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HYPONORMAL DIFFERENTIAL OPERATORS WITH DISCRETE SPECTRUM

Abstract. In this work, we first describe all the maximal hyponormal extensions of a minimal operator generated by a linear differential-operator expression of the first-order in the Hilbert space of vector-functions in a finite interval. Next, we investigate the discreteness of the spectrum and the asymptotical behavior of the modules of the eigenvalues for these maximal hyponormal extensions.

Keywords: hyponormal operators, differential operators, minimal and maximal operators, extension of operators, compact operators, eigenvalues, asymptotes of eigenvalues.

Mathematics Subject Classification: 47A20.

1. INTRODUCTION

The general and spectral theory of linear bounded hyponormal operators in a Hilbert space was founded and developed by P.R. Halmos [8], C.R. Putnam [17], J.G. Stampfli [18, 19], C.R. Williams [20], D. Xia [21].

We know that, all normal extensions and discrete spectrum of the minimal operator generated by a linear differential-operator expression for the first-order in L^2 has been described in terms of boundary conditions in [10–12] and [13]. We work with hyponormal operators instead of normal operators.

A densely defined closed operator N in a Hilbert space \mathcal{H} is called a normal operator if $D(N) = D(N^*)$ and for all $x \in D(N)$ $\|Nx\|_{\mathcal{H}} = \|N^*x\|_{\mathcal{H}}$ (cf. [3]).

A densely defined closed operator T in a Hilbert space \mathcal{H} is called hyponormal if $D(T) \subset D(T^*)$ and for all $x \in D(T)$, $\|T^*x\|_{\mathcal{H}} \leq \|Tx\|_{\mathcal{H}}$.

If a hyponormal operator in \mathcal{H} has no non-trivial hyponormal extension, then it is called a maximal hyponormal operator. It is clear that for the hyponormality of a linear closed operator T in a Hilbert space \mathcal{H} , it is necessary and sufficient to have $D(T) \subset D(T^*)$ and $TT^* \leq T^*T$.

This paper contains two sections. In the first section we investigate all maximal hyponormal extensions of the minimal operator in L^2 in terms of boundary conditions.

In the second section we investigate discreteness of the spectrum and asymptotical behavior of the modules of the eigenvalues for maximal hyponormal extensions of the minimal operator L_0 in L^2 .

Let \mathcal{H} be a separable Hilbert space and let $L^2 = L^2(\mathcal{H}, (a, b))$ be the Hilbert space of vector-functions from the finite interval (a, b) to \mathcal{H} (cf. [7, 9]).

1.1. DESCRIPTION OF MAXIMAL HYPONORMAL EXTENSIONS

In the space L^2 consider a linear differential-operator expression of first order of the form

$$l(u) = u'(t) + Au(t), \quad (1.1)$$

where A is a linear maximal hyponormal operator, $A = A_R + iA_I$, A_R is the real part of A , A_I is the imaginary part of A and A_R is a linear lower positive definite operator in \mathcal{H} . For simplicity, we assume that $A_R \geq E$. E denotes the identical operator in \mathcal{H} .

The formally adjoint expression (1.1) in the Hilbert space L^2 is of the form

$$l(v) = -v'(t) + A^*v(t). \quad (1.2)$$

Let us define the operator L'_0 on the dense L^2 set of vector-functions D'_0 ,

$$D'_0 := \left\{ u(t) \in L^2 : u(t) = \sum_{k=1}^n \varphi_k(t) f_k, \quad \varphi_k \in C_0^\infty(a, b), \quad k = 1, 2, \dots, n, \quad n \in \mathbb{N} \right\},$$

as $L'_0 u = l(u)$. Since the operator $A_R \geq E$, then the L'_0 operator is accretive, that is $Re(L'_0 u, u)_{L^2} \geq 0$, $u \in D'_0$. Hence the operator L'_0 has a closure in L^2 . The closure of L'_0 in L^2 is called the minimal operator, generated by the differential-operator expression (1.1) and is denoted by L_0 .

In a similar way we can construct the minimal operator L_0^+ in L^2 which is generated by the differential-operator expression (1.2) in L^2 . The adjoint operator of L_0^+ (resp. L_0) in L^2 is called the maximal operator, generated by (1.1), (resp. (1.2)) and is denoted by L (resp. L^+) (cf. [1, 7]).

In this section the main purpose is to describe all maximal hyponormal extensions of the minimal operator in L^2 in terms of boundary conditions.

Note that all normal extensions of the minimal operator generated by a linear differential-operator expression for the first-order in L^2 has been described in terms of boundary conditions in [12, 15].

Lemma 1.1. *T is a hyponormal operator in a Hilbert space \mathcal{H} if and only if the following two condition hold:*

- (i) $D(T) \subset D(T^*)$,
- (ii) $Im(T_R x, T_I x) \geq 0$,

where $T_R = \frac{1}{2}(\overline{T + T^*})$ and $T_I = \frac{1}{2i}(\overline{T - T^*})$.

Proof. Let us consider T , a hyponormal operator in \mathcal{H} . Hence for all $x \in D(T)$

$$\begin{aligned} Tx &= T_Rx + iT_Ix \in \mathcal{H}, \\ T^*x &= T_Rx - iT_Ix \in \mathcal{H}, \end{aligned}$$

then $T_Rx \in \mathcal{H}$, $T_Ix \in \mathcal{H}$. On the other hand, for all $x \in D(T)$,

$$\|T^*x\|_{\mathcal{H}} \leq \|Tx\|_{\mathcal{H}}.$$

From this inequality, we can easily show that

$$\text{Im}(T_Rx, T_Ix) \geq 0.$$

Conversely, if $D(T) \subset D(T^*)$ and $\text{Im}(T_Rx, T_Ix) \geq 0$ for all $x \in D(T)$, then it follows immediately from the obvious relation,

$$4\text{Im}(T_Rx, T_Ix)_{\mathcal{H}} = \|Tx\|^2 - \|T^*x\|^2 \text{ for all } x \in D(T).$$

This completes the proof of the theorem. \square

Theorem 1.2. *If the minimal operator L_0 has at least one hyponormal extension in L^2 , then the minimal operator L_0 is hyponormal in L^2 .*

Proof. Let L_h be a hyponormal extension of, that is, $L_0 \subset L_h \subset L$, then from the condition $D(L_h) \subset D(L_h^*)$ and following relation

$$D(L_0) \subset D(L_h) \subset D(L_h^*) \subset D(L^+) = D(L_0^*)$$

we obtain $D(L_0) \subset D(L_0^*)$.

On the other hand, from the inequality $\|L_h^*x\| \leq \|L_hx\|$, $x \in D(L_h)$ for any $u(t) \in D(L_0)$ we have

$$\|L_h^*u\|_{L^2} = \|L_0^*u\|_{L^2} \leq \|L_hu\|_{L^2} = \|L_0u\|_{L^2},$$

that is, $\|L_0^*u\|_{L^2} \leq \|L_0u\|_{L^2}$, $u(t) \in D(L_0)$. \square

Theorem 1.3. *Let A be a linear closed densely defined operator in \mathcal{H} . If the minimal operator L_0 generated by the differential-operator expression $l(u) = u'(t) + Au(t)$ in L^2 is a hyponormal operator, then the operator A is hyponormal in \mathcal{H} .*

Proof. Existence of the minimal operator L_0 in L^2 follows from the result in [7]. On the other hand, since $D(L_0) \subset D(L_0^*) = D(L^+)$ and for vector-functions

$$u(t) = \varphi(t)f, \quad \varphi(t) \in W_2^1, \quad f \in D(A),$$

$$W_2^1(\mathcal{H}, (a, b)) := \{u(t) : u(t) \in L^2, u'(a) = u'(b) = 0\},$$

belonging to $D(L_0)$ and $u(t) \in D(L^+)$, we have

$$L_0^+u = -\varphi'(t)f + \varphi(t)A^*f \in L^2(\mathcal{H}, (a, b)).$$

It follows that $f \in D(A^*)$. Hence

$$D(A) \subset D(A^*). \quad (1.3)$$

From the second condition of hyponormality of the operator L_0 , we have

$$\|L_0^*u\|_{L^2} \leq \|L_0u\|_{L^2}, u(t) \in D(L_0).$$

For the special case of vector-valued functions

$$u(t) = \varphi(t)f, \varphi(t) \in W_2^1(a, b);$$

$f \in D(A)$ in $D(L_0)$ from the last inequality, we have

$$\begin{aligned} \|A^*f\|_{\mathcal{H}}^2 \int_a^b |\varphi(t)|^2 dt &\leq 2(f, A_R f) \left[\int_a^b [\varphi'(t)\overline{\varphi(t)} + \varphi(t)\overline{\varphi'(t)}] dt + \right. \\ &\quad \left. + \|Af\|_{\mathcal{H}}^2 \int_a^b |\varphi(t)|^2 dt \right] \end{aligned}$$

and

$$\begin{aligned} \|A^*f\|_{\mathcal{H}}^2 \int_a^b |\varphi(t)|^2 dt &\leq 2(f, A_R f) \int_a^b (\varphi(t)\overline{\varphi(t)})' dt + \|Af\|_{\mathcal{H}}^2 \int_a^b |\varphi(t)|^2 dt \leq \\ &\leq 2(f, A_R f) (\varphi(t)\overline{\varphi(t)}) \Big|_a^b + \|Af\|_{\mathcal{H}}^2 \int_a^b |\varphi(t)|^2 dt \leq \\ &\leq 2(f, A_R f) [|\varphi|^2(b) - |\varphi|^2(a)] + \|Af\|_{\mathcal{H}}^2 \int_a^b |\varphi(t)|^2 dt \end{aligned}$$

for $\varphi(t) \in W_2^1(a, b)$. Then

$$\int_a^b |\varphi(t)|^2 dt \|A^*f\|_{\mathcal{H}}^2 \leq \int_a^b |\varphi(t)|^2 dt \|Af(t)\|_{\mathcal{H}}^2.$$

Choosing function $\varphi(t) \in W_2^1(a, b)$ with property $\int_a^b |\varphi(t)|^2 dt \neq 0$, from the last relation we obtain,

$$\|A^*f\|_{\mathcal{H}} \leq \|Af\|_{\mathcal{H}}, \quad f \in D(A). \quad (1.4)$$

Hence from (1.3) and (1.4), it is established that operator A is hyponormal in \mathcal{H} . \square

Corollary 1.4. *If the minimal operator L_0 generated by the differential-operator expression $l(u) = u'(t) + Au(t)$ with a linear closed densely defined operator in \mathcal{H} is a normal operator in L^2 , then the operator A is normal in \mathcal{H} (see also [10, 11, 16]).*

Note that furthermore we will take the operator A as a normal operator in \mathcal{H} .

In a similar manner, we can construct the minimal M_0 and the maximal operator M corresponding to the differential-operator expression

$$m(u) = u'(t) + A_R u(t),$$

in the Hilbert space L^2 of vector functions.

Let us introduce the following operator.

$$\begin{cases} U_t'(t, s) f + iA_t U(t, s) f = 0, \\ U(s, s) f = f, \quad f \in D(A), \quad t, s \in [a, b]. \end{cases}$$

The operator $U(t, s)$, $t, s \in [a, b]$, is linear continuous bounded invertible unitary operator in \mathcal{H} and $U^*(t, s) = U(s, t)$, $U^{-1}(t, s) = U(s, t)$ (for detailed analysis of these operators see [2] and [14]).

$$Uz(t) := U(t, a)z(t), \quad U : L^2 \rightarrow L^2.$$

In this case it is easy to see that, for the differentiable vector-function $z(t) \in L^2$ with $z(t) \in D(A)$, $t \in [a, b]$, the following relation holds:

$$\begin{aligned} l(Uz) &= (Uz)'(t) + A(t)Uz(t) = U(z'(t) + A_R z(t) + (U_t' + iA_I(t)U)z(t)) \\ &= Um(z) \in L^2. \end{aligned}$$

From this, then we have

$$U^{-1}lU(z) = m(z).$$

It is clear that, if the operator \tilde{L} is an extension of the minimal operator L_0 , that is, $L_0 \subset \tilde{L} \subset L$, then

$$U^{-1}L_0U = M_0, \quad M_0 \subset U^{-1}\tilde{L}U = \tilde{M} \subset M, \quad U^{-1}LU = M. \quad (1.5)$$

For example we will prove the validity of relation (1.5).

It is known that

$$D(M) = \{u(t) \in L^2 : u(t) \text{ absolutely continuous on } (a, b), m(u) \in L^2\}$$

and

$$D(M_0) = \{u(t) \in D(M) : u(a) = u(b) = 0\}.$$

If $u(t) \in D(M)$, then in this case $Uu(t)$ is absolutely continuous on (a, b) and

$$l(Uz) = (Uz)'(t) + A(t)Uz(t) = Um(z) + (U_t' + iA_I(t)U)z(t) = Um(z) \in L^2, \quad (1.6)$$

that is, $Uu(t) \in D(L)$. Furthermore, from the relation (1.6) we infer that $M \subset U^{-1}LU$.

Contrary, if the vector-function $v(t) \in D(L)$, then the element $U^{-1}v(t)$ is absolutely continuous on (a, b) and

$$\begin{aligned} m(U^{-1}v(t)) &= (U^{-1}v(t))' + A_R(U^{-1}v(t)) = \\ &= U^{-1}[v'(t) + A_R v(t) + iA_I v(t)] = U^{-1}l(v(t)) \in L^2, \end{aligned} \quad (1.7)$$

that is, $U^{-1}v(t) \in D(M)$, and from the relation (1.7)

$$U^{-1}L \subset MU^{-1}, \quad U^{-1}LU \subset M.$$

Hence $U^{-1}LU = M$. Therefore, operator U is a one to one map of $D(M)$ onto $D(L)$.

Here we define a Hilbert scale $\mathcal{H}_j(T)$, $-\infty < j < +\infty$ of the spaces constructed via the operator T^j . Let $\mathcal{H} = \mathcal{H}_0$ be a Hilbert space over the field of complex numbers with inner product $(\cdot, \cdot)_{\mathcal{H}_0}$ and norm

$$\|f\|_{\mathcal{H}_0} = (f, f)_{\mathcal{H}_0}^{1/2}, \quad f \in \mathcal{H}_0.$$

Let T be a linear self-adjoint operator on the Hilbert space \mathcal{H} such that

$$\|Tf\|_{\mathcal{H}_0} \geq \|f\|_{\mathcal{H}_0}.$$

The set $D(T^j)$, $0 < j < +\infty$, under an inner product

$$(f, g)_{\mathcal{H}_{+j}} := (T^j f, T^j g)_{\mathcal{H}_0}, \quad f, g \in D(T^j)$$

is a Hilbert space. We define $\mathcal{H}_{+j} := \mathcal{H}_{+j}(T)$, $0 < j < +\infty$, and it is called a positive space. In a similar way we have $\mathcal{H}_{-j} := \mathcal{H}_{-j}(T)$, $0 < j < +\infty$, and this is called a negative space. It is clear that

$$\mathcal{H}_{+\tau} \subset \mathcal{H}_{+j}, \quad 0 < \tau < j < \infty, \quad \mathcal{H}_{+j} \subset \mathcal{H} = \mathcal{H}_0 \subset \mathcal{H}_{-j}, \quad \mathcal{H}_{+j}^* = \mathcal{H}_{-j}, \quad 0 < j < \infty,$$

and \mathcal{H}_{+j} , $0 < j < \infty$ is dense in \mathcal{H} (for a more detailed analysis of the spaces \mathcal{H}_j , $-\infty < j < +\infty$, see [6] and [7]).

Let $W_2^1(\mathcal{H}, (a, b))$ ($W_2^0(\mathcal{H}, (a, b))$) be the Sobolev space of vector-functions from the finite interval (a, b) into \mathcal{H} (see [7]).

Theorem 1.5. *If the minimal operator M_0 is a hyponormal operator in L^2 , then*

$$D(M_0) \subset W_2^1(\mathcal{H}, (a, b)), \\ A_R D(M_0) \subset L^2(\mathcal{H}, (a, b)).$$

Proof. Indeed, in this case for any vector-functions $u(t)$ from $D(M_0)$ we have

$$u' + A_R u \in L^2(\mathcal{H}, (a, b)), \\ -u' + A_R u \in L^2(\mathcal{H}, (a, b)).$$

From these relations

$$u'(t) \in L^2(\mathcal{H}, (a, b)), \\ A_R u(t) \in L^2(\mathcal{H}, (a, b)), \quad u(t) \in D(M_0) \quad \text{and} \quad u(a) = u(b) = 0,$$

we obtain

$$u(t) \in W_2^1(\mathcal{H}, (a, b)), \\ A_R D(M_0) \subset L^2(\mathcal{H}, (a, b)).$$

□

Theorem 1.6. *If the minimal operator A is normal in \mathcal{H} and the following condition holds*

$$A_R W_2^1(\mathcal{H}, (a, b)) \subset L^2(\mathcal{H}, (a, b)),$$

then the minimal operators M_0 and L_0 are hyponormal in L^2 .

Proof. First, we will prove hyponormality of M_0 in L^2 . Under these conditions for each $u(t) \in D(M_0) \subset W_2^1(\mathcal{H}, (a, b))$, we have

$$\begin{aligned} M_0^* u &= -u'(t) + A_R u(t) = \\ &= -(u'(t) + A_R u(t)) + 2A_R u(t) \in L^2(\mathcal{H}, (a, b)). \end{aligned}$$

That is, $D(M_0) \subset D(M_0^*)$.

On the other hand, for each $u(t) \in D(M_0)$, we have

$$\begin{aligned} \|M_0 u\|_{L^2}^2 - \|M_0^* u\|_{L^2}^2 &= (u'(t) + A_R u, u'(t) + A_R u)_{L^2} - \\ &\quad - (-u'(t) + A_R u, -u'(t) + A_R u)_{L^2} = \\ &= \|u'\|_{L^2}^2 + (u', A_R u)_{L^2} + (A_R u, u')_{L^2} + \|A_R u\|_{L^2}^2 - \|u'\|_{L^2}^2 + \\ &\quad + (u', A_R u)_{L^2} + (A_R u, u')_{L^2} - \|A_R u\|_{L^2}^2 = \\ &= 2[(u', A_R u)_{L^2} + (A_R u, u')_{L^2}] = 2(u, A_R u)_{\mathcal{H}^1_a} = 0, \end{aligned}$$

that is, $\|M_0^* u\| \leq \|M_0 u\|$ for each $u(t) \in D(M_0)$. Hence, the operator M_0 is hyponormal in L^2 . Now we will prove hyponormality of L_0 in L^2 . From the following properties

$$L_0 = U M_0 U^{-1}, \quad L_0^* = U M_0^* U^{-1}$$

and

$$D(L_0) = D(U M_0 U^{-1}) \subset D(U M_0^* U^{-1}) = D(L_0^*)$$

we have $D(L_0) \subset D(L_0^*)$. Furthermore, for each $u(t) \in D(L_0)$

$$\begin{aligned} \|L_0^* u\|_{L^2}^2 &= \|U M_0^* U^{-1} u\|_{L^2}^2 = (U M_0^* U^{-1} u, U M_0^* U^{-1} u)_{L^2} = \\ &= (M_0^* U^{-1} u, U^* U M_0^* U^{-1} u)_{L^2} = \\ &= (M_0^*(U^{-1} u), M_0^*(U^{-1} u))_{L^2} = \|M_0^*(U^{-1} u)\|_{L^2}^2 \leq \\ &\leq \|M_0^*(U^{-1} u)\|_{L^2}^2 = (M(U^{-1} u), M_0(U^{-1} u))_{L^2} = \\ &= (U^* U M_0 U^{-1} u, M_0 U^{-1} u)_{L^2} = (U M_0 U^{-1} u, U M_0 U^{-1} u)_{L^2} \leq \|L_0 u\|_{L^2}^2. \end{aligned}$$

Therefore, operator L_0 is hyponormal. \square

Theorem 1.7. *Let $A_R^{1/2}[D(L) \cap D(L^+)] \subset W_2^1(\mathcal{H}, (a, b))$. Each hyponormal extension L_h of the minimal operator L_0 in L^2 is generated by the differential-operator expression (1.1) with the following boundary condition,*

$$u(a) = V U(a, b) u(b), \tag{1.8}$$

where V is isometric and $A_R^{1/2}VA_R^{-1/2}$ is also a contraction operators in \mathcal{H} . The isometric operator V is determined uniquely by the extensions L_h , i.e. $L_h = L_V$.

Contrary, the restriction of the maximal operator L to the manifold of vector-functions $u(t) \in D(L) \cap D(L^+)$ that satisfies condition (1.8) for some isometric operator V , where $A_R^{1/2}VA_R^{-1/2}$ is also contraction operator in \mathcal{H} , is a maximal hyponormal extension of the minimal operator L_0 in the space L^2 .

Proof. First, we will describe all maximal hyponormal extensions M_h of the minimal operator M_0 in L^2 in terms of boundary values.

Let M_h be a maximal hyponormal extension of M_0 . In this case, for every $u(t) \in D(M_h)$, we have

$$\begin{aligned} M_h u &= u'(t) + A_R u(t) \in L^2, \\ M_h^* u &= -u'(t) + A_R u(t) \in L^2. \end{aligned}$$

From this relation we find that $u'(t) \in L^2$ and $A_R u(t) \in L^2$. In other words, $D(M_h) \subset W_2^1(\mathcal{H}, (a, b))$ and $A_R D(M_h) \subset L^2$.

On the other hand, if $u(t) \in D(M_h) \subset D(M_h^*)$, then we have representations,

$$\begin{aligned} u(t) &= e^{-A_R(t-a)} f + \int_a^t e^{-A_R(t-s)} (M_h u)(s) ds, \\ u(t) &= e^{A_R(t-b)} g + \int_t^b e^{A_R(t-s)} (M_h^* u)(s) ds, \end{aligned}$$

where $f, g \in \mathcal{H}_{-1/2}(A_R)$. Hence every $u(t) \in D(M_h)$ has the property $u(t) \in C(\mathcal{H}_{+1/2}, [a, b])$ (see [7]). Furthermore, from the relation

$$(M_h u, v)_{L^2} = (u(b), v(b))_{\mathcal{H}} - (u(a), v(a))_{\mathcal{H}} + (u, M_h^* v)_{L^2},$$

which holds for every $u(t) \in D(M_h)$ and $v(t) \in D(M_h^*)$, we have

$$\|u(b)\|_{\mathcal{H}} = \|u(a)\|_{\mathcal{H}}.$$

Then there exists an isometric operator V in \mathcal{H} such that

$$u(a) = V u(b). \quad (1.9)$$

On the other hand, for any $u(t) \in D(M_h)$ from the second condition of hyponormality of the extensions M_h we have

$$\begin{aligned} \|M_h^* u\|_{L^2}^2 - \|M_h u\|_{L^2}^2 &= (-u' + A_R u, -u' + A_R u)_{L^2} - (u' + A_R u, u' + A_R u)_{L^2} = \\ &= -2[(u', A_R u)_{L^2} + (A_R u, u')_{L^2}] = -2(u, A_R u)_{\mathcal{H}} I_a^b = \\ &= -2[(u(b), A_R u(b))_{\mathcal{H}} - (u(a), A_R u(a))_{\mathcal{H}}] = \\ &= 2[\|A_R^{1/2} u(a)\|_{\mathcal{H}}^2 - \|A_R^{1/2} u(b)\|_{\mathcal{H}}^2] \leq 0, \end{aligned}$$

that is,

$$\|A_R^{1/2}u(a)\|_{\mathcal{H}}^2 \leq \|A_R^{1/2}u(b)\|_{\mathcal{H}}^2, \quad u(t) \in D(M_h).$$

Hence there exists a contraction operator K in \mathcal{H} such that

$$A_R^{1/2}u(a) = KA_R^{1/2}u(b), \quad u(t) \in D(M_h). \quad (1.10)$$

Now we will prove that, if a hyponormal extension \widetilde{M} , $M_0 \subset \widetilde{M} \subset M$ is maximal, then

$$\mathcal{H}_b(\widetilde{M}) := \{u(b) \in \mathcal{H} : u(t) \in D(\widetilde{M})\} = \mathcal{H}_{+1/2}(A_R).$$

To prove this, we assume that there exists $f \in \mathcal{H}_{+1/2}(A_R)$ such that for each vector-function $u(t) \in D(M)$, $u(b) \neq f$. Now we will look at the vector-function $u_*(t) = f$, $a \leq t \leq b$.

It is clear that

$$f \in D(M) \cap D(M^+), A_R f \in L^2, f \notin \mathcal{H}_b(\widetilde{M})$$

and

$$\|u_*(a)\|_{\mathcal{H}} = \|u_*(b)\|_{\mathcal{H}}, \quad \|A_R^{1/2}u_*(a)\|_{\mathcal{H}} \leq \|A_R^{1/2}u_*(b)\|_{\mathcal{H}}.$$

Now we consider an extension \widetilde{M}_* , $\widetilde{M}_* \subset M$ of the operator \widetilde{M} to the linear manifold

$$D(\widetilde{M}_*) = \text{span}\{D(\widetilde{M}), u_*\}.$$

On the other hand, if we denote by

$$x = \begin{cases} Vx, & x \in D(V) \\ f & x = f \end{cases} \quad \text{or} \quad \begin{cases} V_*(\lambda x + f) := \lambda Vx + f, \\ x \in D(V), \lambda \in \mathbb{C}, \end{cases}$$

then $V_* : D(V_*) \rightarrow \mathcal{H}$, $V \subset V_*$ and an operator V_* is an isometric operator in \mathcal{H} . For the vector-functions $z(t)$ of the manifold $D(\widetilde{M}_*)$ holds. That is, there exists a hyponormal extension of the operator \widetilde{M} to $u_*(t)$. This cannot happen since the extension \widetilde{M} is maximal. Furthermore, from the relation (1.9), (1.10) and $\overline{\mathcal{H}_{+1/2}(A_R)} = \mathcal{H}$ we have

$$V = A_R^{-1/2}KA_R^{1/2}, \quad \text{that is, } K = A_R^{1/2}VA_R^{-1/2}.$$

It is clear that, an isometric operator V is determined uniquely by the extension of M_h

Now let L_h be a maximal hyponormal extension of the minimal operator L_0 in L^2 . It is clear that $M_h = U^{-1}L_hU$, $M_0 \subset M_h \subset M$, is a maximal hyponormal extension of M_0 . Then in the first part of the proof M_h is described by the differential-operator expression $m(u)$ and boundary condition (1.9) with some isometric operator V in \mathcal{H} i.e.

$$v(a) = Vv(b), \quad v(t) \in D(M_h), \quad (1.11)$$

where the operator $K = A_R^{1/2}VA_R^{-1/2}$ is also a contraction operator in \mathcal{H} . Since

$$v(t) = U(a, t)u(t), \quad v(t) \in D(M_h),$$

then the boundary condition (1.11) will be of the form

$$u(a) = VU(a, b)u(b), \quad u(t) \in D(L_h).$$

Now let L_V be an operator generated by the differential-operator expression $l(u)$ with boundary condition (1.8) in L^2 , that is,

$$\begin{aligned} L_V u &= l(u), \\ u(a) &= VU(a, b)u(b), \quad u(t) \in D(L_V), \end{aligned}$$

where V and $K = A_R^{\frac{1}{2}} V A_R^{-\frac{1}{2}}$ are isometric and contraction operators in \mathcal{H} respectively.

In this case the adjoint operator L_V^* is generated by the differential-operator expression $l^*(v)$ with the boundary condition

$$v(b) = U(b, a)V^*v(a), \quad v(t) \in D(L_V^*).$$

It is easy to see that $D(L_V) \subset D(L_V^*)$ and the second condition of the hyponormality extension in L^2 holds. \square

Proposition 1.8. *In order for a densely defined closed operator T to be hyponormal in \mathcal{H} , the necessary and sufficient condition is the hyponormality of $T + \lambda E$, $\lambda \in \mathbb{C}$, in \mathcal{H} .*

Proof. It is clear that for any $\lambda \in \mathbb{C}$

$$\begin{aligned} D(T + \lambda E) &= D(T), \\ D(T^* + \bar{\lambda} E) &= D(T^*). \end{aligned}$$

In addition, it can be verified that for $x \in D(T)$

$$((T + \lambda E)(T^* + \bar{\lambda} E)x, x)_{\mathcal{H}} - ((T^* + \bar{\lambda} E)(T + \lambda E)x, x)_{\mathcal{H}} = (TT^*x, x)_{\mathcal{H}} - (T^*Tx, x)_{\mathcal{H}}. \quad \square$$

Remark 1.9. *If in (1.1) $A_R \geq 0$, then writing (1.1) in the form*

$$\begin{aligned} l(u) &= u'(t) + Au(t) = u'(t) + (A + E)u(t) - u(t) = \\ &= [u'(t) + (A_R + E)u(t) + iA_I(t)u(t)] - u(t), \end{aligned}$$

using Proposition 1.8 and Theorem 1.7 we may describe all maximal hyponormal extension of minimal operator L_0 in L^2 generated by (1.1) and boundary condition (1.8), where V is an isometric and

$$(A_R + E)^{1/2} V (A_R + E)^{-1/2}$$

is a contraction operators in \mathcal{H} .

2. ASYMPTOTICAL BEHAVIOR OF THE MODULES OF THE EIGENVALUES FOR MAXIMAL HYPONORMAL EXTENSIONS

In this section we will investigate discreteness of the spectrum and asymptotical behavior of the modules of the eigenvalues for maximal hyponormal extensions of minimal operator L_0 in L^2 . For the convenience of the reader we give all the proofs which are similar to those used in [13].

First of all it is easy to see that the following result holds.

Theorem 2.1. *If L_h is a maximal hyponormal extension of the minimal operator L_0 and $M_h = U^{-1}L_hU$ is the maximal hyponormal extension of the minimal operator M_0 corresponding to L_h , then on the spectrum of these extensions in L^2 , we have $\sigma(L_h) = \sigma(M_h)$. We denote by $C_p(\mathcal{H})$, $p \geq 1$, the Schatten – von Neumann class of operators in the Hilbert space \mathcal{H} (see [5]) and $B(\mathcal{H})$ is a class of linear bounded operators in \mathcal{H} (see [4]).*

Now we prove the following theorem about the spectrum of maximal hyponormal extensions.

Theorem 2.2. *The spectrum of maximal hyponormal extensions L_V has the form*

$$\sigma(L_V) = \left\{ \lambda \in \mathbb{C} : \lambda = \lambda_0 + \frac{2k\pi i}{b-a}, \text{ where } \lambda_0 \text{ is a set} \right.$$

of solutions on λ for the equation $e^{-\lambda}(b-a) - \mu = 0, \mu \in \sigma(Ve^{-A_R(b-a)}), k \in \mathbb{Z} \left. \right\}$.

Proof. Since $\sigma(L_V) = \sigma(M_V)$, $M_V = U^{-1}L_VU$, then we investigate the spectrum of maximal hyponormal extension M_V in L^2 . Now let us consider a problem for the spectrum of maximal hyponormal extension M_V ,

$$\begin{aligned} u'(t) + A_R u(t) &= \lambda u(t) + f(t), \\ u(a) &= V u(b), \end{aligned}$$

where $\lambda \in \mathbb{C}$, $f(t) \in L^2$, V is an isometric operator and $A_R^{1/2} V A_R^{-1/2}$ is a contraction operator in \mathcal{H} . It is clear that a general solution of a differential equation in L^2 has the form

$$u_\lambda(t) = e^{-(A_R - \lambda)(t-a)} f + \int_a^t e^{-(A_R - \lambda)(t-s)} f(s) ds, \quad f \in \mathcal{H}_{-1/2}(A_R).$$

In this case from the boundary condition, we get the following relation

$$(V e^{-A_R(b-a)} - e^{-\lambda(b-a)}) f = -V \int_a^b e^{-A_R(b-s)} f(s) ds.$$

From this we see that, $\lambda \in \mathbb{C}$ has a point of spectrum of extension M_V it is necessary and sufficient for the following relation to hold:

$$e^{-\lambda(b-a)} = \mu \in \sigma(Ve^{-A_R(b-a)}).$$

Therefore, $\lambda = \lambda_0 + \frac{2k\pi i}{b-a}$, where $\lambda_0 \in \sigma(Ve^{-A_R(b-a)})$ and $k \in \mathbb{Z}$. □

Theorem 2.3. *Since $Ve^{-A_R(b-a)} \in B(\mathcal{H})$, then $\sigma(L_V) \neq \emptyset$ and is infinite.*

It is easy to see that following result holds.

Theorem 2.4. *If $\dim \mathcal{H} < +\infty$, then each maximal hyponormal extension L_V has a pure point spectrum and the modules of the eigenvalues of extensions L_V have the same asymptotics $|\lambda_n(L_V)| \sim \frac{2\pi n}{b-a}$, as $n \rightarrow \infty$.*

Theorem 2.5. *If $A_R^{-1} \in C_\infty(\mathcal{H})$ and the operator L_V is any maximal hyponormal extension of the minimal operator L_0 , then $L_V^{-1} \in C_\infty(L^2)$.*

Proof. Let L_V be any maximal hyponormal extension of the operator L_0 and M_V be a maximal hyponormal extension of the minimal operator M_0 corresponding to L^2 , that is,

$$M_V = U^{-1}L_VU.$$

It can be verified that, for $f(t) \in L^2$

$$\begin{aligned} M_V^{-1}f(t) &= e^{-A_R(t-a)}(E - Ve^{-A_R(b-a)})^{-1}V \int_a^b e^{-A_R(b-s)}f(s)ds + \\ &+ \int_a^t e^{-A_R(t-s)}f(s)ds. \end{aligned}$$

Now we prove that, if $A_R^{-1} \in C_\infty(\mathcal{H})$, then

$$Kf(t) := \int_a^t e^{-A_R(t-s)}f(s)ds \in C_\infty(L^2).$$

In order to prove this, for $\varepsilon > 0$ we define a new operator $K_\varepsilon : L^2 \rightarrow L^2$ of the form

$$K_\varepsilon f(t) := \int_a^{t-\varepsilon} e^{-A_R(t-s)}f(s)ds, \quad f(t) \in L^2, \varepsilon > 0.$$

For each $\varepsilon > 0$, the operator K_ε can be represented in the form

$$K_\varepsilon f(t) := \int_a^b K_\varepsilon(t, s)f(s)ds,$$

where $f(t) \in L^2$ and for each $(t, s) \in [a, b] \times [a, b]$,

$$K_\varepsilon(t, s) = \begin{cases} e^{-A_R(t-s)}, & \text{if } a \leq s < t - \varepsilon, \\ 0, & \text{if } t - \varepsilon \leq s \leq b. \end{cases}$$

Since for each pair $(t, s) \in [a, b] \times [a, b]$, $a \leq s < t - \varepsilon$, satisfies the following property

$$A_R e^{-A_R(t-s)} \in B(\mathcal{H}), e^{-A_R(t-s)} = [A_R e^{-A_R(t-s)}] A_R^{-1} \in C_\infty(\mathcal{H}),$$

then $K_\varepsilon \in C_\infty(L^2)$, $\varepsilon > 0$. On the other hand, the following estimate holds:

$$\begin{aligned} \|(K_\varepsilon - K)f\|_{L^2} &= \left\| \int_{t-\varepsilon}^t e^{-A_R(t-s)} f(s) ds \right\|_{L^2} \leq \int_{t-\varepsilon}^t \|e^{-A_R(t-s)}\| \cdot \|f(s)\|_{\mathcal{H}} ds \leq \\ &\leq \int_{t-\varepsilon}^t \|f(s)\|_{\mathcal{H}} ds \leq \left(\int_{t-\varepsilon}^t \|f(s)\|_{\mathcal{H}}^2 ds \right)^{1/2} \left(\int_{t-\varepsilon}^t 1^2 ds \right)^{1/2} \leq \\ &\leq \left(\int_a^b \|f(s)\|_{\mathcal{H}}^2 ds \right)^{1/2} \varepsilon^{1/2} = \varepsilon^{1/2} \|f\|_{L^2}, \quad f(t) \in L^2, \end{aligned}$$

that is,

$$\|K_\varepsilon - K\| \leq \varepsilon^{1/2}.$$

Therefore, $K_\varepsilon \rightarrow K$, as $\varepsilon \rightarrow 0$.

Hence from the important theorem [5], we have $K \in C_\infty(L^2)$. Thus the representation of M_V implies that $M_V^{-1} \in C_\infty(L^2)$. Hence $L_V^{-1} \in C_\infty(L^2)$. \square

Corollary 2.6. *Let L_V be any maximal hyponormal extension of the minimal operator L_0 and $\lambda \in \rho(L_V)$. Then $R_\lambda(L_V) \in C_\infty(L^2)$.*

This result follows from the relation

$$R_\lambda(L_V) = L_V^{-1} - \lambda R_\lambda(L_V) L_V^{-1}.$$

Using the method in the proof of Theorem 2.5 the following result can be proved.

Corollary 2.7. *If $A_R^{-1} \in C_p(\mathcal{H})$, $p \geq 1$ and L_V is any maximal hyponormal extension of L_0 , then $L_V^{-1} \in C_p(L^2)$.*

Furthermore, from the representation of resolvent $R_\lambda(L_V)$, $\lambda \in \rho(L_V)$, of the operator L_V we have the following corollary.

Corollary 2.8. *Let L_{V_1}, L_{V_2} be two maximal hyponormal extensions of the minimal operator L_0 in L^2 and $\lambda \in \rho(L_{V_1}) \cap \rho(L_{V_2})$. Then we have*

$$R_\lambda(L_{V_1}) - R_\lambda(L_{V_2}) \in C_p(L^2), \quad 1 \leq p,$$

if and only if

$$V_1 - V_2 \in C_p(\mathcal{H}), \quad p \geq 1.$$

Now we prove a result on the structure of the spectrum of the maximal extension of the minimal operator L_0 .

Theorem 2.9. *If $A_R^{-1} \in C_\infty(\mathcal{H})$ and L_V is any maximal hyponormal extension of the minimal operator L_0 in L^2 , then the spectrum of L_V has the form*

$$\sigma(L_V) = \left\{ \lambda_n(A_R) + \frac{i}{a-b} (\arg \lambda_n(V e^{-A_R(b-a)}) + 2k\pi i), n \in \mathbb{N}, k \in \mathbb{Z} \right\}.$$

Proof. Since $\sigma(L_V) = \sigma(M_V) = \sigma_p(M_V)$, then we will investigate the structure of the spectrum of M_V . From Theorem 2.2 we obtain

$$\sigma(L_V) = \left\{ \lambda \in \mathbb{C} : \lambda = \frac{1}{a-b} (\ln |\mu| + i \arg \mu + 2k\pi i), \mu \in \sigma(Ve^{-A_R(b-a)}), k \in \mathbb{Z} \right\}.$$

Since $A_R^{-1} \in C_\infty(\mathcal{H})$, then $Ve^{-A_R(b-a)} = V(A_R e^{-A_R(b-a)})A_R^{-1} \in C_\infty(\mathcal{H})$. For any eigenvector $x_\lambda \in \mathcal{H}$ corresponding to the eigenvalue $\lambda \in \sigma_p(Ve^{-A_R(b-a)})$, we have $Ve^{-A_R(b-a)}x_\lambda = \lambda(Ve^{-A_R(b-a)})x_\lambda$. This implies that

$$\begin{aligned} e^{-A_R(b-a)}V^*Ve^{-A_R(b-a)}x_\lambda &= \lambda(Ve^{-A_R(b-a)})e^{-A_R(b-a)}V^*x = \\ &= \lambda(Ve^{-A_R(b-a)})\overline{\lambda(Ve^{-A_R(b-a)})}x_\lambda, \end{aligned}$$

that is,

$$e^{-2A_R(b-a)}x_\lambda = |\lambda(Ve^{-A_R(b-a)})|^2x_\lambda.$$

Hence $|\lambda(Ve^{-A_R(b-a)})|^2 = \lambda(e^{-2A_R(b-a)}) = e^{-2\lambda A_R(b-a)}$, that is,

$$|\mu| = |\lambda(Ve^{-A_R(b-a)})| = e^{-\lambda(A_R)(b-a)}.$$

From this relation we have $\ln |\mu| = \lambda(A_R)(a-b)$. Thus

$$\sigma(L_V) = \left\{ \lambda \in \mathbb{C} : \lambda = \lambda_n(A_R) + \frac{i}{a-b} (\arg \lambda_n(Ve^{-A_R(b-a)}) + 2k\pi), n \in \mathbb{N}, k \in \mathbb{Z} \right\}. \quad \square$$

Now we can prove the main theorem of this section.

Theorem 2.10. *If $A_R^{-1} \in C_\infty(\mathcal{H})$, $\lambda_n(A_R) \sim cn^\alpha$, $0 < c, \alpha < \infty$, as $n \rightarrow \infty$, then $L_V^{-1} \in C_\infty(L^2)$ and*

$$|\lambda_n(L_V)| \sim dn^\beta, \quad 0 < d < \infty, \beta = \frac{\alpha}{1+\alpha}, \text{ as } n \rightarrow \infty.$$

Proof. Since $A_R^{-1} \in C_\infty(\mathcal{H})$, then $M_V^{-1}, L_V^{-1} = U^{-1}M_V^{-1}U \in C_\infty(L^2)$ and $\lambda_n(L_V) = \lambda_n(M_V), n \in \mathbb{N}$. It is clear that

$$\begin{aligned} |\lambda_m(L_V)| &= \left| \lambda_n(A_R) + \frac{i}{a-b} (\arg \lambda_n(Ve^{-A_R(b-a)}) + 2k\pi) \right| = \\ &= \left| \lambda_n(A_R) + \frac{i}{a-b} (\delta_n + 2k\pi) \right| = \\ &= \left[c^2 n^{2\alpha} + \frac{1}{(b-a)^2} (\delta_n + 2k\pi)^2 \right]^{1/2}, \end{aligned}$$

where $m = m(n, k) \in \mathbb{N}, n \in \mathbb{N}, k \in \mathbb{Z}, \delta_n = \arg \lambda_n(Ve^{-A_R(b-a)})$. Since $0 \leq \delta_n \leq 2\pi$ for each $n \in \mathbb{N}$, then from the last equality we have

$$\left[c^2 n^{2\alpha} + \frac{4\pi^2}{(b-a)^2} k^2 \right]^{1/2} \leq |\lambda(L_V)| \leq \left[c^2 n^{2\alpha} + \frac{4\pi^2}{(b-a)^2} (k+1)^2 \right]^{1/2}, \quad n \in \mathbb{N}, k \in \mathbb{Z}.$$

Therefore, $|\lambda(L_V)| \sim \sqrt{c^2 n^{2\alpha} + h^2 k^2}$, $n \in \mathbb{N}, k \in \mathbb{Z}$, where $h = \frac{4\pi}{b-a}$. On the other hand, we note that $(c^2 n^{2\alpha} + h^2 k^2)^{1/2}$, $n \in \mathbb{N}, k \in \mathbb{Z}$, are modules of eigenvalues of the periodical boundary condition (for the Dirichlet problem), i.e.

$$|\lambda(L_E)| = (c^2 n^{2\alpha} + h^2 k^2), \quad n \in \mathbb{N}, k \in \mathbb{Z}.$$

Therefore, asymptotical behavior of the modules of eigenvalues of each maximal hyponormal extension L_V and Dirichlet extension are the same, that is,

$$|\lambda_m(L_V)| \sim |\lambda_m(L_E)|, \quad \text{as } m \rightarrow \infty.$$

Using the method established in [6, 7] (in our case $k \in \mathbb{Z}$). It can be found that

$$|\lambda_m(L_E)| \sim dm^{\frac{\alpha}{1+\alpha}}, \quad 0 < d < \infty, \quad m \rightarrow \infty,$$

which completes the proof. \square

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