S. Catada-Ghimire, H. Roslan, Y.H. Peng

ON CHROMATIC EQUIVALENCE OF A PAIR OF $K_4$-HOMEOMORPHS

Abstract. Let $P(G, \lambda)$ be the chromatic polynomial of a graph $G$. Two graphs $G$ and $H$ are said to be chromatically equivalent, denoted $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. We write $[G] = \{H | H \sim G\}$. If $[G] = \{G\}$, then $G$ is said to be chromatically unique. In this paper, we discuss a chromatically equivalent pair of graphs in one family of $K_4$-homeomorphs, $K_4(1,2,8,d,e,f)$. The obtained result can be extended in the study of chromatic equivalence classes of $K_4(1,2,8,d,e,f)$ and chromatic uniqueness of $K_4$-homeomorphs with girth 11.

Keywords: chromatic polynomial, chromatic equivalence, $K_4$-homeomorphs.

Mathematics Subject Classification: 05C15.

1. INTRODUCTION

All graphs considered here are simple graphs. For such a graph $G$, let $P(G, \lambda)$ (or simply $P(G)$) denote the chromatic polynomial of $G$. Two graphs $G$ and $H$ are chromatically equivalent (or simply $\chi$-equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$ (or simply $P(G) = P(H)$). A graph $G$ is chromatically unique (or simply $\chi$-unique) if for any graph $H$ such that $H \sim G$, we have $H \cong G$, i.e., $H$ is isomorphic to $G$. A $K_4$-homeomorph is a subdivision of the complete graph $K_4$. Such a homeomorph is denoted by $K_4(a,b,c,d,e,f)$ if the six edges of $K_4$ are replaced by the six paths of length $a$, $b$, $c$, $d$, $e$, $f$, respectively, as shown in Figure 1. So far, the chromaticity of $K_4$-homeomorphs with girth $g$, where $3 \leq g \leq 9$ has been studied by many authors (see [5, 9–11, 18]). In 2004, Peng in [9] published her work on the chromaticity of $K_4$-homeomorphs with girth six by considering her result on the chromatic equivalence pair $K_4(1,2,3,d,e,f)$ and $K_4(1,2,3,d',e',f')$. Dong et. al in [6] summarized the above result. In 2008, Peng [11] investigated the chromatic uniqueness of $K_4(1,3,3,d,e,f)$ with exactly one path of length one and with girth seven. She accomplished this, first by establishing the chromatic equivalence pair of $K_4(1,3,3,d,e,f)$ and $K_4(1,3,3,d',e',f')$ in [12]. She then solved the chromatic equivalence of such families of graphs (see [12–14]) and finally, in [11], she provided the
necessary and sufficient condition for this type of $K_4$-homeomorph to be chromatically unique. S. Catada-Ghimire et al. in [1] investigated the chromaticity of one family of $K_4$-homeomorph with girth 10. For the purpose of completing their on going research on $K_4$-homeomorphs with the said girth, they published their results on three chromatic equivalence pairs of $K_4$-homeomorphs in [2, 3] and [4] which are summarised as follows:

Let $G = K_4(1, b, c, d, e, f)$ and $H = K_4(1, b, c, d', e', f')$ be non-isomorphic but chromatically equivalent. Then $\{G, H\}$ is one of the following pairs:

when $b = b' = 2$ and $c = c' = 7$

$$\{K_4(1, 2, 7, i, i + 8, i + 1), K_4(1, 2, 7, i + 2, i, i + 7)\},$$

$$\{K_4(1, 2, 7, i, i + 1, i + 8), K_4(1, 2, 7, i + 7, i, i + 2)\},$$

$$\{K_4(1, 2, 7, i, i + 1, i + 3), K_4(1, 2, 7, i + 2, i + 2, i)\},$$

when $b = b' = 3$ and $c = c' = 6$

$$\{K_4(1, 3, 6, i, i + 1, i + 4), K_4(1, 3, 6, i + 2, i + 3, i)\},$$

$$\{K_4(1, 3, 6, i, i + 7, i + 1), K_4(1, 3, 6, i + 2, i, i + 6)\},$$

when $b = b' = 4$ and $c = c' = 5$

$$\{K_4(1, 4, 5, i, i + 6, i + 1), K_4(1, 4, 5, i + 2, i, i + 5)\},$$

$$\{K_4(1, 4, 5, i, i + 1, i + 5), K_4(1, 4, 5, i + 2, i + 4, i)\}.$$

Our main aim is to provide a result which can be extended in the study of the chromatic equivalence of $K_4(1, 2, 8, d, e, f)$ (as shown in Fig. 2). Such results are an indispensable tool in the study of the chromatic uniqueness of $K_4$-homeomorphs with girth 11.
2. PRELIMINARY RESULT

In this section, we give the following known result used in the sequel.

Lemma 2.1. Assume that $G$ and $H$ are $\chi$-equivalent. Then:

1. $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ (see [7]).
2. $G$ and $H$ have the same girth and same number of cycles with length equal to their girth (see [15]).
3. If $G$ is a $K_4$-homeomorph, then $H$ must itself be a $K_4$-homeomorph (see [16]).
4. Let $G = K_4(a, b, c, d, e, f)$ and $H = K_4(a', b', c', d', e', f')$, then:
   i. $\min \{a, b, c, d, e, f\} = \min \{a', b', c', d', e', f'\}$ and the number of times that this minimum occurs in the list $\{a, b, c, d, e, f\}$ is equal to the number of times that this minimum occurs in the list $\{a', b', c', d', e', f'\}$ (see [17]);
   ii. if $\{a, b, c, d, e, f\} = \{a', b', c', d', e', f'\}$ as multisets, then $H \cong G$ (see [18]).

3. MAIN RESULT

Lemma 3.1. Let $G \cong K_4(1, 2, 8, d, e, f)$ and $H \cong K_4(1, 2, 8, d', e', f')$, then:

1. $P(G) = (-1)^{x-1}|s/(s-1)^2| \times [-s^x - s^{x-1} - 3s - 2s + 2 + R(G)]$, where
   $R(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{e+2} + s^{e+3} + s^{e+8} + s^{f+9} + s^{d+10} + s^{d+e+f}$,
   $s = 1 - \lambda$, $x$ is the number of edges of $G$.
2. If $P(G) = P(H)$, then $R(G) = R(H)$.

Proof. (1) Let $s = 1 - \lambda$. From [17], the chromatic polynomial of $K_4$-homeomorphs $K_4(a, b, c, d, e, f)$ is as follows:

\[
P(K_4(a, b, c, d, e, f) = (-1)^{x-1}|s/(s-1)^2|[(s^2 + 3s + 2) - (s + 1)(s^a + s^b + s^c + s^d + s^e + s^f) + (s + 3s + s^b + s^c + s^d + s^e + s^f)]
\]

So when $a = 1, b = 2$, and $c = 8$, we have

\[
P(K_4(1, 2, 8, d, e, f) = (-1)^{x-1}|s/(s-1)^2|[(s^2 + 3s + 2) - (s + 1)(s + s^2 + s^8 + s^d + s^e + s^f)]
\]
Theorem 3.2. Let $K_d$-homeomorphs $K_d(1,2,8,d,e,f)$ and $K_d(1,2,8,d',e',f')$ be chromatically equivalent, then we have
\[
K_d(1,2,8,i,i+9,i+1) \sim K_d(1,2,8,i+2,i,i+8),
K_d(1,2,8,i,i+9) \sim K_d(1,2,8,i+8,i,i+2),
K_d(1,2,8,i,i+1,i+3) \sim K_d(1,2,8,i+2,i,2,i),
\]
where $i \geq 1$.

Proof. Let $G \cong K_d(1,2,8,d,e,f)$ and $H \cong K_d(1,2,8,d',e',f')$. We now solve for the equation $R(G) = R(H)$ to find $G$ and $H$ which are not isomorphic. From Lemma 3.1, we have
\[
R(G) = -s^d - s^e - s^f - s^{i+1} - s^{i+1} + s^f + 2 + s^e + s^d + 8 + s^{d/e + f},
R(H) = -s^d - s^e - s^f - s^{i+1} - s^{i+1} + s^f + 2 + s^e + s^d + 8 + s^{d/e + f}.
\]

Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively. From Lemma 2.1 (1), $d + e + f = d' + e' + f'$. We obtain the following after simplification: (Note that our assumption in the following steps of the proof is $R_j(G) = R_j(H)$, where $1 \leq j \leq 18$.)
\[
R_j(G) = -s^d - s^e - s^f - s^{i+1} - s^{i+1} + s^f + 2 + s^e + s^d + 8 + s^{d/e + f},
R_j(H) = -s^d - s^e - s^f - s^{i+1} - s^{i+1} + s^f + 2 + s^e + s^d + 8 + s^{d/e + f}.
\]

Let us consider the h.r.p. in $R_j(G)$ and the h.r.p. in $R_j(H)$. We have max \{e + 8, f + 9, d + 10\} = max \{e' + 8, f' + 9, d' + 10\}. Without loss of generality, we will consider only the following six cases.

Case 1. If max \{e + 8, f + 9, d + 10\} = e + 8 and max \{e' + 8, f' + 9, d' + 10\} = e' + 8, then $e = e'$. Thus, we can cancel the following pairs of terms in the equations $R_j(G)$ and $R_j(H)$:
\[
- s^e \text{ with } - s^e, \quad - s^{i+1} \text{ with } - s^{i+1}, \quad s^d + 8 \text{ with } s^d + s^e + s^d + 8.
\]

Therefore, the l.r.p. in $R_j(G)$ is $d$ or $f$ and the l.r.p. in $R_j(H)$ is $d'$ or $f'$. So, $d = f'$ or $d = d'$ or $f = f'$ or $f = f'$. We have $e = e'$ and $d + e + f = d' + e' + f'$. So, we know that \{d, e, f\} = \{d', e', f'\} as multisets. From Lemma 2.1 (4(ii)), $G \cong H$.

Case 2. If max \{e + 8, f + 9, d + 10\} = f + 9 and max \{e' + 8, f' + 9, d' + 10\} = f' + 9, then $f = f'$. We can deal with this case in the same way as case 1, thus, $G \cong H$.

Case 3. If max \{e + 8, f + 9, d + 10\} = d + 10 and max \{e' + 8, f' + 9, d' + 10\} = d' + 10, then we can deal with this case in the same way as case 1. So, we have $G \cong H$.

Case 4. If max \{e + 8, f + 9, d + 10\} = e + 8 and max \{e' + 8, f' + 9, d' + 10\} = f' + 9, then $e + 8 = f' + 9$, that is
\[
f' = e - 1
\]
(3.1)
from $d + e + f = d' + e' + f'$, we have

$$d + f = d' + e' - 1.$$  \hspace{1cm} (3.2)

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. From Lemma 2.1(4(i)), min \{d, e, f\} = min \{d', e', f'\}. Without loss of generality, let min \{d, e, f\} = d. The following subcases need to be considered.

**Subcase 4.1.** If min \{d, e, f\} = d and min \{d', e', f'\} = d', then $d = d'$. Thus, we can consider this case the same way as case 1. So, $G \cong H$.

**Subcase 4.2.** If min \{d, e, f\} = d and min \{d', e', f'\} = e', then $d = e'$. From Eq. (3.2), we have $d' = f + 1$. Note that $f' = e - 1$ (Eq. (3.1)). We can write $R_1(G)$ and $R_1(H)$ as follows:

$$R_1(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+2} + s^{e+3} + s^{f+9} + s^{d+10}$$

$$R_1(H) = -s^{f+1} - s^d - s^{e-1} - s^{d+1} - s^{e+1} + s^{d+3} + s^{d+8} + s^{e+8} + s^{f+11}.$$  

After simplifying $R_2(G)$ and $R_2(H)$, we have

$$R_2(G) = -s^d - s^e + s^{f+2} + s^{e+3} + s^{f+9} + s^{d+10}$$

$$R_2(H) = -s^{e-1} - s^{d+1} + s^{e+1} + s^{d+3} + s^{d+8} + s^{f+11}.$$  

Consider the term $-s^{d+1}$ in $R_3(H)$. Since the min $d, e, f = d$, $-s^{d+1}$ cannot be cancelled by any of the positive terms in $R_3(H)$. Thus, $-s^{d+1}$ must be equal to $-s^f$ or $-s^{e+1}$ in $R_3(G)$. Note that max $e + 8, f + 9, d + 10 = e + 8$, so $e + 8 \geq d + 10$, that is, $e + 1 \geq d + 3 > d + 1$. Thus, $-s^{e+1} \neq -s^{d+1}$.

If $-s^{d+1} = -s^f$, then $d + 1 = f$. Thus, $R_3(G)$ and $R_3(H)$ can be written as follows:

$$R_3(G) = -s^d + s^e + s^{d+3} + s^{e+3} + s^{d+10}$$

$$R_3(H) = -s^{e-1} - s^{d+1} + s^{e+1} + s^{d+3} + s^{d+8} + s^{d+12}.$$  

After simplifying $R_4(G)$ and $R_4(H)$, we have

$$R_4(G) = -s^e + s^{d+3} + s^{d+10} + s^{d+12}$$

$$R_4(H) = -s^{e-1} + s^{d+1} + s^{d+3} + s^{d+8} + s^{d+12}.$$  

Thus, we have

$$-s^e + s^{d+3} + s^{d+10} + s^{d+12} = -s^{e-1} + s^{d+1} + s^{d+8} + s^{d+12}.$$  

Therefore, we have $e = d + 9$. At this point, we acquire the following equations:

$$e = d + 9, f' = e - 1 = d + 8, d' = f + 1 = d + 2, e' = d.\text{ Let } d = i.\text{ Therefore, we obtain the solution, where } G \text{ is isomorphic to } K_4(1, 2, 8, i, i + 9, i + 1) \text{ and } H \text{ is isomorphic to } K_4(1, 2, 8, i + 2, i, i + 8).$$

**Subcase 4.3.** If min \{d, e, f\} = d and min \{d', e', f'\} = f', then $d = f'$. Note that max \{e' + 8, f' + 9, d' + 10\} = $f' + 9$. So, $f' + 9 \geq d' + 10$. This contradicts min \{d', e', f'\} = $f'$.

**Case 5.** If max \{e + 8, f + 9, d + 10\} = $f + 9$ and max \{e' + 8, f' + 9, d' + 10\} = $d' + 10$, then $f + 9 = d' + 10$, that is,

$$d' = f + 1$$  \hspace{1cm} (3.3)

from $d + e + f = d' + e' + f'$, we have

$$e + d + 1 = e' + f'.\hspace{1cm} (3.4)$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$, where min \{d, e, f\} = min \{d', e', f'\}. Without loss of generality, let min \{d, e, f\} = d. The following subcases need to be considered.
Subcase 5.1. If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = d' \), then we deal with this case the same way with case 1. So, we get \( G \cong H \).

Subcase 5.2. If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = e' \), then \( d = e' \). From Eq. (3.4), we have \( f' = e + 1 \). Thus, we can write \( R_1(G) \) and \( R_1(H) \) as follows:

\[
R_1(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+2} + s^{e+3} + s^{f+9} + s^{d+10},
\]

\[
R_1(H) = -s^{f-1} - s^d - s^{e+1} - s^{d+1} - s^{e+2} + s^{e+3} + s^{d+3} + s^{d+8} + s^{e+10} + s^{f+9}.
\]

After simplifying \( R_1(G) \) and \( R_1(H) \), we have

\[
R_1(G) = -s^d - s^{e+1} + s^{e+2} + s^{e+3} + s^{d+8} + s^{e+10},
\]

\[
R_1(H) = -s^{f-1} - s^{d+1} - s^{e+2} + s^{d+3} + s^{d+8} + s^{e+10}.
\]

Consider the term \(-s^{d+1}\) in \( R_1(H) \). Since \( \max \{e + 8, f + 9, d + 10\} = f + 9 \), we have \( f + 9 \geq d + 10 \), that is, \( f + 1 \geq d + 2 \). So, \( f + 1 \neq d + 1 \). Thus, \(-s^{d+1}\) in \( R_1(H) \) must be equal to \(-s^e\) or \(-s^f\) in \( R_1(G)\). If \(-s^{d+1} = -s^f\), then \( d + 1 = f \).

From Eq. (3.3), we have \( d = d' \) and

\[
R_1(G) = -s^d - s^{e+1} - s^{d+2} + s^{e+3} + s^{d+8} + s^{d+10},
\]

\[
R_1(H) = -s^{f-1} - s^{d+1} - s^{e+2} + s^{d+3} + s^{d+8} + s^{d+11}.
\]

It is easy to see that \( d = e \). Note that \( d = e' \), so \( e = e' \). From \( d + e + f = d' + e' + f' \), we have \( f = f' \). Thus, \( G \cong H \).

If \(-s^{d+1} = -s^e\), then \( d + 1 = e \) and

\[
R_1(G) = -s^d - s^{f-1} - s^f - s^{f+1} + s^{f+2} + s^{d+9} + s^{d+10},
\]

\[
R_1(H) = -s^{f-1} - s^{d+1} - s^{e+2} + s^{d+3} + s^{d+8} + s^{d+11}.
\]

After simplifying, we have

\[
-s^d - s^{f-1} + s^{f+2} + s^{d+9} + s^{d+10} = -s^{f-1} + s^{d+8} + s^{d+11},
\]

Thus, we have \( f = d + 9 \).

Subcase 5.3. If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = f' \), then \( d = f' \). From Eq. (3.4), \( e' = e + 1 \). Note that Eq. (3.3) is \( f' = d + 1 \). We can write \( R_1(G) \) and \( R_1(H) \) as follows:

\[
R_{10}(G) = -s^d - s^e - s^f - s^{f+1} - s^{f+2} + s^{e+3} + s^{f+9} + s^{d+10},
\]

\[
R_{10}(H) = -s^f - s^d - s^e - s^{e+1} - s^{d+2} + s^{e+4} + s^{e+9} + s^{d+9} + s^{f+9}.
\]

After simplifying \( R_{10}(G) \) and \( R_{10}(H) \), we have

\[
R_{11}(G) = -s^d - s^f - s^{f+1} + s^{f+2} + s^{e+3} + s^{e+9} + s^{d+10},
\]

\[
R_{11}(H) = -s^f - s^d - s^e - s^{e+1} + s^{e+4} + s^{e+9} + s^{d+10}.
\]

For the same reasons stated in subcase 5.2, \(-s^{d+3} \) must be equal to \(-s^e\) or \(-s^f\) in \( R_{11}(G) \).

If \(-s^{d+3} = -s^f\), then \( d + 1 = e \). We can write \( R_{11}(G) \) and \( R_{11}(H) \) as follows:

\[
R_{12}(G) = -s^d - s^f - s^{f+1} + s^{f+2} + s^{d+4} + s^{d+9} + s^{d+10},
\]

\[
R_{12}(H) = -s^f - s^d - s^{d+3} + s^{d+2} + s^{d+5} + s^{d+10} + s^{d+9}.
\]

After simplifying, we have

\[
-s^d - s^{f+1} + s^{f+2} + s^{d+4} = -s^f - s^{d+3} + s^{d+2} + s^{d+5}.
\]

So, we get \( f = d + 3 \). We also have \( f' = d, e' = e + 1 = d + 2, d' = f - 1 = d + 2 \). Let \( d = i \), then \( e = i + 1, f = i + 3, d' = i + 2, e' = i + 2, f' = i \).

Therefore, we obtain the solution, where \( G \cong K_4(1, 2, 8, i, i + 1, i + 3) \) and \( H \cong K_4(1, 2, 8, i + 2, i + 2, i) \).
Case 6. If max \( \{ e + 8, f + 9, d + 10 \} = e + 8 \) and max \( \{ e' + 8, f' + 9, d' + 10 \} = d' + 10 \), then \( e + 8 = d' + 10 \), that is, \( d' = e - 2 \) \hspace{1cm} (3.5)

from \( d + e + f = d' + e' + f' \), we have

\[ d + f + 2 = e' + f'. \] \hspace{1cm} (3.6)

Consider the l.r.p. in \( R_1(G) \) and the l.r.p. in \( R_1(H) \). We have min \( \{ d, e, f \} = \min \{ d', e', f' \} \). Without loss of generality, let min \( \{ d, e, f \} = d \). The following subcases need to be considered.

Subcase 6.1. If min \( \{ d, e, f \} = d \) and \( \{ d', e', f' \} = d' \), then \( d = d' \) and we can deal with this case the same way as Case 1. Thus, we get \( G \cong H \).

Subcase 6.2. If min \( \{ d, e, f \} = d \) and \( \{ d', e', f' \} = e' \), then \( d = e' \). From Eq. (3.6), we have \( f' = f + 2 \). Thus, we can write \( R_1(G) \) and \( R_1(H) \) as follows:

\[ R_1(G) = -s^d - s^e - s^f - s^{d+1} - s^{f+2} + s^{e+3} + s^{f+9} + s^{d+10}, \]
\[ R_1(H) = -s^{e-2} - s^{f+2} - s^{d+1} - s^{f+4} + s^{d+3} + s^{d+8} + s^{f+11} + s^{e+3}. \]

After simplifying \( R_1(G) \) and \( R_1(H) \), we have

\[ R_{14}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{e+3} + s^f + 9 + s^{d+10}, \]
\[ R_{14}(H) = -s^{e-2} - s^{f+2} - s^{d+1} - s^{f+4} + s^{d+3} + s^{d+8} + s^f + 11. \]

Consider the term \( -s^{d+1} \) in \( R_{14}(H) \). Since min \( \{ d, e, f \} = d \), \( -s^{d+1} \) cannot cancel any negative term in \( R_{14}(H) \). From max \( \{ e + 8, f + 9, d + 10 \} = e + 8 \), we have \( e + 8 \geq d + 10 \), that is \( e + 1 \geq d + 3 > d + 1 \). So, \( -s^{d+1} \neq -s^{e+1} \). Moreover, \( e \geq d + 2 > d + 1 \), thus \( e \neq d + 1 \), that is \( -s^e \neq -s^{d+1} \). So, \( -s^{d+1} \) must be equal to \( -s^f \) or \( -s^{f+1} \) in \( R_{14}(G) \).

If \( -s^{d+1} = -s^{f+1} \), then \( d = f \). So, we have

\[ R_{15}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{e+3} + s^{f+9} + s^{d+10}, \]
\[ R_{15}(H) = -s^{e-2} - s^{f+2} - s^{d+1} - s^{f+4} + s^{d+3} + s^{d+8} + s^{f+11}. \]

After simplifying, consider the h.r.p. in \( R_{15}(G) \) and the h.r.p. in \( R_{15}(H) \). We have \( s^{e+3} = s^{d+11} \), that is \( e + 3 = d + 11 \). This contradicts \( R_{15}(G) = R_{15}(H) \) since \( -s^e \) cannot be cancelled by \( +s^{f+8} \) in \( R_{15}(H) \).

If \( -s^{d+1} = -s^f \), then \( d + 1 = f \). Thus, we have

\[ R_{16}(G) = -s^e - s^{d+1} - s^{e+1} - s^{d+2} + s^{e+3} + s^{d+10} + s^{d+10}, \]
\[ R_{16}(H) = -s^{e-2} - s^{d+3} - s^{d+1} - s^{d+4} + s^{d+5} + s^{d+8} + s^{d+12}. \]

After simplifying, consider the h.r.p. in \( R_{16}(G) \) and the h.r.p. in \( R_{16}(H) \). We have \( s^{d+3} = s^{d+12} \). The term \( s^{d+8} \) in \( R_{16}(H) \) cannot be cancelled since there is no term equal to it. This contradicts \( R_{16}(G) = R_{16}(H) \).

Subcase 6.3. If min \( \{ d, e, f \} = d \) and \( \{ d', e', f' \} = f' \), then \( d = f' \). From Eq. (3.6), \( e' = f + 2 \) and note that from Eq. (3.5), \( d' = e - 2 \). Thus, we have

\[ R_{17}(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+2} + s^{e+3} + s^{f+9} + s^{d+10}, \]
\[ R_{17}(H) = -s^{e-2} - s^{f+2} - s^{d+1} - s^{f+4} + s^{d+3} + s^{d+5} + s^{f+10} + s^{d+9} + s^{e+8}. \]

After simplifying, consider the term \( -s^{d+1} \) in \( R_{17}(H) \). For the same reasons stated in subcase 4.2, \( -s^{d+1} \) can only be equal to \( -s^f \) or \( -s^{f+1} \) in \( R_{17}(G) \).

If \( -s^{d+1} = -s^f \), then \( d + 1 = f \). So, we have

\[ R_{18}(G) = -s^e - s^{d+1} - s^{e+1} - s^{d+2} + s^{d+3} + s^{d+10} + s^{d+10}, \]
\[ R_{18}(H) = -s^{e-2} - s^{d+3} - s^{d+4} - s^{d+1} + s^{d+2} + s^{d+6} + s^{d+11} + s^{d+9}. \]
After simplifying, consider the h.r.p. in $R_{18}(G)$ and the h.r.p. in $R_{18}(H)$. We have $s^{e+3} = s^{d+11}$. So, $e + 3 = d + 11$, thus $e = d + 8$. There is no term $s^{d+8}$ which is equal to the term $s^e$ in $R_{18}(G)$. This contradicts $R_{18}(G) = R_{18}(H)$. If $-s^{d+1} = -s^{f+1}$, then $d + 1 = f + 1$, that is $d = f = f'$. This case is the same as case 1. So, we get the same result $G \cong H$. At this point, we have solved the equation $R(G) = R(H)$ and the solution is as follows:

$$K_4(1, 2, 8, i, i + 3) \sim K_4(1, 2, 8, i + 2, i, i + 2),$$

where $i \geq 1$. The proof is now complete.

\[\square\]

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REFERENCES


[8] W.M. Li, Almost every $K_4$-homeomorphs is chromatically unique, Ars Combin. 23 (1987), 13–36.


On chromatic equivalence of a pair of $K_4$-homeomorphs


[18] W.M. Li, Almost every $K_4$-homeomorphs is chromatically unique, Ars Combin. 23 (1987), 13–36.

S. Catada-Ghimire
aspa777@gmail.com

Universiti Sains Malaysia
School of Mathematical Sciences
11800 Penang, Malaysia

H. Roslan
hroslan@cs.usm.my

Universiti Sains Malaysia
School of Mathematical Sciences
11800 Penang, Malaysia

Y.H. Peng
yhpeng@fsas.upm.edu.my

Universiti Putra Malaysia
Department of Mathematics and Institute for Mathematical Research
43400UPM Serdang, Malaysia

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