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ON SOME FAMILIES OF ARBITRARILY VERTEX DECOMPOSABLE SPIDERS

Abstract. A graph $G$ of order $n$ is called arbitrarily vertex decomposable if for each sequence $(n_1, \ldots, n_k)$ of positive integers such that $\sum_{i=1}^{k} n_i = n$, there exists a partition $(V_1, \ldots, V_k)$ of the vertex set of $G$ such that for every $i \in \{1, \ldots, k\}$ the set $V_i$ induces a connected subgraph of $G$ on $n_i$ vertices. A spider is a tree with one vertex of degree at least 3. We characterize two families of arbitrarily vertex decomposable spiders which are homeomorphic to stars with at most four hanging edges.

Keywords: arbitrarily vertex decomposable graph, trees.

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1. INTRODUCTION

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $|V(G)| = n$. A sequence $\tau = (n_1, \ldots, n_k)$ of positive integers is called admissible for $G$ if $n_1 + \ldots + n_k = n$. We shall write $(\underbrace{n_1, \ldots, n_i}_{s_1}, \underbrace{n_1, \ldots, n_j}_{s_1})$ for the sequence $(n_1, \ldots, n_i, \ldots, n_j, \ldots, n_k)$. If $\tau = (n_1, \ldots, n_k)$ is an admissible sequence for the graph $G$ and there exists a partition $(V_1, \ldots, V_k)$ of the vertex set $V(G)$ such that for each $i \in \{1, \ldots, k\}$ the subgraph $G[V_i]$ induced by $V_i$ is a connected graph on $n_i$ vertices, then $\tau$ is called $G$-realizable or realizable in $G$ and the sequence $(V_1, \ldots, V_k)$ is said to be a $G$-realization of $\tau$ or a realization of $\tau$ in $G$. Each set $V_i$ will be called a $\tau$-part of a realization of $\tau$ in $G$. A graph $G$ is called arbitrarily vertex decomposable (avd for short) if each admissible sequence for $G$ is realizable in $G$.

Arbitrarily vertex decomposable graphs have been investigated in several papers ([1–5] for example). The problem originated from some applications to computer networks ([1]).

The investigation of avd trees is motivated by the fact that a connected graph is avd if its spanning tree is avd.
In [4] the authors proved that every tree of maximum degree at least 7 is not avd and conjectured that every tree with maximum degree at least 5 is not avd. This conjecture was proved in [2]:

**Theorem 1.1.** If tree \( T \) is arbitrarily vertex decomposable then \( \Delta(T) \leq 4 \). Moreover every vertex of degree four in \( T \) is adjacent to a leaf.

Let \( T = (V(T), E(T)) \) be a tree. A vertex \( v \in V(T) \) is called primary if \( d(v) \geq 3 \). A leaf is a vertex of degree one in \( T \). Let the path \( P \) be a subgraph of \( T \) such that one of its end vertices is a leaf in \( T \), the other one is a primary vertex in \( T \) and all internal vertices of \( P \) have degree two in \( T \). We will call such a path an arm of \( T \). Let \( v \) be a primary vertex of a tree \( T \) such that \( v \) is an end vertex of two arms \( A_1, A_2 \) of \( T \). Let \( y_i \) be the other end vertex of \( A_i \) and \( x_i \in V(A_i) \) the neighbour of \( v \), \( i = 1, 2 \). Define \( T(A_1, A_2) \) to be a tree with \( V(T(A_1, A_2)) = V(T) \) and \( E(T(A_1, A_2)) = E(T) \setminus \{vx_2\} \cup \{y_1y_2\} \).

In [1] and, independently, in [5] the authors observed that:

**Lemma 1.2.** Let \( T \) be an arbitrarily vertex decomposable tree and let \( A_1, A_2 \) be arms of \( T \) that share a primary vertex of \( T \). Then the tree \( T(A_1, A_2) \) is arbitrarily vertex decomposable, too.

That gives a reason for the investigation of avd trees which are homeomorphic to a star \( K_{1,q} \), where \( q \) is three or four. If \( q = 2 \) such a tree is a path which is avd.

A spider is a tree with one primary vertex. Such a tree has \( q \) arms \( A_i \), \( i = 1, \ldots, q \), where \( q \) is the degree of the primary vertex. Let \( a_i \) be the order of \( A_i \), \( i = 1, \ldots, q \). The structure of a spider is determined by the sequence of orders of its arms. Since the ordering of this sequence is not important, we will assume that \( a_1 \leq a_2 \leq \ldots \leq a_q \) and we will denote the above defined spider by \( S(a_1, \ldots, a_q) \).

The first result characterizing the avd spider was found in [1] and, independently, in [5].

We will denote by \( \gcd(a,b) \) the greatest common divisor of two positive integers \( a \) and \( b \).

**Theorem 1.3.** The spider \( S(2, b, c), 2 \leq b \leq c \) is arbitrarily vertex decomposable if and only if \( \gcd(b, c) = 1 \). Moreover, each admissible and non-realizable sequence in \( S(2, b, c) \) is of the form \( (d)^k \), where \( b \equiv c \equiv 0 \pmod{d} \) and \( d \geq 2 \).

In [3] the authors investigated two families of spiders: \( S(2,2,b,c) \) and \( S(3,b,c) \):

**Proposition 1.4.** The spider \( S(2,2,b,c), 2 \leq b \leq c \) is arbitrarily vertex decomposable if and only if the following conditions hold:

1. The spider \( S(3,b,c) \) is arbitrarily vertex decomposable,
2. The numbers \( b, c \) are odd,
3. \( b \not\equiv 2 \pmod{3} \) or \( c \not\equiv 2 \pmod{3} \).

In [3] the authors investigated two families of spiders: \( S(2,2,b,c) \) and \( S(3,b,c) \).
Theorem 1.5. The spider $S(2,2,b,c)$ of order $n$, $3 \leq b \leq c$, is arbitrarily vertex decomposable if and only if the following conditions hold:

1. \( \gcd(b,c) = 1 \),
2. \( \gcd(b+1,c) = 1 \),
3. \( \gcd(b,c+1) = 1 \),
4. \( \gcd(b+1,c+1) = 2 \),
5. \( n \neq ab + \beta(b+1) \) for \( \alpha, \beta \in \mathbb{N} \).

Theorem 1.6. The spider $S(3,b,c)$ of order $n$, $3 \leq b \leq c$, is arbitrarily vertex decomposable if and only if the following conditions hold:

1. \( \gcd(b,c) \leq 2 \),
2. \( \gcd(b+1,c) \leq 2 \),
3. \( \gcd(b,c+1) \leq 2 \),
4. \( \gcd(b+1,c+1) \leq 3 \),
5. \( n \neq ab + \beta(b+1) \) for \( \alpha, \beta \in \mathbb{N} \).

The main result of this paper are Theorems 2.1 and 2.2 of Section 2 which give a complete characterization of avd spiders $S(2,3,b,c)$ and $S(4,b,c)$. To prove them we will also use the following results:

Proposition 1.7 ([1]). The spider $S(a_1,a_2,a_3)$, $a_1 \leq a_2 \leq a_3$, is arbitrarily vertex decomposable if and only if every admissible sequence $((q)^{s_1},(q+1)^{s_2})$, $s_2 > 0$, $q \leq a_1 + a_2 - 2$ and every admissible sequence $(m,(r)^{t_1},(r+1)^{t_2})$, $t_2 > 0$, $1 \leq m \leq r-1$, $r \leq a_1 - 3$, has a realization in $S(a_1,a_2,a_3)$.

Proposition 1.8 ([2]). The spider $S(2,a_1,a_2,a_3)$, $a_1 \leq a_2 \leq a_3$, is arbitrarily vertex decomposable if and only if the following conditions hold:

1. The spider $S(a_1,a_2,a_3)$, $a_1 \leq a_2 \leq a_3$, is arbitrarily vertex decomposable.
2. Every admissible sequence $((q)^{s_1},(q+1)^{s_2})$, $s_2 > 0$, $q \leq a_1 + a_2 - 2$ and every admissible sequence $(m,(r)^{t_1},(r+1)^{t_2})$, $t_2 > 0$, $0 < m \leq r-1$, $r \leq a_1 - 3$, has a realization in $S(2,a_1,a_2,a_3)$.

Proposition 1.9 ([6]). The graph $G$ is arbitrarily vertex decomposable if and only if every admissible sequence $(n_1,\ldots,n_k)$ with $n_i \geq 2$ for each $i = 1,\ldots,k$, has a realization in $G$.

Given an admissible sequence $\tau = (n_1,\ldots,n_k)$ for a graph $G$ of order $n$, we will use the following convention to describe a realization $(V_1,\ldots,V_k)$ of $\tau$ in $G$. We choose an ordering $s = (v_1,\ldots,v_n)$ of the vertex set of $G$. Then we define the $\tau$-parts according to the sequence $s$, that is $V_1 = \{v_1,\ldots,v_{n_1}\}$, $V_2 = \{v_{n_1+1},\ldots,v_{n_1+n_2}\}$ and so on.

2. ARBITRARILY VERTEX DECOMPOSABLE SPIDERS $S(2,3,b,c)$ AND $S(4,b,c)$

Theorem 2.1. The spider $S(2,3,b,c)$ of order $n$, $3 \leq b \leq c$, is arbitrarily vertex decomposable if and only if the following conditions hold:

1. \( \gcd(b,c) = 1 \),
(2) \( \max\{\gcd(b+1,c), \gcd(b,c+1), \gcd(b+1,c+1), \gcd(b+2,c), \gcd(b,c+2)\} \leq 2, \)

(3) \( \max\{\gcd(b+1,c+2), \gcd(b+2,c+1), \gcd(b+2,c+2)\} \leq 3, \)

(4) \( n \neq \alpha b + \beta (b+1) + \gamma (b+2) \) for \( \alpha, \beta, \gamma \in \mathbb{N}, \)

(5) If \( b = 2h, h \in \mathbb{N}, h \geq 3 \) then \( n \neq \alpha h + \beta (h+1) \) for \( \alpha, \beta \in \mathbb{N}. \)

**Proof. Necessity.** If \( d_1 = \gcd(b,c) \geq 2 \) or \( d_2 = \max\{\gcd(b+1,c), \gcd(b,c+1)\} \geq 3 \) or \( d_3 = \max\{\gcd(b+1,c+1), \gcd(b+2,c), \gcd(b,c+2)\} \geq 3 \) or \( d_4 = \max\{\gcd(b+1,c+2), \gcd(b+2,c+1)\} \geq 4 \) or \( d_5 = \gcd(b+2,c+2) \geq 4 \) then the following sequences \((d_1)_{\infty}^{\alpha b}\) or \((d_2)_{\infty}^{\alpha b+1}, d_2 + 1)\) or \((d_3)_{\infty}^{\alpha b}\) or \((d_4)_{\infty}^{\alpha b+1}, d_4 + 1)\) or \((d_5)_{\infty}^{\alpha b+1}, (d_5)_{\infty}^{\alpha b+2}\), respectively, are admissible but not realizable. If \( n = \alpha b + \beta (b+1) + \gamma (b+2) \), where \( \alpha, \beta, \gamma \in \mathbb{N} \) then the sequence \((b(\alpha), (b+1)^{\alpha}, (b+2)^{\alpha})\) is admissible and not realizable. If \( n = \alpha h + \beta (h+1) \), where \( h = \frac{n}{2} \in \mathbb{N}, h \geq 3 \) then the sequence \((b(\alpha), (h+1)^{\beta})\) is admissible and not realizable.

**Sufficiency.** Let \( A_i, i = 1, \ldots, 4 \) be arms of \( S(2,3,b,c), \) \( 3 \leq b \leq c, \) of orders \( 2, 3, b, \) and \( c, \) respectively. Let \( v \) be a primary vertex of \( S(2,3,b,c). \) Set \( A_1 = \{v, v_1^1\}, A_2 = \{v, v_1^2, v_2^2\}, A_3 = \{v, v_1^3, \ldots, v_{c-1}^1\}, \) and \( A_4 = \{v, v_1^3, \ldots, v_{c-1}^2\}, \) such that \( v v_1^1, v v_1^2, v v_2^2, v v_1^3, v v_{c-1}^1, v v_{c-1}^2, v v_1^3 v_{c+1}^1 \) are edges of \( S(2,3,b,c), i = 1, \ldots, b-2, j = 1, \ldots, c-2. \) Let \( \tau = (n_1, \ldots, n_k) \) be an admissible sequence for \( S(2,3,b,c). \) We assume that \( n_1 \leq \ldots \leq n_k. \)

By Proposition 1.8, Proposition 1.9 and Theorem 1.6 we may assume that \( \tau = ((n_1)k_1, (n_1+1)k_2), \) where \( k_1, k_2 \in \mathbb{N} \) and \( 2 \leq n_1 \leq b + 1. \)

If \( n_1 = 2 \) then by Theorem 1.3 there is the realization \((V_2, \ldots, V_k)\) of the sequence \((n_2, \ldots, n_k)\) in \( S(2,3,b,c) \) and hence \(((v_1^1, v_2^2), V_2, \ldots, V_k)\) is a realization of \( \tau \) in \( S(2,3,b,c). \) We may assume that \( n_1 \geq 3. \)

Since \( \max\{\gcd(b+1,c+1), \gcd(b+2,c), \gcd(b,c+2)\} \leq 2, \) we have \( \tau \neq ((3)^k) \) and hence especially \( n_k \geq 4. \) Since \( n_k \leq b + 2, \) by the condition (4), we obtain that \( n_1 \leq b - 2, n_k \leq b \). We define the sequence \((V_1, V_2, \ldots, V_k)\) of \( \tau \)-parts according to \( s^1 = (v_b^1, v_{b-1}^1, \ldots, v_2^1, v_1^2, v_2^2, v_3^3, v_4^3, v_5^3). \) Suppose that the construction does not give a realization of \( \tau \) in \( S(2,3,b,c). \) It follows that there is \( i_0 \) such that \( \gamma_{b-1}^1, v_{c-1}^1 \in V_{i_0}. \) Since \( n_k \leq b, n_1 \leq b - 1, \) we have \( 2 \leq i_0 \leq k - 1. \)

If \( |V_{i_0} \cap V(A_3)| \leq n_k - 4 \) then we modify the ordering of elements of \( \tau, \) we obtain \( \tau = (n_{i_0}, n_{i_0+1}, n_{i_0+2}, n_{i_0+3}, \ldots, n_{i_0+1}, n_{i_0+1}, n_{i_0+1}-1) \) and we define the sequence of \( \tau \)-parts according to \( s^2 = (v_{c-1}^1, v_{c-2}^1, \ldots, v_1^2, v_0^2, v_1^3, v_2^4, v_3^5, v_4^5, \ldots, v_{i_0}^5) \) and we obtain a realization of \( \tau \) in \( S(2,3,b,c). \) Hence we may assume that \( |V_{i_0} \cap V(A_3)| \geq n_k - 3. \)

We will use the following notation: \( d = n_k - n_{i_0}, r = |V_{i_0} \cap V(A_3)| - (n_k - 4). \) It is easily seen that \( d + r + |V_{i_0} \cap V(A_4)| = 4. \) Since \( |V_{i_0} \cap V(A_4)| \geq 1, d \leq 1, \) we obtain that \( 1 \leq r \leq 3 \) or \( 1 \leq r \leq 2 \) for \( d = 0 \) or \( d = 1, \) respectively. Observe that \( b = \sum_{i=1}^{k-1} n_i + 1 + r + (n_k - 4) = \sum_{i=1}^{k-1} n_i + n_k + r - 3 \) and \( c = \sum_{i=0}^{k-1} n_i + 1 - r. \)

Let us suppose that \( n_{k-1} - n_1 \geq r. \) We modify the ordering of elements of \( \tau \) and we consider \( \tau = (n_{k-1}, n_2, \ldots, n_{k-2}, n_1, n_k). \) We define the sequence of \( \tau \)-parts according to \( s^1 \) and, since \( 0 \leq |V_{i_0} \cap V(A_3)| - (n_{k-1} - n_1) \leq n_k - 4, \) either we obtain a realization of \( \tau \) or \( v_{c-1}^1 \in V_{i_0}, \) where \( j_0 = i_0 \) for \( i_0 < k - 1 \) and \( j_0 = 1 \) for \( i_0 = k - 1. \) In the second case we modify the ordering of elements of \( \tau \) such that \( \tau = (n_{i_0}, n_{i_0+1}, \ldots, n_1, n_k, n_{k-1}, n_2, \ldots, n_{i_0-1}) \) if \( i_0 < k - 1 \)
or \( \tau = (n_1, n_k, n_{k-1}, n_2, \ldots, n_{k-2}) \) if \( i_0 = k - 1 \) and we define the sequence of \( \tau \)-parts according to \( s^2 \). Since \( |V_{j_0} \cap V(A_3)| \leq n_k - 4 \), we obtain a realization of \( \tau \). Hence we may assume that \( n_{k-1} = n_1 > r \).

If \( \tau = ((n_1)^k) \) then \( b = i_0(n_1 + r - 3, c = (k - i_0)n_1 + 1 - r \) and hence \( \max\{\gcd(b, c + 2), \gcd(b + 1, c + 1), \gcd(b + 2, c)\} \geq n_1 \geq 3 \), contrary to (2). If \( \tau = ((n_1)^{k-1}, n_1 + 1) \) then \( d = 1 \) and hence \( r \in \{1, 2\} \). Since \( b = i_0(n_1 + r - 2, c = (k - i_0)n_1 + 1 - r \), we obtain that \( \max\{\gcd(b, c + 1), \gcd(b + 1, c)\} \geq n_1 \geq 3 \), contrary to (2). Therefore we may assume that \( n_{k-1} = n_1 + 1 \).

Let us suppose that \( \tau = (n_1, (n_1 + 1)^{k-1}) \). Then \( r \in \{2, 3\} \). Since \( b = i_0(n_1 + 1) + r - 4 \) and \( c = (k - i_0)(n_1 + 1) + 1 - r \), we obtain that \( \max\{\gcd(b + 1, c + 2), \gcd(b + 2, c + 1)\} \geq n_1 + 1 \geq 4 \), contrary to (3). Hence we may assume that \( n_2 = n_1 \).

Let us suppose that \( i_0 = 2 \). Then \( d = 1, r = 2 \) and \( b = 2n_1 \), contrary to (5). We may assume that \( i_0 \geq 3 \), and hence \( k \geq 4 \).

If \( i_0 = k - 1 \) then \( b = \sum_{i=1}^{k-2} n_i + n_k + r - 3 \geq n_k + r \) and \( c = n_k + 1 - r \), which contradicts the assumption \( b \leq c \). Hence we may assume that \( i_0 \leq k - 2 \) and hence \( k \geq 5 \).

Let us suppose that \( (n_{k-1} + n_{k-2}) - (n_1 + n_2) \geq r \). We modify the ordering of elements of \( \tau \) and we consider \( \tau = (n_{k-1}, n_{k-2}, n_3, \ldots, n_{k-3}, n_2, n_1, n_k) \). We define the sequence of \( \tau \)-parts according to \( s^1 \). Combining condition \( n_{k-1} - n_1 < r \) with the values of \( d \) and \( n_i \), \( i = 2, k - 2, k - 1 \) we obtain that \( 0 \leq |V_{i_0} \cap V(A_3)| - |(n_{k-1} + n_k) - (n_1 + n_2)| \leq n_k - 4 \). Then either we obtain a realization of \( \tau \) or \( v_{i-1}^j \in V_{j_0} \), where \( j_0 = i_0 \) for \( i_0 < k - 2 \) and \( j_0 = 2 \) for \( i_0 = k - 2 \). In the second case we modify the ordering of elements of \( \tau \) such that \( \tau = (n_{i_0}, n_{i_0 + 1}, \ldots, n_k, n_1, n_k, n_{k-1}, n_{k-2}, n_3, \ldots, n_{i_0 - 1}) \) if \( i_0 < k - 2 \) or \( \tau = (n_2, n_1, n_k, n_{k-1}, n_{k-2}, n_3, \ldots, n_{k-3}) \) if \( i_0 = k - 2 \) and we define the sequence of \( \tau \)-parts according to \( s^2 \). Since \( |V_{j_0} \cap V(A_3)| \leq n_k - 4 \), we obtain a realization of \( \tau \). Hence we may assume that \( (n_{k-1} + n_{k-2}) - (n_1 + n_2) < r \).

It is not difficult to check that then we have two possibilities: either \( \tau = ((n_1)^{k-2}, (n_1 + 1)^2), r = 2 \) or \( n_1 = n_2, n_{k-2} = n_{k-1} = n_k = n_1 + 1, r = 3 \).

If \( \tau = ((n_1)^{k-2}, (n_1 + 1)^2) \) and \( r = 2 \) then \( b = i_0(n_1, c = (k - i_0)n_1 \) and hence \( \gcd(b, c) \geq n_1 \geq 3 \), contrary to (1). Hence \( n_1 = n_2, n_{k-2} = n_{k-1} = n_k = n_1 + 1 \) and \( r = 3 \). If \( \tau = ((n_1)^2, (n_1 + 1)^{k-2}) \) then \( b = i_0(n_1 + 1) - 2, c = (k - i_0)(n_1 + 1) - 2 \) and hence \( \gcd(b + 2, c + 2) \geq n_1 + 1 \geq 4 \), contrary to (3). Therefore we may assume that \( k \geq 6 \) and \( n_3 = n_1 \).

If \( i_0 = 3 \) then \( d = 1 \) and hence \( r \leq 2 \), a contradiction. Hence \( 4 \leq i_0 \). If \( i_0 = k - 2 \) then \( 4n_1 + 1 \leq b \leq c = 2n_1 \), a contradiction. Hence \( i_0 \leq k - 3 \) and \( k \geq 7 \). We obtain that \( n_1 = n_2 = n_3, n_{k-2} = n_{k-1} = n_k = n_1 + 1 \), \( r = 3 \) and \( 4 \leq i_0 \leq k - 3 \). Then \( d = 0 \) and hence \( n_k = n_1 + 1 \). We modify the ordering of elements of \( \tau \) and we consider \( \tau = (n_{k-1}, n_{k-2}, n_{k-3}, n_4, \ldots, n_{k-4}, n_3, n_2, n_1, n_k) \). We define the sequence of \( \tau \)-parts according to \( s^1 \). Let us suppose that the construction does not give a realization of \( \tau \). Then we modify the ordering of elements of \( \tau \) and we consider \( \tau = (n_{i_0}, n_{i_0 + 1}, \ldots, n_{k-4}, n_3, n_2, n_1, n_k, n_{k-1}, n_{k-2}, n_{k-3}, n_4, \ldots, n_{i_0 - 1}) \) if \( i_0 < k - 3 \) or \( \tau = (n_3, n_2, n_1, n_k, n_{k-1}, n_{k-2}, n_{k-3}, n_4, \ldots, n_{k-4}) \) if \( i_0 = k - 3 \). We define the sequence of \( \tau \)-parts according to \( s^2 \) and obtain a realization of \( \tau \). \( \square \)
Theorem 2.2. The spider $S(4,b,c)$ of order $n$, $4 \leq b \leq c$, is arbitrarily vertex decomposable if and only if the following conditions hold:

1. $\gcd(b,c) = 1$ or $\gcd(b,c) = 3$,
2. $\max\{\gcd(b+1,c), gcd(b+2,c)\}$
3. $\max\{\gcd(b+1,c), gcd(b+2,c)\} \leq 3$,
4. $n \neq ab + \beta(b+1) + \gamma(b+2)$ for $\alpha, \beta, \gamma \in \mathbb{N}$,
5. If $b = 2h$, $h \in \mathbb{N}$, $h \geq 4$ then $n \neq \alpha h + \beta(h+1)$ for $\alpha, \beta \in \mathbb{N}$.

Proof. We will use the similar method to that in the proof of Theorem 2.1.

Necessity. If $d_1 = \gcd(b,c) \notin \{1, 3\}$ or $d_2 = \max\{\gcd(b+1,c), gcd(b+1,c)\}$
3. $d_3 = \gcd(b+1,c+1), gcd(b+2,c+1) \geq 4$ or $d_4 = \max\{\gcd(b+1,c), gcd(b+2,c)\} \geq 5$ then the following sequences $(2, (d_1) \frac{n-1}{d_1}-1$, $d_2 + 1)$ or $(d_3) \frac{n-1}{d_3}$ or $(d_4) \frac{n-1}{d_4}$, respectively, are admissible but not realizable. If $n = \alpha b + \beta(b+1) + \gamma(b+2)$, where $\alpha, \beta, \gamma \in \mathbb{N}$ then the sequence $((b)^{\alpha}, (b)^{\beta}, (b)^{\gamma})$ is admissible and not realizable.

Sufficiency. Let $A_1, i = 1, 2, 3$ be arms of $S(4,b,c)$, $4 \leq b \leq c$, of orders $4, b$ and $c$, respectively. Let $v$ be a primary vertex of $S(4,b,c)$. Set $A_1 = \{v, v_1^4, v_1^3, v_1^3\}$, $A_2 = \{v, v_1^4, v_1^3, v_0^3\}$, $A_3 = \{v, v_1^4, v_1^3, v_0^3\}$, $v_1^4, v_1^3, v_0^3$ are edges of $S(4,b,c)$, $i = 1, 2, j = 1, \ldots, b-1, l = 1, \ldots, c-2$. Let $\tau = (n_1, \ldots, n_k)$ be an admissible sequence for $S(4,b,c)$. We assume that $n_k \leq \ldots \leq n_k$.

If there is $i_0 \in \{1, \ldots, k\}$ such that $n_{i_0} = 3$ then we set $V_{i_0} = \{v_1^4, v_1^4, v_1^3\}$ and obtain a realization of $\tau$ in $S(4,b,c)$. Hence we may assume that $n_i \neq 3$ for $i \in \{1, \ldots, k\}$.

Let us suppose that $n_{i_0} = 2$ for any $i_0 \in \{1, \ldots, k\}$. Since $\tau \neq (2, (3)^{k-1})$, if we set $V_{i_0} = \{v_1^4, v_1^4, v_1^3\}$ then by Theorem 1.3 we obtain a realization of $\tau$ in $S(4,b,c)$. Hence we may assume that $n_i \neq 2$ for $i \in \{1, \ldots, k\}$. Then by Proposition 1.9 and Proposition 1.7 we have that $\tau = ((n_1)k_1, (n_1+1)k_2)$, where $k_1, k_2 \in \mathbb{N}$ and $4 \leq n_1 \leq b+2$. If $n_k = b+3+1$ then the sequence $V_1, V_2, V_3$ such that $[V(A_1) \cup V(A_2)] \subseteq V_k$ and for $i = 1, \ldots, k-1, V_i \subseteq [V(A_3) \setminus \{v\}]$ is a realization of $\tau$ in $S(4,b,c)$. We may assume that $n_k \leq b+2$. By the condition (4) we obtain that $n_1 \leq b-1, n_k \leq b$. We define the sequence $(V_1, \ldots, V_k)$ of $\tau$-parts according to $s^i = (v_1^4, v_2^b, v_0^3, v_0^3, v_1^4, v_1^3, v_3^4)$. Suppose that the construction does not give a realization of $\tau$ in $S(4,b,c)$. It follows that there is $i_0$ such that $b_{i_0}^b, v_{i_0}^b \in V_{i_0}$. Since $n_k \leq b$ and $n_1 \leq b-1$, we have $2 \leq i_0 \leq k-1$. Using similar arguments to that in the proof of Theorem 2.1 we may assume that $|V_{i_0} \cap V(A_2)| \geq n_k - 3$. We will use the following notation: $d = n_k - n_{i_0}, r = |V_{i_0} \cap V(A_2)| - (n_k - 4)$. It is easily seen that $d + r + |V_{i_0} \cap V(A_3)| = 4$.

Since $|V_{i_0} \cap V(A_3)| \geq 1, d \leq 1$, we obtain that $1 \leq r \leq 3$ or $1 \leq r \leq 2$ for $d = 0$ or $d = 1$, respectively. Observe that $b = \sum_{i=1}^{n_k-1} n_i + n_k + r - 3, c = \sum_{i=i_0}^{n_k-1} n_i + 1 - r$.
If \( \tau = (n_1)^k \) then \( \max\{\gcd(b + 2, c), \gcd(b + 1, c + 1), \gcd(b + c + 2)\} \geq n_1 \geq 4 \), contrary to (2). If \( \tau = ((n_1)^{k-1}, n_1 + 1) \) then \( d = 1 \) and hence \( r \in \{1, 2\} \) and \( \max\{\gcd(b + 1, c), \gcd(b + c + 1)\} \geq n_1 \), contrary to (2). If \( \tau = (n_1, (n_1 + 1)^{k-1}) \) then \( r \in \{2, 3\} \) and hence \( \max\{\gcd(b + 2, c + 1), \gcd(b + 1, c + 2)\} \geq n_1 + 1 \geq 5 \), contrary to (3). Hence we may assume that \( k \geq 4 \) and \( n_1 = n_2, n_k = n_{k-1} = n_1 + 1 \).

Using similar method to that in the proof of Theorem 2.1 we may assume that \( k - 2 \geq i_0 \geq 3 \) and that \( (n_k - 1 + n_k - 2) - (n_1 + n_2) < r \). Then we obtain that either \( \tau = ((n_1)^{k-2}, (n_1 + 1)^2) \), \( r = 2 \) or \( n_1 = n_2, n_k = n_{k-1} = n_k = n_1 + 1, r = 3 \). In the first case \( b = i_0 n_1, c = (k - i_0) n_1 \) and \( \gcd(b, c) \geq n_1 \geq 4 \) contrary to (1). We may assume that \( n_1 = n_2, n_k = n_{k-1} = n_k = n_1 + 1 \) and \( r = 3 \).

If \( \tau = ((n_1)^2, (n_1 + 1)^{k-2}) \) then \( b = i_0 (n_1 + 1) - 2, c = (k - i_0) (n_1 + 1) - 2 \) and \( \gcd(b + 2, c + 2) \geq n_1 + 1 \geq 5 \), contrary to (3). Hence we may assume that \( k \geq 6 \) and \( n_3 = n_1 \). Since \( r = 3 \), we obtain that \( d = 0 \) and hence \( i_0 \geq 4 \). If \( i_0 = k - 2 \) then \( 4n_1 + 1 \leq b \leq c = 2n_1 \), a contradiction. Hence \( i_0 \leq k - 3 \) and \( k \geq 7 \).

Since \( r = 3 \), we have \( n_{i_0} = n_k = n_1 + 1 \) and especially \( n_{k-3} = n_1 + 1 \). Then, similarly to the proof of Theorem 2.1, we obtain a realization of \( \tau \) in \( S(4, b, c) \).

**Corollary 2.3.** The number of arbitrarily vertex decomposable spiders \( S(2, 3, b, c) \) and \( S(4, b, c) \) is infinite.

**Proof.** It is not difficult to check that for \( b \) and \( c \) such that \( b \in \{60s + 1, 60s + 13, 60s + 49, s \geq 0\} \), \( c = b + 3 \) the assumptions (1)–(5) of Theorem 2.1 and assumptions (1)–(5) of Theorem 2.2 hold.

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**REFERENCES**


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