} f(x_1, s_j)\|}{|\Delta s_j|}\right] |\Delta s_j|, \tag{1.2}$$

where the supremum is taken over the set of all partitions $\Pi_2 = \{s_j\}_{j=0}^n$ $(n \in \mathbb{N})$ of the interval $[x_2, y_2]$.

(c) The φ-bidimensional variation in the sense of Riesz is defined by the formula

$$V^R_{\varphi}(f) := \sup_{\Pi_1,\Pi_2} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi\left[\frac{\|\Delta_{11} f(t_i, s_j)\|}{|\Delta t_i||\Delta s_j|}\right] \cdot |\Delta t_i||\Delta s_j|, \tag{1.3}$$

where the supremum is taken over the set of all partitions (Π_1, Π_2) of the rectangle $I_a^b \subset \mathbb{R}^2$.

(d) The total φ-bidimensional variation in the sense of Riesz of the function $f : I_a^b \longrightarrow X$ is denoted by $TV^R_{\varphi}(f)$ and is defined as:

$$TV^R_{\varphi}(f) := TV^R_{\varphi}(f, I_a^b) := V^R_{\varphi,[a_1,b_1]}(f(\cdot, a_2)) + V^R_{\varphi,[a_2,b_2]}(f(a_1,\cdot)) + V^R_{\varphi}(f). \tag{1.4}$$

Definition 1.2. Let $\varphi \in \mathcal{F}$. We say that φ satisfies condition ∞_1 if

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty. \tag{1.5}$$

A function $\varphi \in \mathcal{F}$ is said to be in the Δ_2 class, if there exists a constant $t_0 \geq 0$ and $K > 0$ such that

$$\varphi(2t) \leq K\varphi(t) \quad \text{for all} \quad t > t_0.$$

Remark 1.3. It is easy to show that if $\varphi \in \mathcal{N}$ satisfies condition ∞_1, then the following equality holds:

$$\lim_{r\to 0} r\varphi^{-1}(1/r) = \lim_{\rho\to\infty} \rho/\varphi(\rho) = 0. \tag{1.6}$$

For $\varphi \in \mathcal{N} \cap \Delta_2$, we denote by $BV^R_{\varphi}(I_a^b, X)$ the vector space (see [3])

$$BV^R_{\varphi}(I_a^b, X) = \left\{ f \in X^{I_a^b} : \exists\, \lambda > 0,\, TV^R_{\varphi}(\lambda f) < \infty \right\}.$$

Just as in the one dimensional situation (cf. [1]), in $BV^R_{\varphi}(I_a^b, X)$ one can define the so called Luxemburg-Nakano-Orlicz seminorm [3, 7, 16, 17] by means of:

$$p_{\varphi}(f) := \inf\left\{ \epsilon > 0 : TV^R_{\varphi}\big(f/\epsilon\big) \leq 1 \right\},$$

and therefore $BV_\varphi^R(I_a^b, X)$ may be equipped with the norm

$$\|f\|_\varphi := |f(a)| + p_\varphi(f).$$

Also, if $C \subseteq X$ we use the notation $BV_\varphi^R(I_a^b, C)$ for the set

$$\{f \in BV_\varphi^R(I_a^b, X) : f(I_a^b) \subset C\}.$$

The following lemma exhibits some properties of p_φ.

For $(t, s), (t', s') \in I_a^b$, we put

$$\Omega_{t,t',s,s'} := \Big\{|t - t'|, |s - s'|, |t - t'||s - a_2|, |a_1 - t'||s - s'|\Big\}.$$

Lemma 1.4 ([3]). *For $\phi \in \mathcal{F}$ and $f \in BV_\varphi^R(I_a^b; X)$, we have:*

(a) *If $(t, s), (t', s') \in I_a^b$, then*

$$\|f(t, s) - f(t', s')\| \le 4M\varphi^{-1}(1/m)p_\varphi(f),$$

where $M := \max \Omega_{t,t',s,s'}$ and $m := \min \Omega_{t,t',s,s'}$.

(b) *If $p_\varphi(f) > 0$, then $TV_\varphi^R(f/p_\varphi(f)) \le 1$.*

(c) *If $r > 0$, then $TV_\varphi^R(f/r) \le 1$ if, and only if, $p_\varphi(f) \le r$.*

(d) *If $r > 0$ y $TV_\varphi^R(f/p_\varphi(f)) = 1$, then $p_\varphi(f) = r$.*

Theorem 1.5 ([3]). *If $\varphi \in \mathcal{F} \cap \Delta_2$ and X is a Banach space, then $BV_\varphi^R(I_a^b, X)$ is a Banach space.*

2. MAIN RESULT

The following theorem is the main result of this work, and it extends the results of Matkowski and others (see [1, 11]) in the case when the Nemytskij operator is defined on the space $BV_\varphi^R([a, b]; \mathbb{R})$. The techniques used for the proof are based on those of [1].

Theorem 2.1. *Assume that $I_a^b \subset \mathbb{R}^2$ is a rectangle, φ, ψ are $\mathcal{N}$-functions that satisfy the ∞_1 condition, $(X, \|\cdot\|_X)$ is a real normed space, $(Y, \|\cdot\|_Y)$ is a real Banach space and C is a closed and convex set in X. If a composition operator H generated by a function $h : I_a^b \times C \longrightarrow Y$, maps the set $BV_\varphi^R(I_a^b, C)$ into the space $BV_\psi^R(I_a^b, Y)$ and is uniformly continuous, then there exist a function $A : I_a^b \to \mathcal{L}(X, Y)$ and $B \in BV_\psi^R(I_a^b, Y)$ such that*

$$h(t, s, u) = A(t, s)u + B(t, s), \quad (t, s) \in I_a^b, \quad u \in C.$$

Proof. It is readily seen that for each $u \in C$, the constant function $f(t, s) := u$ belongs to $BV_\varphi^R(I_a^b, C)$; thus, since H maps $BV_\varphi^R(I_a^b, C)$ into $BV_\psi^R(I_a^b, Y)$, it follows that, for each $u \in C$, the function $h_u : I_a^b \to Y$ defined as

$$h_u(t, s) := h(t, s, u)$$

belongs to $BV_\psi^R(I_a^b, Y)$.

On the other hand, taking into account the uniform continuity of H, and denoting by $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ the *modulus of continuity*[1)] operator, we have

$$\|H(f_1) - H(f_2)\|_\psi \leq \omega\big(\|f_1 - f_2\|_\varphi\big), \quad \text{for} \quad f_1, f_2 \in BV^R_\varphi(I, C). \tag{2.1}$$

Also, from the definition of the norm $\|\cdot\|_\psi$, we obtain

$$p_\psi\big(H(f_1) - H(f_2)\big) \leq \|H(f_1) - H(f_2)\|_\psi, \quad \text{for} \quad f_1, f_2 \in BV_\varphi(I, C). \tag{2.2}$$

Hence, by Lemma 1.4(c) we deduce that if $\omega\big(\|f_1 - f_2\|_\varphi\big) > 0$, then the last inequality is equivalent to

$$V^R_\psi\left(\frac{(H(f_1) - H(f_2))}{\omega\big(\|f_1 - f_2\|_\varphi\big)}\right) \leq TV^R_\psi\left(\frac{H(f_1) - H(f_2)}{\omega\big(\|f_1 - f_2\|_\varphi\big)}\right) \leq 1. \tag{2.3}$$

Now, by definition of V^R_ψ, it follows that for any rectangle $[t_1, t_2] \times [s_1, s_2] \subseteq I^b_a$, with $t_1 < t_2$ and $s_1 < s_2$:

$$\psi\left(\frac{\Big\|\sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2}(-1)^{i+j}(H(f_1) - H(f_2))(t_i, s_j)\Big\|}{\omega\big(\|f_1 - f_2\|_\varphi\big)(t_2 - t_1)(s_2 - s_1)}\right)(t_2 - t_1)(s_2 - s_1) \leq 1. \tag{2.4}$$

Let us define now, for arbitrarily fixed $\alpha, \beta \in \mathbb{R}$, with $\alpha < \beta$:

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0 & \text{for} \quad t \leq \alpha, \\ \dfrac{t - \alpha}{\beta - \alpha} & \text{for} \quad \alpha \leq t \leq \beta, \\ 1 & \text{for} \quad t \geq \beta. \end{cases} \tag{2.5}$$

Observe that $\eta_{\alpha,\beta} : \mathbb{R} \longrightarrow [0, 1)$.

Next, consider two auxiliary functions: $\eta_i : [a_i, b_i] \longrightarrow [0, 1]$, $i = 1, 2$, defined in the following way:

$$\eta_1(t) := \begin{cases} 0 & \text{for} \quad a_1 \leq t \leq t_1, \\ \eta_{t_1,t_2}(t) & \text{for} \quad t_1 \leq t \leq t_2, \\ 1 & \text{for} \quad t_2 \leq t, \end{cases} \tag{2.6}$$

$$\eta_2(s) := \begin{cases} 0 & \text{for} \quad a_2 \leq s \leq s_1, \\ \eta_{s_1,s_2}(s) & \text{for} \quad s_1 \leq s \leq s_2, \\ 1 & \text{for} \quad s_2 \leq s. \end{cases} \tag{2.7}$$

Finally, for arbitrary points $y_1, y_2 \in C$, $y_1 \neq y_2$, define the functions $f_1, f_2 : I^b_a \longrightarrow C$ as follows:

$$f_j(t, s) := \frac{1}{2}\Big[\big(\eta_1(t) \cdot \eta_2(s)\big)(y_1 - y_2) + y_j + y_2\Big], \quad (t, s) \in I^b_a,\ j = 1, 2.$$

[1)] $\omega(\rho) := \sup\Big\{\|H(f_1) - H(f_2)\|_\psi : \|f_1 - f_2\|_\varphi \leq \rho;\ f_1, f_2 \in BV^R_\psi(I^b_a, C)\Big\}$, for $\rho > 0$.

Observe, that

$$f_1(t_1,s_1)=f_1(t_1,s_2)=f_1(t_2,s_1)=\frac{y_1+y_2}{2};\quad f_1(t_2,s_2)=y_1,$$
$$f_2(t_1,s_1)=f_2(t_1,s_2)=f_2(t_2,s_1)=y_2;\quad f_2(t_2,s_2)=\frac{y_1+y_2}{2},$$

$$f_1(\cdot)-f_2(\cdot)=\frac{y_1-y_2}{2},\quad \text{and consequently}\quad \|f_1-f_2\|_\varphi=\frac{|y_1-y_2|}{2}>0.$$

Also, by the definition of H:

$$(H(f_1)-H(f_2))(t_1,s_1)=h\big((t_1,s_1),\frac{y_1+y_2}{2}\big)-h((t_1,s_1),y_2), \tag{2.8}$$
$$(H(f_1)-H(f_2))(t_1,s_2)=h\big((t_1,s_2),\frac{y_1+y_2}{2}\big)-h((t_1,s_2),y_2), \tag{2.9}$$
$$(H(f_1)-H(f_2))(t_2,s_1)=h\big((t_2,s_1),\frac{y_1+y_2}{2}\big)-h((t_2,s_1),y_2), \tag{2.10}$$
$$(H(f_1)-H(f_2))(t_2,s_2)=h((t_2,s_2),y_1)-h((t_2,s_2),\frac{y_1+y_2}{2}). \tag{2.11}$$

Now notice that, by applying the inverse function ψ^{-1} to both sides of (2.4), one gets

$$\Big\|\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}(H(f_1)-H(f_2))(t_i,s_j)\Big\|\le \Delta t\,\Delta s\cdot\omega\Big(\frac{|y_1-y_2|}{2}\Big)\psi^{-1}\Big(\frac{1}{\Delta t\Delta s}\Big). \tag{2.12}$$

Taking into account the fact that for any $u\in C$, the function $h_u\in BV_\psi^R(I_a^b,Y)$, the identities (2.8)–(2.11), that ψ satisfies the condition ∞_1, and passing to the limit in (2.12) as $\Delta t\,\Delta s\longrightarrow 0$, in such a way that $(t,s)\in[t_1,t_2]\times[s_1,s_2]\subseteq I_a^b$, with $t_1<t_2$ and $s_1<s_2$, we obtain, after simplification (the first two summands cancel out each other), for all $(t,s)\in I_a^b$, $y_1,y_2\in C$:

$$h\Big((t,s),\frac{y_1+y_2}{2}\Big)=\frac{1}{2}\big(h((t,s),y_1)+h(x,y_2)\big). \tag{2.13}$$

Therefore, the function $h((t,s),\cdot)$ is a solution of Jensen equation in C for $(t,s)\in I_a^b$. Thus, by a slight modification of a standard argument (see Kuczma [6]), we get, for each $(t,s)\in I_a^b$ the existence of an additive function $A:I_a^b\longrightarrow Y^X$ and $B:I_a^b\longrightarrow Y$ such that

$$h(\cdot,y)=A(\cdot)y+B(\cdot),\quad y\in C. \tag{2.14}$$

The uniform continuity of the operator $H:BV_\varphi^R(I_a^b,C)\longrightarrow BV_\psi^R(I_a^b,Y)$ implies the continuity of the function $A(\cdot)$ which implies that $A(t,s)\in\mathcal{L}(X,Z)$. Finally, notice that $A(t,s)(0)=0$, for every $(t,s)\in I_a^b$. Therefore, putting $y=0$ in (2.14), we get

$$h(t,s,0)=B(t,s),\quad (t,s)\in I_a^b,$$

which implies that $B\in BV_\psi^R(I_a^b,Y)$. □

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