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ON ELLIPTIC PROBLEMS WITH A NONLINEARITY DEPENDING ON THE GRADIENT

Abstract. We investigate the solvability of the Neumann problem (1.1) involving the nonlinearity depending on the gradient. We prove the existence of a solution when the right hand side f of the equation belongs to $L^m(\Omega)$ with $1 \leq m < 2$.

Keywords: Neumann problem, nonlinearity depending on the gradient, L^1 data.

Mathematics Subject Classification: 35D05, 35J25, 35J60.

1. INTRODUCTION

In this paper we investigate the solvability of the nonlinear Neumann problem with a nonlinearity depending on the gradient. First we consider the following problem

$$\begin{cases} -\Delta u + |\nabla u|^q + \lambda u = f(x) & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $1 \leq q \leq 2$ and $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with a smooth boundary $\partial\Omega$. It is assumed that $f \in L^1(\Omega)$. If $f > 0$ on Ω, then solutions, if they exist, are positive. In Section 3 we consider problem (1.1) with $|\nabla u|^q$ replaced by a nonlinearity satisfying a sign condition. The boundary value problems with data in L^1 has been studied quite extensively in recent years. The Dirichlet problem with a nonlinearity depending only on u has been considered in papers [7, 10]. Some extensions to the Neumann problem can be found in paper [12]. These results has been extended to the case where a nonlinearity depends on the gradient. In particular, more general elliptic operators with more general nonlinearities with $f \in L^1(\Omega)$ or being a Radon measure have been investigated in [3–6, 11]. Further extensions to the Dirichlet problem with L^2 boundary data can be found in [11]. We refer to paper [2] for the bibliographical references. It seems that less is known for the Neumann problem.

By $W^{1,p}(\Omega)$, $1 \le p < \infty$, we denote the Sobolev space equipped with norm

$$\|u\|_{W^{1,p}}^p = \int\limits_\Omega \left(|\nabla u|^p + |u|^p\right) dx.$$

Throughout this paper, in a given Banach space X, we denote strong convergence by "$\to$" and weak convergence by "$\rightharpoonup$". The norms in the Lebesgue spaces $L^p(\Omega)$, $1 \le p < \infty$, are denoted by $\|\cdot\|_{L^p}$.

The paper is organized as follows. In Section 2 we prove the existence of positive solutions of (1.1) assuming that f is positive and belongs to $L^1(\Omega)$. Section 3 is devoted to the problem with a nonlinearity satisfying a sign condition, where we do not assume that f is positive. The crucial point in our approach are estimates of $W^{1,q}$ - norm of solutions of (1.1) in terms of L^m – norm of f (see Lemmas 2.1, 3.1, 3.3). The estimates in terms of L^m norm of f (see Lemmas 3.1, 3.3) in a linear case were given in [8] and are extended in this paper to solutions of (1.1). In these two lemmas the important assumption is that $q \neq \frac{N}{N-1}$, which is due to the use of special test functions in the proofs. We were unable to show whether these lemmas continue to hold for $q = \frac{N}{N-1}$. In Section 4 we establish the higher integrability property for positive solutions of (1.1).

The main results of this paper are Theorems 2.2, 3.2, 3.4. In the proofs we use some ideas from paper [4].

2. EXISTENCE OF POSITIVE SOLUTIONS

In this section consider problem (1.1) assuming that $f > 0$ on Ω. Then a solution, if it exists, is positive on Ω. We need the following definition of a solution of (1.1): let $f \in L^1(\Omega)$, then a function $u \in W^{1,q}(\Omega)$ is a solution of (1.1) if

$$\int\limits_\Omega \nabla u \nabla v\, dx + \int\limits_\Omega |\nabla u|^q v\, dx + \lambda \int\limits_\Omega uv\, dx = \int\limits_\Omega fv\, dx \tag{2.1}$$

for every function $v \in W^{1,\infty}(\Omega)$.

Lemma 2.1. *Let $1 \le q \le 2$ and $f \in L^\infty(\Omega)$ with $f > 0$ on Ω. If $u \in W^{1,2}(\Omega)$ is a positive solution of* (1.1)*, then*

$$\int\limits_\Omega \left(|\nabla u|^q + u^q\right) dx \le C_1 \int\limits_\Omega f\, dx + C_2 \left(\int\limits_\Omega f\, dx\right)^q, \tag{2.2}$$

where $C_1, C_2 > 0$ are constants independent of u and f.

Proof. Testing (2.1) with the constant function 1 we get

$$\int\limits_\Omega |\nabla u|^q\, dx + \lambda \int\limits_\Omega u\, dx = \int\limits_\Omega f\, dx. \tag{2.3}$$

It is clear that equality (2.3) yields (2.2) if $q = 1$. To proceed further we use a decomposition $W^{1,2}(\Omega) = V \oplus \operatorname{span} 1$, where

$$V = \{v \in W^{1,2}(\Omega);\ \int_\Omega v\,dx = 0\}.$$

Then $u = v + t$, with $v \in V$ and $t = \frac{1}{|\Omega|}\int_\Omega u\,dx > 0$, because u is positive. From (2.3) we deduce

$$t \le \frac{1}{\lambda|\Omega|}\int_\Omega f\,dx. \tag{2.4}$$

We now observe that the Poincaré inequality is valid in V, that is, there exists a constant $C(\Omega) > 0$ such that

$$\int_\Omega |v|^q\,dx \le C(\Omega)\int_\Omega |\nabla v|^q\,dx$$

for every $v \in V$. Consequently, using (2.4), we can estimate the norm of u in $W^{1,q}(\Omega)$ as follows

$$\begin{aligned}\int_\Omega \left(|\nabla u|^q + u^q\right)dx &\le \int_\Omega |\nabla v|^q\,dx + 2^{q-1}\int_\Omega \left(v^q + t^q\right)dx \le \\ &\le \int_\Omega |\nabla v|^q\,dx + 2^{q-1}C(\Omega)\int_\Omega |\nabla v|^q\,dx + 2^{q-1}|\Omega|t^q.\end{aligned}$$

This combined with (2.4) and (2.3) implies (2.2). □

We are now in a position to formulate the first existence result.

Theorem 2.2. *Let* $1 \le q \le 2$ *and* f *be a positive function in* $L^1(\Omega)$. *Then problem* (1.1) *admits a positive solution in* $W^{1,q}(\Omega)$.

Proof. The proof will be given in 2 steps.
Step 1. Assume $f \in L^\infty(\Omega)$. Consider the problem

$$\begin{cases} -\Delta u + \lambda u = f(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{on } \Omega. \end{cases} \tag{2.5}$$

This problem has a unique positive solution $v \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ (see [1]). We now use some ideas from papers [5] and [6]. For each $n \in \mathbb{N}$ we consider the following problem

$$\begin{cases} -\Delta w_n + \dfrac{|\nabla w_n|^q}{1 + \frac{1}{n}|\nabla w_n|^q} + \lambda w_n = f(x) & \text{in } \Omega, \\ \dfrac{\partial w_n}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ w_n > 0 & \text{on } \Omega. \end{cases} \tag{2.6}$$

It is clear that v is a super-solution to problem (2.6) and 0 is a sub-solution. Thus problem (2.6) admits a solution $0 \leq w_n \leq v$. This fact is known for equation (2.6) with the Dirichlet boundary conditions (see [5]). The result from [5] can be easily extended to the Neumann problem (2.6). The sequence $\{w_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Testing (2.6) with w_n we obtain

$$\int_\Omega \left(|\nabla w_n|^2 + \lambda w_n^2\right) dx \leq \|f\|_{L^2} \|w_n\|_{L^2},$$

which shows that the sequence $\{w_n\}$ is bounded in $W^{1,2}(\Omega)$. We may assume that $w_n \rightharpoonup w$ in $W^{1,2}(\Omega)$, $w_n \to w$ in $L^2(\Omega)$ and $w_n \to w$ a.e. on Ω. We now show that $w_n \to w$ in $W^{1,2}(\Omega)$. We put $\phi(s) = s\exp(\frac{s^2}{4})$ for $s \in \mathbb{R}$. We introduce notation $H_n(s) = \frac{|s|^q}{1+\frac{1}{n}|s|^q}$. The function ϕ satisfies $\phi'(s) - |\phi(s)| \geq \frac{1}{2}$ for $s \in \mathbb{R}$. Testing (2.6) with $\phi(w_n - w)$ we obtain

$$\begin{aligned} \int_\Omega \nabla w_n \phi'(w_n - w)\nabla(w_n - w)\, dx + \int_\Omega H_n(|\nabla w_n|)\phi(w_n - w)\, dx + \\ + \lambda \int_\Omega w_n \phi(w_n - w)\, dx = \int_\Omega f(x)\phi(w_n - w)\, dx. \end{aligned} \tag{2.7}$$

It is easy to check that

$$\int_\Omega \nabla w_n \phi'(w_n - w)\nabla(w_n - w)\, dx = \int_\Omega |\nabla(w_n - w)|^2 \phi'(w_n - w)\, dx + o(1). \tag{2.8}$$

To estimate the second term on the left side of (2.7) we use the inequality: if $1 \leq q < 2$, then for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$s^q \leq \epsilon s^2 + C_\epsilon \quad \text{for every} \quad s \geq 0. \tag{2.9}$$

We then have

$$\begin{aligned} \int_\Omega H_n(|\nabla w_n|)|\phi(w_n - w)|\, dx &\leq \epsilon \int_\Omega |\nabla w_n|^2 |\phi(w_n - w)|\, dx + C_\epsilon \int_\Omega |\phi(w_n - w)|\, dx = \\ &= \epsilon \int_\Omega |\nabla(w_n - w)|^2 |\phi(w_n - w)|\, dx - \\ &\quad - \epsilon \int_\Omega |\nabla w|^2 |\phi(w_n - w)|\, dx + \\ &\quad + 2\epsilon \int_\Omega \nabla w_n \nabla w |\phi(w_n - w)|\, dx + \\ &\quad + C_\epsilon \int_\Omega |\phi(w_n - w)|\, dx. \end{aligned} \tag{2.10}$$

Since

$$\int_\Omega |\nabla w|^2|\phi(w_n-w)|\,dx \to 0, \quad \int_\Omega \nabla w_n\nabla w|\phi(w_n-w)|\,dx \to 0$$

and

$$\int_\Omega |\phi(w_n-w)|\,dx \to 0$$

as $n\to\infty$, we derive from (2.10) that

$$\int_\Omega H_n(|\nabla w_n|)|\phi(w_n-w)|\,dx \le \epsilon\int_\Omega |\nabla w_n-\nabla w|^2|\phi(w_n-w)|\,dx + o(1). \tag{2.11}$$

If $q=2$, then instead of (2.10) we have

$$\int_\Omega H_n(|\nabla w_n|)|\phi(w_n-w)|\,dx \le \int_\Omega |\nabla w_n|^2\phi(w_n-w)\,dx$$

and (2.11) holds with $\epsilon=1$. We also have

$$\int_\Omega f(x)\phi(w_n-w)\,dx\to 0 \text{ and } \int_\Omega w_n\phi(w_n-w)\,dx\to 0 \tag{2.12}$$

as $n\to\infty$. If $1\le q<2$ we derive from (2.7), (2.8), (2.11) and (2.12) that

$$\frac{1}{2}\int_\Omega |\nabla(w_n-w)|^2\,dx \le \int_\Omega \big(\phi'(w_n-w)-\epsilon|\phi(w_n-w)|\big)|\nabla(w_n-w)|^2\,dx = o(1).$$

Thus $w_n\to w$ in $W^{1,2}(\Omega)$. If $q=2$, the above inequality continues to hold with $\epsilon=1$. In this case we also have that $w_n\to w$ in $W^{1,2}(\Omega)$. Since $1\le q\le 2$, $\nabla w_n\to\nabla w$ in $L^q(\Omega)$. For each $\phi\in W^{1,2}(\Omega)\cap L^\infty(\Omega)$ and for each n we have

$$\int_\Omega \nabla w_n\nabla\phi\,dx + \int_\Omega \frac{|\nabla w_n|^q}{1+\frac{1}{n}|\nabla w_n|^q}\phi\,dx + \lambda\int_\Omega w_n\phi\,dx = \int_\Omega f\phi\,dx.$$

Letting $n\to\infty$ we get

$$\int_\Omega \nabla w\nabla\phi\,dx + \int_\Omega |\nabla w|^q\phi\,dx + \lambda\int_\Omega w\phi\,dx = \int_\Omega f\phi\,dx.$$

So $w\in W^{1,2}(\Omega)\cap L^\infty(\Omega)$ is a weak solution of (1.1).

Step 2. First we consider the case $1 \leq q < 2$. Let $f \in L^1(\Omega)$ and let $\{f_n\} \subset L^\infty(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$. By Step 1 for each $n \in \mathbb{N}$ there exists a solution $u_n \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ to problem (1.1) with $f = f_n$. For each $k > 1$ we put $T_k(s) = \min(s,k)$ for $0 \leq s$. Taking $T_k u_n$ as a test function in (1.1) we get

$$\int_\Omega |\nabla T_k u_n|^2\,dx + \lambda \int_\Omega |T_k u_n|^2\,dx \leq \int_\Omega f_n T_k u_n\,dx \leq k\|f_n\|_{L^1}.$$

Consequently, $\{T_k u_n\}$ is bounded in $W^{1,2}(\Omega)$. By Lemma 2.1 we may assume that $u_n \rightharpoonup u$ in $W^{1,q}(\Omega)$. We may also assume that $T_k u_n \rightharpoonup T_k u$ in $W^{1,2}(\Omega)$ and $T_k u_n \to T_k u$ in $L^2(\Omega)$. Let $G_k(s) = s - T_k(s)$ and put $\psi_{k-1}(s) = T_1(G_{k-1}(s))$. Thus

$$\psi_{k-1}(u_n)|\nabla u_n|^q \geq |\nabla u_n|^q \chi_{(u_n > k)}.$$

Using $\psi_{k-1}(u_n)$ as a test function in (2.1) (with $f = f_n$) we get

$$\int_\Omega |\nabla \psi_{k-1}(u_n)|^2\,dx + \int_\Omega \psi_{k-1}(u_n)|\nabla u_n|^q\,dx + \lambda \int_\Omega u_n \psi_{k-1}(u_n)\,dx = \int_\Omega f_n \psi_{k-1}(u_n)\,dx.$$

Since $\{u_n\}$ is bounded in $L^p(\Omega)$ for each $p \leq q^* = \frac{Nq}{N-q}$ we see that

$$|\{x \in \Omega;\ k-1 < u_n(x) < k\}| \to 0 \ \text{ and } \ |\{x \in \Omega;\ k < u_n(x)\}| \to 0$$

as $k \to \infty$ uniformly in n. So

$$\lim_{k\to\infty} \int_{u_n > k} |\nabla u_n|^q\,dx = 0 \tag{2.13}$$

uniformly in n. Using as a test function $\phi(T_k u_n - T_k u)$ and repeating the argument from Step 1 we show that $T_k u_n \to T_k u$ in $W^{1,2}(\Omega)$. We now use this to show that the sequence $\{|\nabla u_n|^q\}$ is equi-integrable. This follows from (2.13) and the following inequality: for every measurable subset $E \subset \Omega$ we have

$$\int_E |\nabla u_n|^q\,dx \leq \int_E |\nabla T_k u_n|^q\,dx + \int_{(u_n \geq k)\cap E} |\nabla u_n|^q\,dx.$$

Indeed, given $\epsilon > 0$, according to (2.13), we can find k large enough such that

$$\int_{u_n \geq k} |\nabla u_n|^q\,dx < \frac{\epsilon}{2}$$

for all n. Since $\nabla T_k(u_n) \to T_k(u)$ in $L^2(\Omega)$ there exists $\delta > 0$ such that

$$\int_E |\nabla T_k(u_n)|^q\,dx < \frac{\epsilon}{2}$$

provided $|E| \leq \delta$ and for all n. By Vitali's theorem $\nabla u_n \to \nabla u$ in $L^q(\Omega)$. Thus u is a weak solution of (1.1). If $q = 2$, then by Lemma 2.1 the sequence $\{u_n\}$ is bounded in $W^{1,2}(\Omega)$. An obvious modification of Step 2 completes the proof. □

3. NONLINEARITY WITH A SIGN CONDITION

In this section we discuss the solvability of the following problem

$$\begin{cases} -\Delta u + g(x,u,\nabla u) + \lambda u = f(x) & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

We assume that the nonlinearity $g:\ \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, that is, $g(\cdot,s,\xi)$ is measurable on Ω for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$ and $g(x,\cdot,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}^N$ for a.e. $x\in\Omega$. Moreover, we assume that

(g_1) there exist an increasing and continuous function $b:[0,\infty)\to[0,\infty)$ with $b(0)=0$ and a positive function $a\in L^1(\Omega)$ such that

$$|g(x,s,\xi)| \le b(|s|)\big(|\xi|^q + a(x)\big)$$

for a.e. $x\in\Omega$ and for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$.

(g_2) $g(x,s,\xi)$ sgn $s\ge 0$ for a.e. $x\in\Omega$ and for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$.

A typical example of a nonlinearity satisfying (g_1) and (g_2) is $g(x,s,\xi)=s|\xi|^q$.

We now consider equation (3.1) without assumption that f is positive on Ω. Obviously, it is assumed that $f\not\equiv 0$ on Ω. We assume that $\frac{N}{N-1}<q<2$. Then there exists $1<m<\frac{2N}{N+q}$ such that $q=m^*=\frac{Nm}{N-m}$. In this case m is given by $m=\frac{Nq}{N+q}$. We also use notation $q^*=\frac{Nq}{N-q}$. With these notations we establish the estimates of norms $\|u\|_{L^{q^*}}$ and $\|u\|_{W^{1,q}}$ of a solution u of (1.1) in terms of the norm $\|f\|_{L^m}$.

Lemma 3.1. *Let $f\in L^\infty(\Omega)$ and $\frac{N}{N-1}<q<2$. If $u\in W^{1,2}(\Omega)\cap L^\infty(\Omega)$ is a solution of* (3.1), *then*

$$\begin{aligned}\int_\Omega |u|^{q^*}\,dx &\le C_1\bigg(\int_\Omega \big(|\nabla u|^q+|u|^q\big)\,dx\bigg)^{\frac{q^*}{q}} \le \\ &\le C_2\|f\|_{L^m}^{\frac{q^*}{2}}\bigg(\int_\Omega |u|^{q^*}\,dx\bigg)^{\frac{(1-r)}{2}}\bigg(\int_\Omega \big(1+u^2\big)^{\frac{q^*}{2}}\,dx\bigg)^{\frac{r}{2}},\end{aligned} \tag{3.2}$$

where $r=\frac{N(2-q)}{N-q}$ and $C_1>0$ and $C_2>0$ are constants independent of u and f.

Proof. We follow some ideas from [8], where the same estimate was proved for the linear problem. Put $\varphi(x) = \frac{u}{\left(1+u^2\right)^{\frac{r}{2}}}$. Since $\frac{N}{N-1} < q < 2$, we have $0 < r < 1$. Since $u \in L^\infty(\Omega)$, φ is a legitimate test function. Upon the substitution we obtain

$$\begin{aligned}(1-r)\int_\Omega \frac{|\nabla u|^2}{\left(1+u^2\right)^{\frac{r}{2}}}\,dx + \lambda\int_\Omega \frac{u^2}{\left(1+u^2\right)^{\frac{r}{2}}}\,dx \le \int_\Omega \frac{|fu|}{\left(1+u^2\right)^{\frac{r}{2}}}\,dx \le \\ \le \|f\|_{L^m}\left(\int_\Omega |u|^{(1-r)m'}\,dx\right)^{\frac{1}{m'}},\end{aligned} \tag{3.3}$$

where $m' = \frac{m}{m-1}$. Here we used the fact that

$$\int_\Omega \frac{ug(x,u,\nabla u)}{\left(1+u^2\right)^{\frac{r}{2}}}\,dx \ge 0$$

due to assumption (g_2). In what follows we denote by $C > 0$ a constant which is independent of u and f and may vary from line to line. By the Sobolev inequality we have

$$\begin{aligned}\left(\int_\Omega |u|^{q^*}\,dx\right)^{\frac{q}{q^*}} &\le C\int_\Omega \left(|\nabla u|^q + |u|^q\right)dx = \\ &= C\int_\Omega \frac{|\nabla u|^q}{\left(1+u^2\right)^{\frac{rq}{4}}}\left(1+u^2\right)^{\frac{rq}{4}}\,dx + \\ &\quad + C\int_\Omega \frac{|u|^q}{\left(1+u^2\right)^{\frac{rq}{4}}}\left(1+u^2\right)^{\frac{rq}{4}}\,dx \le \\ &\le C\left(\int_\Omega \frac{|\nabla u|^2}{\left(1+u^2\right)^{\frac{r}{2}}}\,dx\right)^{\frac{q}{2}}\left(\int_\Omega \left(1+u^2\right)^{\frac{rq}{2(2-q)}}\,dx\right)^{\frac{2-q}{2}} + \\ &\quad + C\left(\int_\Omega \frac{u^2}{\left(1+u^2\right)^{\frac{r}{2}}}\,dx\right)^{\frac{q}{2}}\left(\int_\Omega \left(1+u^2\right)^{\frac{rq}{2(2-q)}}\,dx\right)^{\frac{2-q}{2}}.\end{aligned} \tag{3.4}$$

Inserting (3.3) into (3.4) we derive

$$\begin{aligned}\left(\int_\Omega |u|^{q^*}\,dx\right)^{\frac{q}{q^*}} &\le C\int_\Omega \left(|\nabla u|^q + |u|^q\right)dx \le \\ &\le C\|f\|_{L^m}^{\frac{q}{2}}\left(\int_\Omega |u|^{(1-r)m'}\,dx\right)^{\frac{q}{2m'}}\left(\int_\Omega \left(1+u^2\right)^{\frac{rq}{2(2-q)}}\,dx\right)^{\frac{2-q}{2}}.\end{aligned}$$

Since $r = \frac{N(2-q)}{N-q}$, we have $\frac{rq}{2-q} = q^*$ and $(1-r)m' = q^*$. Therefore the above inequality becomes

$$\int_\Omega |u|^{q^*}\,dx \le C\bigg(\int_\Omega \big(|\nabla u|^q + |u|^q\big)\,dx\bigg)^{\frac{q^*}{q}} \le$$
$$\le C\|f\|_{L^m}^{\frac{q^*}{2}}\bigg(\int_\Omega |u|^{q^*}\,dx\bigg)^{\frac{q^*}{2m'}}\bigg(\int_\Omega \big(1+u^2\big)^{\frac{q^*}{2}}\,dx\bigg)^{\frac{(2-q)q^*}{2q}}.$$

Since $\frac{q^*}{2m'} = \frac{1-r}{2}$ and $\frac{(2-q)q^*}{2q} = \frac{r}{2}$, the result follows. □

We are now in a position to formulate the second existence result.

Theorem 3.2. *Let $\frac{N}{N-1} < q < 2$ and $f \in L^m(\Omega)$ with $m = \frac{Nq}{N+q}$. Suppose that assumptions (g_1) and (g_2) hold. Then problem (1.1) admits a solution in $W^{1,q}(\Omega)$.*

Proof. The proof is similar to that of Theorem 2.2 except some technical modifications. First we assume that $f \in L^\infty(\Omega)$. For every $n \in \mathbb{N}$ we put

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$$

and consider the following problem

$$\begin{cases} -\Delta u + g_n(x,u,\nabla u) + \lambda u = f(x) & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.5}$$

Then the functions $v_1 = \frac{\|f\|_\infty}{\lambda}$ and $v_2 = -\frac{\|f\|_\infty}{\lambda}$ are a super-solution and a sub-solution to problem (3.5), respectively. For every n problem (3.5) has a solution w_n satisfying $v_1 \le w_n \le v_2$ on Ω. Hence the sequence $\{w_n\}$ is bounded in $L^\infty(\Omega)$, that is, $\|w_n\|_\infty \le M$ for some constant $M > 0$ and for all $n \in \mathbb{N}$. Testing (3.5) with w_n we show that $\{w_n\}$ is bounded in $W^{1,2}(\Omega)$. So we may assume that $w_n \rightharpoonup w$ in $W^{1,2}(\Omega)$, $w_n \to w$ in $L^2(\Omega)$ and $w_n \to w$ a.e. on Ω. Let ϕ be a function introduced in the proof of Theorem 2.2. Testing (3.5) with $\phi(w_n - w)$ we obtain

$$\begin{aligned} \int_\Omega \nabla w_n \phi'(w_n - w)\nabla(w_n - w)\,dx + \int_\Omega g_n(x,w_n,\nabla w_n)\phi(w_n - w)\,dx+ \\ + \lambda\int_\Omega w_n\phi(w_n - w)\,dx = \int_\Omega f(x)\phi(w_n - w)\,dx. \end{aligned} \tag{3.6}$$

It is clear that

$$\int_\Omega \nabla w_n \phi'(w_n - w)\nabla(w_n - w)\,dx = \int_\Omega |\nabla(w_n - w)|^2\phi'(w_n - w)\,dx + o(1). \tag{3.7}$$

We use inequality (2.9) and assumption (g_1) to estimate the second integral on the left side of (3.6)

$$\int_\Omega |g_n\phi(w_n-w)|\,dx \le b(M)\int_\Omega |\nabla w_n|^q|\phi(w_n-w)|\,dx + \int_\Omega a(x)|\phi(w_n-w)|\,dx \le$$
$$\le b(M)\epsilon\int_\Omega |\nabla w_n|^2|\phi(w_n-w)|\,dx + C_\epsilon\int_\Omega |\phi(w_n-w)|\,dx+$$
$$+\int_\Omega a(x)|\phi(w_n-w)|\,dx.$$

Since $\phi(w_n-w)\to 0$ a.e. on Ω and $\sup_n|\phi(w_n-w)|<\infty$ by the Lebesgue dominated convergence theorem we get

$$\int_\Omega |g_n\phi(w_n-w)|\,dx \le b(M)\epsilon\int_\Omega |\nabla w_n|^2|\phi(w_n-w)|\,dx + o(1).$$

As in the proof of Theorem 2.2 we deduce from this that

$$\int_\Omega |g_n\phi(w_n-w)|\,dx \le b(M)\epsilon\int_\Omega |\nabla w_n-\nabla w|^2|\phi(w_n-w)|\,dx + o(1). \tag{3.8}$$

Taking $\epsilon b(M)\le 1$ we deduce from (3.6), (3.7) and (3.8) that

$$\int_\Omega |\nabla w_n-\nabla w|^2\,dx \le \int_\Omega \big(\phi'(w_n-w)-\epsilon b(M)|\phi(w_n-w)|\big)|\nabla w_n-\nabla w|^2\,dx = o(1).$$

Thus $w_n\to w$ in $W^{1,2}(\Omega)$. It is clear that w is a solution of (3.1). In the final step we choose a sequence $\{f_n\}\subset L^\infty(\Omega)$ such that $f_n\to f$ in $L^m(\Omega)$. Then for every $n\in\mathbb{N}$ problem (3.1) with $f=f_n$ admits a solution $u_n\in W^{1,2}(\Omega)\cap L^\infty(\Omega)$. We now define a sequence of truncations $\{T_k(u_n)\}$ for every $k>0$, where $T_k=\max(-k,\min(s,k))$. Let $G_k(s)=s-T_k(s)$ and put $\psi_{k-1}(s)=T_1(G_{k-1}(s))$. Thus

$$\psi_{k-1}(u_n)|\nabla u_n|^2 \ge |\nabla u_n|^2\chi_{|u_n|\ge k}.$$

As in the proof of Theorem 2.2 we show that the sequence $\{T_k(u_n)\}$ is bounded in $W^{1,2}(\Omega)$. Hence we can assume that $T_k(u_n)\rightharpoonup T_ku$ in $W^{1,2}(\Omega)$, $T_k(u_n)\to T_ku$ in $L^2(\Omega)$ and $T_k(u_n)\to T_k(u)$ a.e. on Ω. By Lemma 3.1 we may also assume that $u_n\rightharpoonup u$ in $W^{1,q}(\Omega)$. Using as a test function $\psi_{k-1}(u_n)$ we show that $\nabla u_n\to\nabla u$ in $L^q(\Omega)$ and u is a weak solution of (3.1). □

We now turn our attention to positive solutions of (3.1). If $f > 0$ on Ω, then a solution obtained in Theorem 4.3 is positive. In this case we can also consider the interval $1 \le q < \frac{N}{N-1}$. We commence with an apriori estimate.

Lemma 3.3. *Suppose that* $1 \le q < \frac{N}{N-1}$, $f > 0$ *on* Ω *and* $f \in L^\infty(\Omega)$. *If* $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ *is a positive solution of problem* (3.1), *then*

$$\int_\Omega u^{q^*}\,dx \le C_1\Big(\int_\Omega (|\nabla u|^q + u^q)dx\Big)^{\frac{q^*}{q}} \le$$
$$\le C_2\Big(\int_\Omega (1+u)^{q^*}\,dx\Big)^{\frac{(2-q)q^*}{2q}}\Big(\|f\|_{L^1}^{\frac{q^*}{2}} + \|f\|_{L^1}^{\frac{(2-r)q^*}{2}}\Big)$$

where $C_1, C_2 > 0$ *are constants independent of* f *and* u *and* $r = \frac{N(2-q)}{N-q}$.

Proof. The proof is a modification of the argument used in the proof of Lemma 2.5 in [8]. We take as a test function $\phi(x) = (1+u)^{1-r}$. Since $q < \frac{N}{N-1}$, we have $r > 1$. Also $r < 2$ because $N \ge 3$. Hence $\phi(x) \le 1$ on Ω and upon a substitution we obtain

$$\begin{aligned}(r-1)\int_\Omega \frac{|\nabla u|^2}{(1+u)^r}\,dx &= \int_\Omega g(x,u,\nabla u)(1+u)^{1-r}dx + \\ &\quad + \lambda\int_\Omega u(1+u)^{1-r}dx - \\ &\quad - \int_\Omega f(1+u)^{1-r}dx \le \\ &\le \int_\Omega g(x,u,\nabla u)dx + \lambda\int_\Omega u\,dx.\end{aligned} \tag{3.9}$$

Testing equation (3.1) with a constant function 1 we obtain

$$\int_\Omega g(x,u,\nabla u)\,dx + \lambda\int_\Omega u\,dx = \int_\Omega f\,dx. \tag{3.10}$$

From (3.9) and (3.10) we derive

$$\int_\Omega \frac{|\nabla u|^2}{(1+u)^r}\,dx \le \frac{1}{r-1}\int_\Omega f\,dx \quad\text{and}\quad \int_\Omega u\,dx \le \frac{1}{\lambda}\int_\Omega f\,dx. \tag{3.11}$$

By the Sobolev inequality we obtain

$$\begin{aligned}
\Big(\int_\Omega u^{q^*}\,dx\Big)^{\frac{q}{q^*}} &\le C\int_\Omega \big(|\nabla u|^q + u^q\big)\,dx = \\
&= C\int_\Omega \frac{|\nabla u|^q}{(1+u)^{\frac{rq}{2}}}(1+u)^{\frac{rq}{2}}\,dx + C\int_\Omega \frac{u^q}{(1+u)^{\frac{rq}{2}}}(1+u)^{\frac{rq}{2}}\,dx \le \\
&\le C\Big(\int_\Omega \frac{|\nabla u|^2}{(1+u)^r}\,dx\Big)^{\frac{q}{2}}\Big(\int_\Omega (1+u)^{\frac{rq}{2-q}}\,dx\Big)^{\frac{2-q}{2}} + \\
&\quad + C\Big(\int_\Omega \frac{u^2}{(1+u)^r}\,dx\Big)^{\frac{q}{2}}\Big(\int_\Omega (1+u)^{\frac{rq}{2-q}}\,dx\Big)^{\frac{2-q}{2}} \le \\
&\le C\Big(\int_\Omega \frac{|\nabla u|^2}{(1+u)^r}\,dx\Big)^{\frac{q}{2}}\Big(\int_\Omega (1+u)^{\frac{rq}{2-q}}\,dx\Big)^{\frac{2-q}{2}} + \\
&\quad + C\Big(\int_\Omega u^{2-r}\,dx\Big)^{\frac{q}{2}}\Big(\int_\Omega (1+u)^{\frac{rq}{2-q}}\,dx\Big)^{\frac{2-q}{2}} \le \\
&\le C\Big(\int_\Omega \frac{|\nabla u|^2}{(1+u)^r}\,dx\Big)^{\frac{q}{2}}\Big(\int_\Omega (1+u)^{\frac{rq}{2-q}}\,dx\Big)^{\frac{2-q}{2}} + \\
&\quad + C|\Omega|^{\frac{q(r-1)}{2}}\Big(\int_\Omega |u|\,dx\Big)^{\frac{(2-r)q}{2}}\Big(\int_\Omega (1+u)^{\frac{rq}{2-q}}\,dx\Big)^{\frac{2-q}{2}}.
\end{aligned}$$

We now observe that $q^* = \frac{rq}{2-q}$. Hence combining the above estimate with (3.11) the result follows. □

It is clear that Lemma 3.3 leads to the following existence result.

Theorem 3.4. *Suppose that* $1 \le q < \frac{N}{N-1}$, $f > 0$ *on* Ω *and* $f \in L^1(\Omega)$. *The problem* (3.1) *has a positive solution* $u \in W^{1,q}(\Omega)$.

4. HIGHER INTEGRABILITY PROPERTY FOR SOLUTIONS OF (1.1)

The method used in the proof of Lemma 2.1 allows only to estimate the norm $W^{1,q}$ of a positive solution, where q is the exponent appearing in the equation. In the case $1 \le q < 2$, a question arises whether a solution to (1.1) belongs to $W^{1,\bar{q}}(\Omega)$ with $q < \bar{q}$. We distinguish two cases: (i) $1 \le q < \frac{N}{N-1}$ and (ii) $\frac{N}{N-1} < q < 2$. In the case (i) assuming that $f \in L^1(\Omega)$ we show that a solution belongs to $W^{1,\bar{q}}(\Omega)$ or every $q < \bar{q} < \frac{N}{N-1}$. In the case (ii) we show that a solution belongs $W^{1,\bar{q}}(\Omega)$ for some $q < \bar{q} < 2$ under some additional assumption on f. According to Step 1 of the proof of Theorem 2.2, if $f \in L^\infty(\Omega)$, then problem (1.1) has a solution $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Lemma 4.1. *Suppose that $f > 0$ on Ω, $f \in L^\infty(\Omega)$ and $1 \le q < \bar{q} < \frac{N}{N-1}$. If $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ is a positive solution of* (1.1)*, then there exist constants $C_1, C_2 > 0$, independent of u and f such that*

$$\int_\Omega u^{\bar{q}^*}\,dx \le C_1\Big(\int_\Omega \big(|\nabla u|^{\bar{q}} + u^{\bar{q}}\big)\,dx\Big)^{\frac{\bar{q}^*}{\bar{q}}} \le$$

$$\le C_2\Big(\int_\Omega \big(1 + u^{\bar{q}^*}\big)\,dx\Big)^{\frac{(2-\bar{q})\bar{q}^*}{2\bar{q}}}\Big(\|f\|_{L^1}^{\frac{\bar{q}^*}{2}} + \|f\|_{L^1}^{\frac{(2-\bar{r})\bar{q}^*}{2}}\Big),$$

where $\bar{r} = \frac{N(2-\bar{q})}{N-\bar{q}}$ and $\bar{q}^* = \frac{N\bar{q}}{N-\bar{q}}$.

Proof. As in the proof of Lemma 3.3 we take as a test function $\phi(x) = (1+u)^{1-\bar{r}}$. Since $\bar{q} < \frac{N}{N-1}$, we have $\bar{r} > 1$. Also $\bar{r} < 2$ because $N \ge 3$. Hence $\phi(x) \le 1$ on Ω and upon a substitution we obtain

$$\begin{aligned}(\bar{r}-1)\int_\Omega \frac{|\nabla u|^2}{(1+u)^{\bar{r}}}\,dx &= \int_\Omega |\nabla u|^q (1+u)^{1-\bar{r}}\,dx + \lambda\int_\Omega u(1+u)^{1-\bar{r}}\,dx - \\ &- \int_\Omega f(1+u)^{1-\bar{r}}\,dx \le \int_\Omega |\nabla u|^q\,dx + \lambda\int_\Omega u\,dx.\end{aligned} \tag{4.1}$$

Testing (1.1) with a constant function 1 we obtain

$$\int_\Omega |\nabla u|^q + \lambda\int_\Omega u\,dx = \int_\Omega f\,dx. \tag{4.2}$$

By the Sobolev inequality we obtain

$$\begin{aligned}\Big(\int_\Omega u^{\bar{q}^*}\,dx\Big)^{\frac{\bar{q}}{\bar{q}^*}} &\le C\int_\Omega \big(|\nabla u|^{\bar{q}} + u^{\bar{q}}\big)\,dx = \\ &= C\int_\Omega \frac{|\nabla u|^{\bar{q}}}{(1+u)^{\frac{\bar{r}\bar{q}}{2}}}(1+u)^{\frac{\bar{r}\bar{q}}{2}}\,dx + C\int_\Omega \frac{u^{\bar{q}}}{(1+u)^{\frac{\bar{r}\bar{q}}{2}}}(1+u)^{\frac{\bar{r}\bar{q}}{2}}\,dx \le \\ &\le C\Big(\int_\Omega \frac{|\nabla u|^2}{(1+u)^{\bar{r}}}\,dx\Big)^{\frac{\bar{q}}{2}}\Big(\int_\Omega (1+u)^{\frac{\bar{r}\bar{q}}{2-\bar{q}}}\,dx\Big)^{\frac{2-\bar{q}}{2}} + \\ &+ C\Big(\int_\Omega \frac{u^2}{(1+u)^{\bar{r}}}\,dx\Big)^{\frac{\bar{q}}{2}}\Big(\int_\Omega (1+u)^{\frac{\bar{r}\bar{q}}{2-\bar{q}}}\,dx\Big)^{\frac{2-\bar{q}}{2}}.\end{aligned}$$

Combining the above inequality with (4.1) and (4.2) we obtain

$$\begin{aligned}\left(\int_\Omega u^{\bar q^*}\,dx\right)^{\frac{\bar q}{\bar q^*}} &\le C\int_\Omega \left(|\nabla u|^{\bar q}+u^{\bar q}\right)dx \le \\ &\le C\left(\int_\Omega f\,dx\right)^{\frac{\bar q}{2}}\left(\int_\Omega (1+u)^{\bar q^*}\,dx\right)^{\frac{2-\bar q}{2}}+ \\ &\quad + C\left(\int_\Omega (1+u)^{\bar q^*}\,dx\right)^{\frac{2-\bar q}{2}}\left(\int_\Omega u^{2-\bar r}\,dx\right)^{\frac{\bar q}{2}} \le \\ &\le C\left(\int_\Omega (1+u)^{\bar q^*}\,dx\right)^{\frac{2-\bar q}{2}}\left[\|f\|_{L^1}^{\frac{\bar q}{2}}+\|f\|_{L^1}^{(2-\bar r)\frac{\bar q}{2}}\right].\end{aligned}$$

This yields the desired estimate. □

Lemma 4.2. *Let $f>0$ on Ω, $f\in L^\infty(\Omega)$ and $\frac{N}{N-1}<q<\bar q<2$. If $u\in W^{1,2}(\Omega)\cap L^\infty(\Omega)$ is a positive solution of (1.1), then*

$$\begin{aligned}\int_\Omega u^{\bar q^*}\,dx &\le C_1\left(\int_\Omega \left(|\nabla u|^{\bar q}+u^{\bar q}\right)dx\right)^{\frac{\bar q^*}{\bar q}} \le \\ &\le C_2\|f\|_{L^{\bar m}}^{\frac{\bar q^*}{2}}\left(\int_\Omega u^{\bar q^*}\,dx\right)^{\frac{1-\bar r}{2}}\left(\int_\Omega \left(1+u^2\right)^{\frac{\bar q^*}{2}}\,dx\right)^{\frac{\bar r}{2}},\end{aligned}$$

where $C_1, C_2>0$ are positive constants independent of u and f, and $\bar r=\frac{N(2-\bar q)}{N-\bar q}$, $\bar m=\frac{N\bar q}{N+\bar q}$.

The proof is similar to that of Lemma 3.1 and is omitted.

These two lemmas yield the following result.

Theorem 4.3. *Suppose that $f>0$ on Ω.*

(i) *If $f\in L^1(\Omega)$ and $1\le q<\frac{N}{N-1}$, then problem (1.1) has a solution that belongs to $W^{1,\bar q}(\Omega)$ for every $q\le\bar q<\frac{N}{N-1}$.*

(ii) *If $f\in L^{\bar m}(\Omega)$ with $\bar m=\frac{N\bar q}{N+\bar q}$, $\frac{N}{N-1}\le q<\bar q<2$, then problem (1.1) has a solution belonging to $W^{1,\bar q}(\Omega)$.*

Higher integrability property can also be established to solutions of problem (3.1).

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