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ON ELLIPTIC PROBLEMS WITH A NONLINEARITY DEPENDING ON THE GRADIENT

Abstract. We investigate the solvability of the Neumann problem (1.1) involving the nonlinearity depending on the gradient. We prove the existence of a solution when the right hand side $f$ of the equation belongs to $L^m(\Omega)$ with $1 \leq m < 2$.

Keywords: Neumann problem, nonlinearity depending on the gradient, $L^1$ data.

Mathematics Subject Classification: 35D05, 35J25, 35J60.

1. INTRODUCTION

In this paper we investigate the solvability of the nonlinear Neumann problem with a nonlinearity depending on the gradient. First we consider the following problem

\begin{equation}
\begin{cases}
-\Delta u + |\nabla u|^q + \lambda u = f(x) & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega,
\end{cases}
\end{equation}

where $\lambda > 0$ is a parameter, $1 \leq q \leq 2$ and $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with a smooth boundary $\partial\Omega$. It is assumed that $f \in L^1(\Omega)$. If $f > 0$ on $\Omega$, then solutions, if they exist, are positive. In Section 3 we consider problem (1.1) with $|\nabla u|^q$ replaced by a nonlinearity satisfying a sign condition. The boundary value problems with data in $L^1$ has been studied quite extensively in recent years. The Dirichlet problem with a nonlinearity depending only on $u$ has been considered in papers [7, 10]. Some extensions to the Neumann problem can be found in paper [12]. These results has been extended to the case where a nonlinearity depends on the gradient. In particular, more general elliptic operators with more general nonlinearities with $f \in L^1(\Omega)$ or being a Radon measure have been investigated in [3–6, 11]. Further extensions to the Dirichlet problem with $L^2$ boundary data can be found in [11]. We refer to paper [2] for the bibliographical references. It seems that less is known for the Neumann problem.
By $W^{1,p}(\Omega)$, $1 \leq p < \infty$, we denote the Sobolev space equipped with norm

$$
\|u\|_{W^{1,p}(\Omega)} = \int_\Omega (|\nabla u|^p + |u|^p) \, dx.
$$

Throughout this paper, in a given Banach space $X$, we denote strong convergence by “$\rightarrow$” and weak convergence by “$\rightharpoonup$”. The norms in the Lebesgue spaces $L^p(\Omega)$, $1 \leq p < \infty$, are denoted by $\| \cdot \|_{L^p}$.

The paper is organized as follows. In Section 2 we prove the existence of positive solutions of (1.1) assuming that $f$ is positive and belongs to $L^1(\Omega)$. Section 3 is devoted to the problem with a nonlinearity satisfying a sign condition, where we do not assume that $f$ is positive. The crucial point in our approach are estimates of $W^{1,q}$-norm of solutions of (1.1) in terms of $L^m$-norm of $f$ (see Lemmas 2.1, 3.1, 3.3). The estimates in terms of $L^m$ norm of $f$ (see Lemmas 3.1, 3.3) in a linear case were given in [8] and are extended in this paper to solutions of (1.1). In these two lemmas the important assumption is that $q \neq \frac{N}{N-1}$, which is due to the use of special test functions in the proofs. We were unable to show whether these lemmas continue to hold for $q = \frac{N}{N-1}$. In Section 4 we establish the higher integrability property for positive solutions of (1.1).

The main results of this paper are Theorems 2.2, 3.2, 3.4. In the proofs we use some ideas from paper [4].

2. EXISTENCE OF POSITIVE SOLUTIONS

In this section consider problem (1.1) assuming that $f > 0$ on $\Omega$. Then a solution, if it exists, is positive on $\Omega$. We need the following definition of a solution of (1.1): let $f \in L^1(\Omega)$, then a function $u \in W^{1,q}(\Omega)$ is a solution of (1.1) if

$$
\int_\Omega \nabla u \nabla v \, dx + \int_\Omega |\nabla u|^q v \, dx + \lambda \int_\Omega uv \, dx = \int_\Omega fv \, dx,
$$

(2.1)

for every function $v \in W^{1,\infty}(\Omega)$.

**Lemma 2.1.** Let $1 \leq q \leq 2$ and $f \in L^\infty(\Omega)$ with $f > 0$ on $\Omega$. If $u \in W^{1,2}(\Omega)$ is a positive solution of (1.1), then

$$
\int_\Omega (|\nabla u|^q + u^q) \, dx \leq C_1 \int_\Omega f \, dx + C_2 \left( \int_\Omega f \, dx \right)^q,
$$

(2.2)

where $C_1, C_2 > 0$ are constants independent of $u$ and $f$.

**Proof.** Testing (2.1) with the constant function 1 we get

$$
\int_\Omega |\nabla u|^q \, dx + \lambda \int_\Omega u \, dx = \int_\Omega f \, dx.
$$

(2.3)
It is clear that equality (2.3) yields (2.2) if \( q = 1 \). To proceed further we use a decomposition \( W^{1,2}(\Omega) = V \oplus \text{span} \ 1 \), where

\[
V = \{ v \in W^{1,2}(\Omega); \int_\Omega v \, dx = 0 \}.
\]

Then \( u = v + t \), with \( v \in V \) and \( t = \frac{1}{|\Omega|} \int_\Omega u \, dx > 0 \), because \( u \) is positive. From (2.3) we deduce

\[
t \leq \frac{1}{\lambda |\Omega|} \int_\Omega f \, dx.
\]

(2.4)

We now observe that the Poincaré inequality is valid in \( V \), that is, there exists a constant \( C(\Omega) > 0 \) such that

\[
\int_\Omega |v|^q \, dx \leq C(\Omega) \int_\Omega |\nabla v|^q \, dx
\]

for every \( v \in V \). Consequently, using (2.4), we can estimate the norm of \( u \) in \( W^{1,q}(\Omega) \) as follows

\[
\int_\Omega (|\nabla u|^q + u^q) \, dx \leq \int_\Omega |\nabla v|^q \, dx + 2^{q-1} \int_\Omega (v^q + t^q) \, dx \leq \int_\Omega |\nabla v|^q \, dx + 2^{q-1} C(\Omega) \int_\Omega |\nabla v|^q \, dx + 2^{q-1} |\Omega| t^q.
\]

This combined with (2.4) and (2.3) implies (2.2).

We are now in a position to formulate the first existence result.

**Theorem 2.2.** Let \( 1 \leq q \leq 2 \) and \( f \) be a positive function in \( L^1(\Omega) \). Then problem (1.1) admits a positive solution in \( W^{1,q}(\Omega) \).

**Proof.** The proof will be given in 2 steps.

**Step 1.** Assume \( f \in L^{\infty}(\Omega) \). Consider the problem

\[
\begin{cases}
-\Delta u + \lambda u = f(x) & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \\
u > 0 & \text{on } \Omega.
\end{cases}
\]

(2.5)

This problem has a unique positive solution \( v \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega) \) (see [1]). We now use some ideas from papers [5] and [6]. For each \( n \in \mathbb{N} \) we consider the following problem

\[
\begin{cases}
-\Delta w_n + \frac{|\nabla w_n|^q}{1 + \frac{1}{n}|\nabla w_n|^q} + \lambda w_n = f(x) & \text{in } \Omega, \\
\frac{\partial w_n}{\partial \nu} = 0 & \text{on } \partial \Omega, \\
w_n > 0 & \text{on } \Omega.
\end{cases}
\]

(2.6)
It is clear that $v$ is a super-solution to problem (2.6) and $0$ is a sub-solution. Thus problem (2.6) admits a solution $0 \leq w_n \leq v$. This fact is known for equation (2.6) with the Dirichlet boundary conditions (see [5]). The result from [5] can be easily extended to the Neumann problem (2.6). The sequence $\{w_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Testing (2.6) with $w_n$ we obtain

$$
\int_\Omega (|\nabla w_n|^2 + \lambda w_n^2) \, dx \leq \|f\|_{L^2} \|w_n\|_{L^2},
$$

which shows that the sequence $\{w_n\}$ is bounded in $W^{1,2}(\Omega)$. We may assume that $w_n \to w$ in $W^{1,2}(\Omega)$, $w_n \to w$ in $L^2(\Omega)$ and $w_n \to w$ a.e. on $\Omega$. We now show that $w_n \to w$ in $W^{1,2}(\Omega)$. We put $\phi(s) = s \exp(-s^2/4)$ for $s \in \mathbb{R}$. We introduce notation $H_n(s) = \frac{|s|^q}{1 + |s|^q}$. The function $\phi$ satisfies $\phi'(s) - \phi(s) \geq \frac{1}{2}$ for $s \in \mathbb{R}$. Testing (2.6) with $\phi(w_n - w)$ we obtain

$$
\int_\Omega \nabla w_n \phi'(w_n - w) \nabla (w_n - w) \, dx + \int_\Omega H_n(|\nabla w_n|) \phi(w_n - w) \, dx + \\
+ \lambda \int_\Omega w_n \phi(w_n - w) \, dx = \int_\Omega f(x) \phi(w_n - w) \, dx.
$$

(2.7)

It is easy to check that

$$
\int_\Omega \nabla w_n \phi'(w_n - w) \nabla (w_n - w) \, dx = \int_\Omega |\nabla (w_n - w)|^2 \phi'(w_n - w) \, dx + o(1). \quad (2.8)
$$

To estimate the second term on the left side of (2.7) we use the inequality: if $1 \leq q < 2$, then for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$
s^q \leq \epsilon s^2 + C_\epsilon \quad \text{for every} \quad s \geq 0. \quad (2.9)
$$

We then have

$$
\int_\Omega H_n(|\nabla w_n|) \phi(w_n - w) \, dx \leq \epsilon \int_\Omega |\nabla w_n|^2 \phi(w_n - w) \, dx + C_\epsilon \int_\Omega |\phi(w_n - w)| \, dx = \\
= \epsilon \int_\Omega |\nabla (w_n - w)|^2 \phi(w_n - w) \, dx + \\
- \epsilon \int_\Omega |\nabla w|^2 \phi(w_n - w) \, dx + \\
+ 2\epsilon \int_\Omega \nabla w_n \nabla w \phi(w_n - w) \, dx + \\
+ C_\epsilon \int_\Omega |\phi(w_n - w)| \, dx.
$$

(2.10)
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Since

\[ \int_{\Omega} |\nabla w|^2 |\phi(w_n - w)| \, dx \to 0, \quad \int_{\Omega} \nabla w_n \nabla |\phi(w_n - w)| \, dx \to 0 \]

and

\[ \int_{\Omega} |\phi(w_n - w)| \, dx \to 0 \]

as \( n \to \infty \), we derive from (2.10) that

\[ \int_{\Omega} H_n(|\nabla w_n|)|\phi(w_n - w)| \, dx \leq \epsilon \int_{\Omega} |\nabla w_n - \nabla w|^2 |\phi(w_n - w)| \, dx + o(1). \quad (2.11) \]

If \( q = 2 \), then instead of (2.10) we have

\[ \int_{\Omega} H_n(|\nabla w_n|)|\phi(w_n - w)| \, dx \leq \int_{\Omega} |\nabla w_n|^2 |\phi(w_n - w)| \, dx \]

and (2.11) holds with \( \epsilon = 1 \). We also have

\[ \int_{\Omega} f(x) \phi(w_n - w) \, dx \to 0 \quad \text{and} \quad \int_{\Omega} w_n \phi(w_n - w) \, dx \to 0 \quad (2.12) \]

as \( n \to \infty \). If \( 1 \leq q < 2 \) we derive from (2.7), (2.8), (2.11) and (2.12) that

\[ \frac{1}{2} \int_{\Omega} |\nabla (w_n - w)|^2 \, dx \leq \int_{\Omega} \left( \phi'(w_n - w) - \epsilon|\phi(w_n - w)| \right) |\nabla (w_n - w)|^2 \, dx = o(1). \]

Thus \( w_n \to w \) in \( W^{1,2}(\Omega) \). If \( q = 2 \), the above inequality continues to hold with \( \epsilon = 1 \). In this case we also have that \( w_n \to w \) in \( W^{1,2}(\Omega) \). Since \( 1 \leq q \leq 2, \nabla w_n \to \nabla w \) in \( L^q(\Omega) \). For each \( \phi \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \) and for each \( n \) we have

\[ \int_{\Omega} \nabla w_n \nabla \phi \, dx + \int_{\Omega} \frac{|\nabla w_n|^q}{1 + \frac{1}{q} |\nabla w_n|^q} \phi \, dx + \lambda \int_{\Omega} w_n \phi \, dx = \int_{\Omega} f \phi \, dx. \]

Letting \( n \to \infty \) we get

\[ \int_{\Omega} \nabla w \nabla \phi \, dx + \int_{\Omega} |\nabla w|^q \phi \, dx + \lambda \int_{\Omega} w \phi \, dx = \int_{\Omega} f \phi \, dx. \]

So \( w \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \) is a weak solution of (1.1).
Step 2. First we consider the case $1 \leq q < 2$. Let $f \in L^1(\Omega)$ and let $\{f_n\} \subset L^\infty(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$. By Step 1 for each $n \in \mathbb{N}$ there exists a solution $u_n \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ to problem (1.1) with $f = f_n$. For each $k > 1$ we put $T_k(s) = \min(s, k)$ for $0 \leq s$. Taking $T_k u_n$ as a test function in (1.1) we get

$$
\int_\Omega |\nabla T_k u_n|^2 \, dx + \lambda \int_\Omega |T_k u_n|^2 \, dx \leq \int_\Omega f_n T_k u_n \, dx \leq k\|f_n\|_{L^1}.
$$

Consequently, $\{T_k u_n\}$ is bounded in $W^{1,2}(\Omega)$. By Lemma 2.1 we may assume that $u_n \to u$ in $W^{1,q}(\Omega)$. We may also assume that $T_k u_n \to T_k u$ in $W^{1,2}(\Omega)$ and $T_k u_n \to T_k u$ in $L^2(\Omega)$. Let $G_k(s) = s - T_k(s)$ and put $\psi_k^{-1}(s) = T_1(G_k^{-1}(s))$. Thus

$$
\psi_k^{-1}(u_n)|\nabla u_n|^q \geq |\nabla u_n|^q \chi_{(\max(u_n) < k)}.
$$

Using $\psi_k^{-1}(u_n)$ as a test function in (2.1) (with $f = f_n$) we get

$$
\int_\Omega |\nabla \psi_k^{-1}(u_n)|^2 \, dx + \int_\Omega \psi_k^{-1}(u_n)|\nabla u_n|^q \, dx + \lambda \int_\Omega u_n \psi_k^{-1}(u_n) \, dx = \int_\Omega f_n \psi_k^{-1}(u_n) \, dx.
$$

Since $\{u_n\}$ is bounded in $L^p(\Omega)$ for each $p \leq q^* = \frac{Nq}{N-q}$ we see that

$$
|x \in \Omega; k-1 < u_n(x) < k| \to 0 \text{ and } |\{x \in \Omega; k < u_n(x)\}| \to 0
$$

as $k \to \infty$ uniformly in $n$. So

$$
\lim_{k \to \infty} \int_{u_n > k} |\nabla u_n|^q \, dx = 0 \tag{2.13}
$$

uniformly in $n$. Using as a test function $\phi(T_k u_n - T_k u)$ and repeating the argument from Step 1 we show that $T_k u_n \to T_k u$ in $W^{1,2}(\Omega)$. We now use this to show that the sequence $\{|\nabla u_n|^q\}$ is equi-integrable. This follows from (2.13) and the following inequality: for every measurable subset $E \subset \Omega$ we have

$$
\int_E |\nabla u_n|^q \, dx \leq \int_E |\nabla T_k u_n|^q \, dx + \int_{\{u_n > k\} \cap E} |\nabla u_n|^q \, dx.
$$

Indeed, given $\epsilon > 0$, according to (2.13), we can find $k$ large enough such that

$$
\int_{u_n > k} |\nabla u_n|^q \, dx < \frac{\epsilon}{2}
$$

for all $n$. Since $\nabla T_k (u_n) \to \nabla T_k (u)$ in $L^2(\Omega)$ there exists $\delta > 0$ such that

$$
\int_E |\nabla T_k (u_n)|^q \, dx < \frac{\epsilon}{2}
$$

provided $|E| \leq \delta$ and for all $n$. By Vitali’s theorem $\nabla u_n \to \nabla u$ in $L^q(\Omega)$. Thus $u$ is a weak solution of (1.1). If $q = 2$, then by Lemma 2.1 the sequence $\{u_n\}$ is bounded in $W^{1,2}(\Omega)$. An obvious modification of Step 2 completes the proof. \qed
3. NONLINEARITY WITH A SIGN CONDITION

In this section we discuss the solvability of the following problem

\[\begin{cases}
-\Delta u + g(x, u, \nabla u) + \lambda u = f(x) & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}\] (3.1)

We assume that the nonlinearity \( g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) is a Carathéodory function, that is, \( g(\cdot, s, \xi) \) is measurable on \( \Omega \) for every \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^N \) and \( g(x, \cdot, \cdot) \) is continuous on \( \mathbb{R} \times \mathbb{R}^N \) for a.e. \( x \in \Omega \). Moreover, we assume that

1. \((g_1)\) there exist an increasing and continuous function \( b : [0, \infty) \to [0, \infty) \) with \( b(0) = 0 \) and a positive function \( a \in L^1(\Omega) \) such that
   \[|g(x, s, \xi)| \leq b(|s|)(|\xi|^q + a(x))\]
   for a.e. \( x \in \Omega \) and for every \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\).

2. \((g_2)\) \( g(x, s, \xi) \) \( \text{sgn } s \geq 0 \) for a.e. \( x \in \Omega \) and for every \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\).

A typical example of a nonlinearity satisfying \((g_1)\) and \((g_2)\) is \( g(x, s, \xi) = s|\xi|^q \).

We now consider equation (3.1) without assumption that \( f \) is positive on \( \Omega \). Obviously, it is assumed that \( f \neq 0 \) on \( \Omega \). We assume that \( \frac{N}{N-1} < q < 2 \). Then there exists \( 1 < m < \frac{2N}{N+q} \) such that \( q = m^* = \frac{Nm}{N-m} \). In this case \( m \) is given by \( m = \frac{Nq}{N+q} \).

We also use notation \( q^* = \frac{Nq}{N+q} \). With these notations we establish the estimates of norms \( \|u\|_{L^{q^*}} \) and \( \|u\|_{W^{1,q}} \) of a solution \( u \) of (1.1) in terms of the norm \( \|f\|_{L^m} \).

**Lemma 3.1.** Let \( f \in L^\infty(\Omega) \) and \( \frac{N}{N-1} < q < 2 \). If \( u \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \) is a solution of (3.1), then

\[\int_{\Omega} |u|^{q^*} \, dx \leq C_1 \left( \int_{\Omega} (|\nabla u|^q + |u|^q) \, dx \right)^{\frac{q}{q^*}} \leq C_2 \|f\|_{L^m}^{\frac{q^*}{q}} \left( \int_{\Omega} |u|^{q^*} \, dx \right)^{\frac{1-r}{2r}} \left( \int_{\Omega} (1 + u^2)^{q^*} \, dx \right)^{\frac{r}{2}}, \] (3.2)

where \( r = \frac{N(2-q)}{N-q} \) and \( C_1 > 0 \) and \( C_2 > 0 \) are constants independent of \( u \) and \( f \).
Proof. We follow some ideas from [8], where the same estimate was proved for the linear problem. Put \( \varphi(x) = \frac{u}{(1+u^2)^{\frac{q}{2}}} \). Since \( \frac{N}{N-1} < q < 2 \), we have \( 0 < r < 1 \). Since \( u \in L^\infty(\Omega) \), \( \varphi \) is a legitimate test function. Upon the substitution we obtain

\[
(1-r) \int_\Omega \frac{|\nabla u|^2}{(1+u^2)^{\frac{q}{2}}} \, dx + \lambda \int_\Omega \frac{u^2}{(1+u^2)^{\frac{q}{2}}} \, dx \leq \int_\Omega \frac{|fu|}{(1+u^2)^{\frac{q}{2}}} \, dx \leq \|f\|_{L^m} \left( \int_\Omega |u|^{(1-r)m'} \, dx \right)^{\frac{1}{m'}} ,
\]

(3.3)

where \( m' = \frac{m}{m-1} \). Here we used the fact that

\[
\int_\Omega \frac{ug(x,u,\nabla u)}{(1+u^2)^{\frac{q}{2}}} \, dx \geq 0
\]
due to assumption \((g_2)\). In what follows we denote by \( C > 0 \) a constant which is independent of \( u \) and \( f \) and may vary from line to line. By the Sobolev inequality we have

\[
\left( \int_\Omega |u|^q \, dx \right)^{\frac{1}{q}} \leq C \int_\Omega \left( |\nabla u|^q + |u|^q \right) \, dx =
\]

\[
= C \int_\Omega \frac{|\nabla u|^q}{(1+u^2)^{\frac{q}{2}}} (1+u^2)^{\frac{q}{4}} \, dx +
\]

\[
+ C \int_\Omega \frac{|u|^q}{(1+u^2)^{\frac{q}{2}}} (1+u^2)^{\frac{q}{4}} \, dx \leq
\]

(3.4)

Inserting (3.3) into (3.4) we derive

\[
\left( \int_\Omega |u|^q \, dx \right)^{\frac{1}{q}} \leq C \int_\Omega \left( |\nabla u|^q + |u|^q \right) \, dx \leq
\]

\[
\leq C \|f\|_{L^m} \left( \int_\Omega |u|^{(1-r)m'} \, dx \right)^{\frac{1}{m'}} \left( \int_\Omega (1+u^2)^{\frac{q}{4m'-q}} \, dx \right)^{\frac{2-q}{2}} .
\]
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Since \( r = \frac{N(2-q)}{N-q} \), we have \( \frac{rq}{2} = q^* \) and \( (1-r)m' = q^* \). Therefore the above inequality becomes

\[
\int_{\Omega} |u|^{q^*} \, dx \leq C \left( \int_{\Omega} \left( |\nabla u|^q + |u|^q \right) \, dx \right)^{\frac{q^*}{2}} \leq C \|f\|_{L^m(\Omega)}^{\frac{q^*}{2}} \left( \int_{\Omega} \left( |u|^{q^*} \, dx \right)^{\frac{q^*}{2-m}} \left( \int_{\Omega} \left( 1 + u^2 \right)^{\frac{q^*}{2}} \, dx \right)^{\frac{(2-q)e^*}{4}} \right).
\]

Since \( \frac{q^*}{2m} = \frac{1-r}{2} \) and \( \frac{(2-q)e^*}{4} = \frac{r}{2} \), the result follows.

We are now in a position to formulate the second existence result.

**Theorem 3.2.** Let \( \frac{N}{N-1} < q < 2 \) and \( f \in L^m(\Omega) \) with \( m = \frac{Nq}{N+q} \). Suppose that assumptions \((g_1)\) and \((g_2)\) hold. Then problem (1.1) admits a solution in \( W^{1,q}(\Omega) \).

**Proof.** The proof is similar to that of Theorem 2.2 except some technical modifications. First we assume that \( f \in L^\infty(\Omega) \). For every \( n \in \mathbb{N} \) we put

\[
g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{n}|g(x,s,\xi)|}
\]

and consider the following problem

\[
\begin{cases}
-\Delta u + g_n(x,u,\nabla u) + \lambda u = f(x) & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega.
\end{cases}
\]

Then the functions \( v_1 = \frac{\|f\|_\infty}{\lambda} \) and \( v_2 = -\frac{\|f\|_\infty}{\lambda} \) are a super-solution and a sub-solution to problem (3.5), respectively. For every \( n \) problem (3.5) has a solution \( w_n \) satisfying \( v_1 \leq w_n \leq v_2 \) on \( \Omega \). Hence the sequence \( \{w_n\} \) is bounded in \( L^\infty(\Omega) \), that is, \( \|w_n\|_\infty \leq M \) for some constant \( M > 0 \) and for all \( n \in \mathbb{N} \). Testing (3.5) with \( w_n \) we show that \( \{w_n\} \) is bounded in \( W^{1,2}(\Omega) \). So we may assume that \( w_n \rightharpoonup w \) in \( W^{1,2}(\Omega) \), \( w_n \rightarrow w \) in \( L^2(\Omega) \) and \( w_n \rightarrow w \) a.e. on \( \Omega \). Let \( \phi \) be a function introduced in the proof of Theorem 2.2. Testing (3.5) with \( \phi(w_n - w) \) we obtain

\[
\int_{\Omega} \nabla w_n \phi'(w_n - w) \nabla(w_n - w) \, dx + \int_{\Omega} g_n(x,w_n,\nabla w_n)\phi(w_n - w) \, dx + \\
+ \lambda \int_{\Omega} w_n \phi(w_n - w) \, dx = \int_{\Omega} f(x)\phi(w_n - w) \, dx.
\]

It is clear that

\[
\int_{\Omega} \nabla w_n \phi'(w_n - w) \nabla(w_n - w) \, dx = \int_{\Omega} |\nabla(w_n - w)|^2 \phi'(w_n - w) \, dx + o(1).
\]
We use inequality (2.9) and assumption \((g_1)\) to estimate the second integral on the left side of (3.6)

\[
\int_{\Omega} |g_n\phi(w_n - w)| \, dx \leq b(M) \int_{\Omega} |\nabla w_n|^2 |\phi(w_n - w)| \, dx + \int_{\Omega} a(x)|\phi(w_n - w)| \, dx \leq
\]

\[
\leq b(M)e \int_{\Omega} |\nabla w_n|^2 |\phi(w_n - w)| \, dx + C_1 \int_{\Omega} |\phi(w_n - w)| \, dx + \int_{\Omega} a(x)|\phi(w_n - w)| \, dx.
\]

Since \(\phi(w_n - w) \to 0\) a.e. on \(\Omega\) and \(\sup_n |\phi(w_n - w)| < \infty\) by the Lebesgue dominated convergence theorem we get

\[
\int_{\Omega} |g_n\phi(w_n - w)| \, dx \leq b(M)e \int_{\Omega} |\nabla w_n|^2 |\phi(w_n - w)| \, dx + o(1).
\]

As in the proof of Theorem 2.2 we deduce from this that

\[
\int_{\Omega} |g_n\phi(w_n - w)| \, dx \leq b(M)e \int_{\Omega} |\nabla w_n - \nabla w|^2 |\phi(w_n - w)| \, dx + o(1). \tag{3.8}
\]

Taking \(eb(M) \leq 1\) we deduce from (3.6), (3.7) and (3.8) that

\[
\int_{\Omega} |\nabla w_n - \nabla w|^2 \, dx \leq \int_{\Omega} \left( |\phi'(w_n - w) - eb(M)| |\phi(w_n - w)| \right) |\nabla w_n - \nabla w|^2 \, dx = o(1).
\]

Thus \(w_n \rightharpoonup w\) in \(W^{1,2}(\Omega)\). It is clear that \(w\) is a solution of (3.1). In the final step we choose a sequence \(\{f_n\} \subset L^\infty(\Omega)\) such that \(f_n \to f\) in \(L^m(\Omega)\). Then for every \(n \in \mathbb{N}\) problem (3.1) with \(f = f_n\) admits a solution \(u_n \in W^{1,2}(\Omega) \cap L^\infty(\Omega)\). We now define a sequence of truncations \(\{T_k(u_n)\}\) for every \(k > 0\), where \(T_k = \max(-k, \min(s,k))\). Let \(G_k(s) = s - T_k(s)\) and put \(\psi_{k-1}(s) = T_k(G_{k-1}(s))\). Thus

\[
\psi_{k-1}(u_n) |\nabla u_n|^2 \geq |\nabla u_n|^2 \chi_{|u_n| \geq k}.
\]

As in the proof of Theorem 2.2 we show that the sequence \(\{T_k(u_n)\}\) is bounded in \(W^{1,2}(\Omega)\). Hence we can assume that \(T_k(u_n) \to T_ku\) in \(W^{1,2}(\Omega)\), \(T_k(u_n) \to T_ku\) in \(L^2(\Omega)\) and \(T_k(u_n) \to T_k(u)\) a.e. on \(\Omega\). By Lemma 3.1 we may also assume that \(u_n \rightharpoonup u\) in \(W^{1,q}(\Omega)\). Using as a test function \(\psi_{k-1}(u_n)\) we show that \(\nabla u_n \rightharpoonup \nabla u\) in \(L^q(\Omega)\) and \(u\) is a weak solution of (3.1). \(\square\)
On elliptic problems with a nonlinearity depending on the gradient

We now turn our attention to positive solutions of (3.1). If \( f > 0 \) on \( \Omega \), then a solution obtained in Theorem 4.3 is positive. In this case we can also consider the interval \( 1 \leq q < \frac{N}{N-1} \). We commence with an apriori estimate.

**Lemma 3.3.** Suppose that \( 1 \leq q < \frac{N}{N-1} \), \( f > 0 \) on \( \Omega \) and \( f \in L^\infty(\Omega) \). If \( u \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \) is a positive solution of problem (3.1), then

\[
\int_\Omega u^q \, dx \leq C_1 \left( \int_\Omega \left( |\nabla u|^q + u^q \right) \, dx \right)^{\frac{q}{q^*}} \leq C_2 \left( \int_\Omega (1 + u)^{q^*} \, dx \right)^{\frac{(2-q)q^*}{2q}} \left( \|f\|_{L^1}^{q^*} + \|f\|_{L^{(2-q)q^*}}^{(2-q)q^*} \right)
\]

where \( C_1, C_2 > 0 \) are constants independent of \( f \) and \( u \) and \( r = \frac{N(2-q)}{N-1} \).

**Proof.** The proof is a modification of the argument used in the proof of Lemma 2.5 in [8]. We take as a test function \( \phi(x) = (1 + u)^{1-r} \). Since \( q < \frac{N}{N-1} \), we have \( r > 1 \). Also \( r < 2 \) because \( N \geq 3 \). Hence \( \phi(x) \leq 1 \) on \( \Omega \) and upon a substitution we obtain

\[
(r-1) \int_\Omega \frac{|\nabla u|^2}{(1+u)^r} \, dx = \int_\Omega g(x,u,\nabla u)(1+u)^{1-r} \, dx + \lambda \int_\Omega u(1+u)^{1-r} \, dx - \int_\Omega f(1+u)^{1-r} \, dx \leq \int_\Omega g(x,u,\nabla u) \, dx + \lambda \int_\Omega u \, dx.
\]

Testing equation (3.1) with a constant function 1 we obtain

\[
\int_\Omega g(x,u,\nabla u) \, dx + \lambda \int_\Omega u \, dx = \int_\Omega f \, dx.
\]

From (3.9) and (3.10) we derive

\[
\int_\Omega \frac{|\nabla u|^2}{(1+u)^r} \, dx \leq \frac{1}{r-1} \int_\Omega f \, dx \quad \text{and} \quad \int_\Omega u \, dx \leq \frac{1}{\lambda} \int_\Omega f \, dx.
\]
By the Sobolev inequality we obtain
\[
\left( \int_\Omega u^{q^*} \, dx \right)^{\frac{1}{q^*}} \leq C \left( \int_\Omega (|\nabla u|^q + u^q) \, dx \right) = C \left( \int_\Omega \frac{|\nabla u|^q}{(1 + u)^{\frac{q^*}{q}}} (1 + u)^{\frac{2q^*}{q}} \, dx + C \int_\Omega \frac{u^q}{(1 + u)^{\frac{q^*}{q}}} (1 + u)^{\frac{2q^*}{q}} \, dx \leq \right.
\]
\[
\leq C \left( \int_\Omega \frac{|\nabla u|^2}{(1 + u)^r} \, dx \right)^{\frac{q^*}{2}} \left( \int_\Omega \frac{u^{2q^*}}{(1 + u)^{\frac{q^*}{r}}} \, dx \right)^{\frac{2-q^*}{2}} + \]
\[
+ C \left( \int_\Omega \frac{u^2}{(1 + u)^r} \, dx \right)^{\frac{q^*}{2}} \left( \int_\Omega \frac{u^{2q^*}}{(1 + u)^{\frac{q^*}{r}}} \, dx \right)^{\frac{2-q^*}{2}} \leq \]
\[
\leq C \left( \int_\Omega \frac{|\nabla u|^2}{(1 + u)^r} \, dx \right)^{\frac{q^*}{2}} \left( \int_\Omega \frac{u^{2q^*}}{(1 + u)^{\frac{q^*}{r}}} \, dx \right)^{\frac{2-q^*}{2}} + \]
\[
+ C \left( \int_\Omega \frac{u^{2-q}}{\bar{q}} \, dx \right)^{\frac{q^*}{2}} \left( \int_\Omega \frac{u^{2q^*}}{(1 + u)^{\frac{q^*}{r}}} \, dx \right)^{\frac{2-q^*}{2}} \leq \]
\[
\leq C \left( \int_\Omega \frac{|\nabla u|^2}{(1 + u)^r} \, dx \right)^{\frac{q^*}{2}} \left( \int_\Omega \frac{u^{2q^*}}{(1 + u)^{\frac{q^*}{r}}} \, dx \right)^{\frac{2-q^*}{2}} + \]
\[
+ C |\Omega|^{\frac{q^*}{2}} \left( \int_\Omega |u| \, dx \right)^{\frac{(2-q^*)q^*}{2}} \left( \int_\Omega (1 + u)^{\frac{2q^*}{q}} \, dx \right)^{\frac{2-q^*}{2}}.
\]

We now observe that $q^* = \frac{rq}{2-q}$. Hence combining the above estimate with (3.11) the result follows. 

It is clear that Lemma 3.3 leads to the following existence result.

**Theorem 3.4.** Suppose that $1 \leq q < \frac{N}{N-1}$, $f > 0$ on $\Omega$ and $f \in L^1(\Omega)$. The problem (3.1) has a positive solution $u \in W^{1,q}(\Omega)$.

4. HIGHER INTEGRABILITY PROPERTY FOR SOLUTIONS OF (1.1)

The method used in the proof of Lemma 2.1 allows only to estimate the norm $W^{1,q}$ of a positive solution, where $q$ is the exponent appearing in the equation. In the case $1 \leq q < 2$, a question arises whether a solution to (1.1) belongs to $W^{1,q}(\Omega)$ with $q < \bar{q}$. We distinguish two cases: (i) $1 \leq q < \frac{N}{N-1}$ and (ii) $\frac{N}{N-1} < q < 2$. In the case (i) assuming that $f \in L^1(\Omega)$ we show that a solution belongs to $W^{1,q}(\Omega)$ or every $q < \bar{q} < q < \frac{N}{N-1}$. In the case (ii) we show that a solution belongs $W^{1,q}(\Omega)$ for some $q < \bar{q} < 2$ under some additional assumption on $f$. According to Step 1 of the proof of Theorem 2.2, if $f \in L^\infty(\Omega)$, then problem (1.1) has a solution $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$. 


Lemma 4.1. Suppose that \( f > 0 \) on \( \Omega \), \( f \in L^\infty(\Omega) \) and \( 1 \leq q < \bar{q} < \frac{N}{N-1} \). If \( u \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \) is a positive solution of (1.1), then there exist constants \( C_1, C_2 > 0 \), independent of \( u \) and \( f \) such that

\[
\int_\Omega u^{q^*} \, dx \leq C_1 \left( \int_\Omega (|\nabla u|^q + u^q) \, dx \right)^\frac{q^*}{q} \leq C_2 \left( \int_\Omega (1 + u^{q^*}) \, dx \right)^\frac{(2-q)q^*}{2-q} \left( \|f\|_{L^1_{\bar{q}}} + \|f\|_{L^1_{(\bar{q})^*}} \right),
\]

where \( \bar{r} = \frac{N(2-q)}{N-q} \) and \( \bar{q}^* = \frac{Nq}{N-\bar{q}} \).

Proof. As in the proof of Lemma 3.3 we take as a test function \( \phi(x) = (1 + u)^{1-r} \). Since \( \bar{q} < \frac{N}{N-1} \), we have \( \bar{r} > 1 \). Also \( \bar{r} < 2 \) because \( N \geq 3 \). Hence \( \phi(x) \leq 1 \) on \( \Omega \) and upon a substitution we obtain

\[
\bar{r} - 1 \int_\Omega \frac{|\nabla u|^2}{(1 + u)^{\bar{r}}} \, dx = \int_\Omega |\nabla u|^q (1 + u)^{1-r} \, dx + \lambda \int_\Omega u(1 + u)^{1-r} \, dx - \int_\Omega f(1 + u)^{1-r} \, dx \leq \int_\Omega |\nabla u|^q \, dx + \lambda \int_\Omega u \, dx. \tag{4.1}
\]

Testing (1.1) with a constant function 1 we obtain

\[
\int_\Omega |\nabla u|^q + \lambda \int_\Omega u \, dx = \int_\Omega f \, dx. \tag{4.2}
\]

By the Sobolev inequality we obtain

\[
\left( \int_\Omega u^{q^*} \, dx \right)^\frac{q}{q^*} \leq C \left( \int_\Omega (|\nabla u|^q + u^q) \, dx \right) = C \left( \int_\Omega \frac{|\nabla u|^q}{(1 + u)^{\bar{r}}} (1 + u)^{\frac{\bar{r}q}{2}} \, dx + C \int_\Omega \frac{u^q}{(1 + u)^{\bar{r}}} (1 + u)^{\frac{\bar{r}q}{2}} \, dx \leq C \left( \int_\Omega \frac{|\nabla u|^2}{(1 + u)^{\bar{r}}} \, dx \right)^\frac{\frac{\bar{r}q}{2}}{\frac{\bar{r}q}{2}} \left( \int_\Omega (1 + u)^{\frac{\bar{r}q}{2}} \, dx \right)^{\frac{\bar{r}q}{\bar{r}q}} + C \left( \int_\Omega \frac{u^2}{(1 + u)^{\bar{r}}} \, dx \right)^\frac{\frac{\bar{r}q}{2}}{\frac{\bar{r}q}{2}} \left( \int_\Omega (1 + u)^{\frac{\bar{r}q}{2}} \, dx \right)^{\frac{\bar{r}q}{\bar{r}q}}.
\]
Combining the above inequality with (4.1) and (4.2) we obtain
\[
\left( \int_{\Omega} u^{\bar{q}^*} \, dx \right)^{\frac{2}{\bar{q}}^*} \leq C \int_{\Omega} (|\nabla u|^{\bar{q}} + u^{\bar{q}}) \, dx \leq \\
\leq C \left( \int_{\Omega} f \, dx \right)^{\frac{2}{\bar{q}}^*} \left( \int_{\Omega} (1 + u)^{\bar{q}^*} \, dx \right)^{\frac{2-\bar{q}}{\bar{q}}} + \\
+ C \left( \int_{\Omega} (1 + u)^{\bar{q}^*} \, dx \right)^{\frac{2-\bar{q}}{\bar{q}}} \left( \int_{\Omega} u^{2-r} \, dx \right)^{\frac{2}{r}} \leq \\
\leq C \left( \int_{\Omega} (1 + u)^{\bar{q}^*} \, dx \right)^{\frac{2-\bar{q}}{\bar{q}}} \left[ \| f \|_{L_1}^{\frac{2}{\bar{q}}^*} + \| f \|_{L_1}^{(2-r)^{\frac{2}{r}}} \right].
\]

This yields the desired estimate. \( \square \)

**Lemma 4.2.** Let \( f > 0 \) on \( \Omega \), \( f \in L^{\infty}(\Omega) \) and \( \frac{N}{N-1} < q < \bar{q} < 2 \). If \( u \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega) \) is a positive solution of (1.1), then
\[
\int_{\Omega} u^{\bar{q}^*} \, dx \leq C_1 \left( \int_{\Omega} (|\nabla u|^{\bar{q}} + u^{\bar{q}}) \, dx \right)^{\frac{2}{\bar{q}}^*} \leq \\
\leq C_2 \| f \|_{L_{\bar{m}}}^{\frac{2}{\bar{q}}^*} \left( \int_{\Omega} u^{\bar{q}^*} \, dx \right)^{\frac{1}{\bar{q}}} \left( \int_{\Omega} (1 + u)^{\bar{q}^*} \, dx \right)^{\frac{2}{\bar{q}}},
\]
where \( C_1, C_2 > 0 \) are positive constants independent of \( u \) and \( f \), and \( \bar{r} = \frac{N(2-q)}{N-q} \), \( \bar{m} = \frac{Nq}{N+q} \).

The proof is similar to that of Lemma 3.1 and is omitted.

These two lemmas yield the following result.

**Theorem 4.3.** Suppose that \( f > 0 \) on \( \Omega \).

(i) If \( f \in L^1(\Omega) \) and \( 1 \leq q < \frac{N}{N-1} \), then problem (1.1) has a solution that belongs to \( W^{1,q}(\Omega) \) for every \( q \leq \bar{q} < \frac{N}{N-1} \).

(ii) If \( f \in L^{\bar{m}}(\Omega) \) with \( \bar{m} = \frac{Nq}{N+q} \), \( \frac{N}{N-1} \leq q < \bar{q} < 2 \), then problem (1.1) has a solution belonging to \( W^{1,q}(\Omega) \).

Higher integrability property can also be established to solutions of problem (3.1).

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Received: July 29, 2009.
Accepted: August 17, 2009.