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**EXTENSIONS OF SOLUTIONS
OF A FUNCTIONAL EQUATION IN TWO VARIABLES**

Abstract. An extension theorem for the functional equation of several variables

$$f(M(x, y)) = N(f(x), f(y)),$$

where the given functions M and N are left-side autodistributive, is presented.

Keywords: functional equation, autodistributivity, strict mean, extension theorem.

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1. AN EXTENSION THEOREM

Recently Zs. Páles [2] considered the extension problem for the functional equation of the form

$$f(x) = M(f(m_1(x, y)), \dots, f(m_k(x, y)))$$

where M is a k -variable bisymmetric operation and m_1, \dots, m_k some binary commuting operations.

In this note we deal with the extension problem for the functional equation

$$f(M(x, y)) = N(f(x), f(y))$$

where the given functions M and N are left-side autodistributive and M is a strict mean. The functions satisfying this functional equation are called (M, N) -affine (cf. [1] where the case when both M and N are means is considered).

We prove the following

Theorem 1.1. *Let $I, J \subset \mathbb{R}$ be open intervals. Suppose that $M : I^2 \rightarrow I$, $N : J^2 \rightarrow J$ are continuous strictly increasing with respect to the first variable and such that*

$$M(M(x, y), z) = M(M(x, z), M(y, z)), \quad x, y, z \in I, \quad (1.1)$$

$$N(N(x, y), z) = N(N(x, z), N(y, z)), \quad x, y, z \in J. \quad (1.2)$$

We also assume that M is a strict mean, that is

$$\min(x, y) < M(x, y) < \max(x, y), \quad x, y \in I, \quad x \neq y.$$

If $f : I_0 \rightarrow J$ satisfies the functional equation

$$f(M(x, y)) = N(f(x), f(y)), \quad x, y \in I_0, \quad (1.3)$$

for a nontrivial interval $I_0 \subset I$, then there exists a unique function $F : I \rightarrow J$ such that $F|_{I_0} = f$ and

$$F(M(x, y)) = N(F(x), F(y)), \quad x, y \in I.$$

Proof. Assume that $I_0 \subset I$ is a maximal subinterval of I on which the function f can be extended to satisfy equation (1.3). Suppose first that

$$b := \sup I_0 < \sup I.$$

Take an arbitrary $a \in I_0$, $a < b$. Then

$$f(M(x, y)) = N(f(x), f(y)), \quad x, y \in [a, b]. \quad (1.4)$$

Since M is a continuous and strict mean, there is a $c \in I$, $c > b$ such that $M(a, c) < b$. Hence, as M is strictly increasing with respect to the first variable,

$$M(x, c) < b, \quad x \in [a, c].$$

Setting $y := a$ in (1.4) we have

$$f(M(x, a)) = N(f(x), f(a)), \quad x \in [a, c].$$

Let $N_{f(a)}^{-1}$ denote the inverse function of $N(\cdot, f(a))$. Define $F_a : [a, c] \rightarrow J$ by

$$F_a(x) := N_{f(a)}^{-1}(f(M(x, a))), \quad x \in [a, c].$$

Note that by (1.4) the function F_a is correctly defined,

$$F_a(x) = f(x), \quad x \in [a, b],$$

and

$$f(M(x, a)) = N(F_a(x), f(a)), \quad x \in [a, c]. \quad (1.5)$$

Now making use respectively of the definition of F_a , the property (1.1) of M , equation (1.4), equation (1.5), property (1.2) of N , for all $x, y \in [a, c]$ we have

$$\begin{aligned} F_a(M(x, y)) &:= N_{f(a)}^{-1}(f(M(M(x, y), a))) = N_{f(a)}^{-1}(f(M(M(x, a), M(y, a)))) = \\ &= N_{f(a)}^{-1}(N(f(M(x, a)), f(M(y, a)))) = \\ &= N_{f(a)}^{-1}(N(N(F_a(x), f(a)), N(F_a(y), f(a)))) = \\ &= N_{f(a)}^{-1}(N(N(F_a(x), F_a(y)), f(a))) = N(F_a(x), F_a(y)), \end{aligned}$$

that is

$$F_a(M(x, y)) = N(F_a(x), F_a(y)), \quad x, y \in [a, c]. \quad (1.6)$$

In the case when $\inf I_0 \in I_0$ we can take $a = \inf I_0$. Putting $F := F_a$ we get

$$F(M(x, y)) = N(F(x), F(y)), \quad x, y \in [a, c].$$

Since $F|_{I_0} = f$ and $I_0 \subsetneq [a, c]$, this contradicts the maximality of the interval I_0 .

In the case when $\inf I_0 \notin I$, we take a decreasing sequence $(a_n : n \in \mathbb{N})$ such that $\inf I_0 = \lim_{n \rightarrow \infty} a_n$ and define $F : (\inf I_0, c] \rightarrow J$ by

$$F(x) := F_{a_n}(x), \quad x \in [a_n, c], \quad n \in \mathbb{N}.$$

This definition is correct because, by (1.3),

$$m < n \implies F_{a_m} = F_{a_n}|_{[a_m, c]}.$$

In view of (1.6), we have

$$F(M(x, y)) = N(F(x), F(y)), \quad x, y \in (\inf I_0, c].$$

Since $F|_{I_0} = f$ and $I_0 \subsetneq (\inf I_0, c]$, this contradicts the maximality of the interval I_0 .

The obtained contradiction proves that $\sup I_0 = \sup I$. In a similar way we can show that $\inf I_0 = \inf I$. This completes the proof. \square

REFERENCES

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