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MATRICES DEFINED BY FRAMES

Abstract. Matrix representations of bounded Hilbert space operators are considered. The matrices in question are defined with respect to frames, rather than bases. The frames, a generalisation of bases, used extensively in applied harmonic analysis, are overcomplete sequences. We consider some properties related to tight frames, where, up to some multiplicative constant, a form of Parseval Identity takes place. We also describe parts of spectra of operators in terms of their matrices.

Keywords: frames, operators, spectrum.

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1. INTRODUCTION

The matrix representation of bounded linear operators on a Hilbert space, although generalising the standard finite-dimensional procedure, is not quite popular among operator theorists, at the same time forming one of the most investigated objects in numerical analysis of linear equations. Recent advances in harmonic analysis and signal theory related to wavelet bases and frames provide a motivation to study frame coefficients of operators. In [2] Peter Balazs introduces matrices with respect to pairs of frames for bounded linear operators on a Hilbert space, showing the homomorphic nature of such representations and a couple of properties collected in Theorem 3.1 below.

But essential in the study of operators – the (Hermitian) conjugation operation seemed not well represented in this general setting. It turns out however, that in the presence of additional assumption of tightness, the matrix representation is star-homomorphic (Theorem 3.2). As a consequence, the crucial properties of operators, like self-adjointness, or normality, are shared by their matrices of with respect to tight frames.

In Section 4 we apply this preservation of the involution to the Schatten–von Neumann ideal membership criterion. We also consider the class of Bessel multipliers and the related question raised in [3].

Section 5 is related to a part of the first-named Author's Master Thesis [1]. Here some basic subsets of spectra, like the point spectrum, continuous spectrum and the essential spectrum of T are considered. In some non-trivial way these spectra are related to appropriate parts of the spectrum of the corresponding matrix, which (at least numerically) are easier to localise.

2. TIGHT FRAMES

To a sequence $\mathcal{G} = (g_j), j \in \mathbb{N}$ of vectors in a Hilbert space H one associates the *analysis operator* $C = C_{\mathcal{G}}$. It assigns to a given vector $f \in H$ the sequence

$$Cf := (\langle f, g_j \rangle)$$

of its inner products with the members of $\mathcal{G}$. The Bessel condition with constant K holds for $\mathcal{G}$, if each Cf is square-summable and the operator norm of $C : H \to \ell^2$ satisfies $\|C\|^2 \leq K$. If C is also bounded from below (say, by $\kappa^{\frac{1}{2}}$), then $\mathcal{G}$ is called a *frame*. The frame bound is any pair (κ, K) of positive numbers such that

$$\kappa \|f\|^2 \leq \sum_{j=1}^{\infty} |\langle f, g_j \rangle|^2 \leq K \|f\|^2. \tag{2.1}$$

These frame bounds may be equal, in which case we speak of a *tight frame with bound* $\kappa = K$. *Parseval Frames* are defined as tight frames with bound 1. For such frames the relation (2.1) becomes just one equality.

We shall assume from now on the frame condition (2.1). Along with the analysis operator comes its adjoint, $D = D_{\mathcal{G}} := C_{\mathcal{G}}^*$ mapping ℓ^2 into H, called *the synthesis* operator, which sends the square-summable sequence $(\alpha_j) \in \ell^2$ to the vector $\sum \alpha_j g_j \in H$. (The norm-convergence of the series follows easily from the Bessel condition – cf. for example [6], Theorem 3.2.3.)

Finally, their composition $S = S_{\mathcal{G}}$, called *the frame operator*, given by

$$S := DC, \quad Sf = \sum \langle f, g_j \rangle g_j$$

is self-adjoint, positive, with $\kappa I_H \leq S \leq K I_H$.

Now $\widetilde{\mathcal{G}} := (S^{-1} g_j)$ is again a frame with bounds K^{-1}, κ^{-1}, called *the canonical dual frame for* $\mathcal{G}$. The canonical dual frame for $\widetilde{\mathcal{G}}$ (i.e. $\mathcal{G}$'s second canonical dual) is again $\mathcal{G}$. The equality $S^{-1} S = I_H$ (the identity on H) yields

$$D_{\widetilde{\mathcal{G}}} \circ C_{\mathcal{G}} = I_H, \tag{2.2}$$

that applied to vectors $f \in H$ becomes the *Reconstruction Formula:*

$$f = \sum_{j=1}^{\infty} \langle f, g_j \rangle \widetilde{g_j}, \quad \text{where} \quad \widetilde{g_j} := S^{-1} g_j. \tag{2.3}$$

The dual frame is biorthogonal to the sequence (g_j) only in one case: if (and only if) the later is a Riesz basis. The latter occurs iff the system $\mathcal{G}$ is *ω-independent* (equivalently in the case of frames – *ω-minimal*), so that the above infinite series decomposition (2.3) of f is unique for any vector $f \in H$. In the opposite case there exist other frames (called *non-canonical dual frames* $\bar{\mathcal{G}} = (\bar{g}_j)$ that still reconstruct f as in (2.3), when put in place of the $\widetilde{g_j}$'s. Here and whenever no confusion may result, we suppress the index denoting frames putting tilde to indicate the canonical dual frame operators and bar – for any other dual frame (rather than complex conjugation, or closure!). Note that apart of $\widetilde{D}C = I$ we have also $\bar{D}C = I$.

For tight frames, normalised if necessary by multiplying each g_j by appropriate constant (namely, by $\kappa^{-\frac{1}{2}}$), one gets $\widetilde{g_j} = g_j$ (self-duality), simplifying the Reconstruction Formula which then says that D is a left inverse to C.

In any case, the (self-adjoint) isomorphism $S^{-\frac{1}{2}}$ applied to the frame sequence yields a normalised tight frame $(S^{-\frac{1}{2}}g_j)$. Hence (in some cases) the tightness assumption for frames is not so stringent, although the mentioned isomorphism can involve complications. Being tight, a frame can still be overcomplete, as the classical example of three unit-length vectors in $\mathbb{R}^2$ positioned at equal angles $\frac{2}{3}\pi$ shows.

From the signal engineering point of view, overcompleteness is often an advantage due to the better stability of the Reconstruction Formula than in the case of bases. If the signal comes contaminated by a white noise, part of the noise is killed (if the overcomplete vectors are "evenly positioned"), since unlike the original signal, the noise has no "directional polarisation" ([6], Prop. 5.9.1, [8]). Good sources of references on frames are the books [6] by Ole Christensen, [8] by Stéphane Mallat or the review paper [5].

3. MATRIX REPRESENTATION OF OPERATORS

If $\mathcal{G}_k = (g_{jk})_{j=1}^{\infty}$, $k = 1, 2$ are frames in Hilbert spaces H_k, the coefficients of a bounded linear operator $T : H_1 \longrightarrow H_2$ are defined simply as

$$T_{nm} := \langle T g_{m1}, g_{n2} \rangle.$$

The so obtained matrix (T_{nm}) will be denoted as $\mathrm{Matr}(T)$, or $\mathrm{Matr}^{(\mathcal{G}_2,\mathcal{G}_1)}(T)$. It belongs to the algebra $\mathcal{B}(\ell^2)$ of those infinite matrices that correspond to bounded linear endomorphisms of ℓ^2 (cf. (iii) below).

In the opposite direction, the operator $\mathcal{O}^{(\mathcal{G}_2,\mathcal{G}_1)}(M) = \mathcal{O}(A)$ assigned to a bounded matrix $A = (A_{nk})_{n,k \in \mathbb{N}}$ maps $f \in H_1$ into the (norm convergent) sum

$$\mathcal{O}^{(\mathcal{G}_2,\mathcal{G}_1)}(A)f := \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} A_{kn} \langle f, g_{n1} \rangle g_{k2}.$$

Theorem 3.1, Proposition 3.2 and Corollary 3.3 in [2] are summarised as follows.

Theorem 3.1 (P. Balazs)**.**

(i) *Treated as an operator on* ℓ^2*, the matrix* $\mathrm{Matr}(T)$ *is bounded and*

$$\mathrm{Matr}^{(\mathcal{G}_2,\mathcal{G}_1)}(T) = C_{\mathcal{G}_2} \circ T \circ D_{\mathcal{G}_1} \quad \textit{with} \quad \|\mathrm{Matr}^{(\mathcal{G}_2,\mathcal{G}_1)}(T)\| \leq \sqrt{K_1 K_2}\|T\|.$$

(ii) *Operator* $\mathcal{O}^{(\mathcal{G}_2,\mathcal{G}_1)}(A) : H_1 \longrightarrow H_2$ *is bounded and satisfies*

$$\mathcal{O}^{(\mathcal{G}_2,\mathcal{G}_1)}(A) = D_{\mathcal{G}_2} \circ A \circ C_{\mathcal{G}_1} \quad \textit{with} \quad \|\mathcal{O}(A)\| \leq \sqrt{K_1 K_2}\|A\|.$$

(iii) *For any frames* $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ *and operators* $T : H_1 \to H_2$*,* $L : H_2 \to H_3$ *the Product Formula holds:*

$$\mathrm{Matr}^{(\mathcal{G}_3,\mathcal{G}_2)}(L) \cdot \mathrm{Matr}^{(\widetilde{\mathcal{G}_2},\mathcal{G}_1)}(T) = \mathrm{Matr}^{(\mathcal{G}_3,\mathcal{G}_1)}(LT),$$

which together with $\mathcal{G} = \widetilde{\widetilde{\mathcal{G}}}$ *and (i) implies that* $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}$ *is a (non unital) continuous homomorphism of Banach algebra* $\mathcal{B}(H)$ *into* $\mathcal{B}(\ell^2)$*.*

(iv) *We have the following Operator Reconstruction Formula:*

$$\mathcal{O}^{(\mathcal{G}_2,\mathcal{G}_1)}(\mathrm{Matr}^{(\widetilde{\mathcal{G}_2},\widetilde{\mathcal{G}_1})}(T)) = T,$$
$$\mathcal{O}^{(\mathcal{G},\widetilde{\mathcal{G}})}(I_{\ell^2}) = I_H.$$

To simplify and shorten the formulation of the proofs of (iii) and (iv) one can use the equalities of (i), (ii) together with the Reconstruction Formula in its operator form (2.2), making calculations on vectors unnecessary.

Note, that (iv) means only one-sided inverse, due to one-sided character of (2.2). In particular, for overcomplete frames the operation $\mathrm{Matr}^{(\mathcal{G}_2,\mathcal{G}_1)}$ of assigning matrices to operators is injective, but its range differs from $\mathcal{B}(\ell^2)$. Even worse, $\mathcal{O}^{(\mathcal{G},\widetilde{\mathcal{G}})}$, although unit-preserving, may not be a homomorphism, as it fails to obey the product formula in the overcomplete frames case (cf. [3], Cor. 5.3). Only in the case of pairs of Riesz bases both $\mathrm{Matr}^{(\mathcal{G}_2,\mathcal{G}_1)}$ and $\mathcal{O}^{(\mathcal{G}_2,\mathcal{G}_1)}$ are (unital) isomorphisms.

Now let us see what happens with the involution.

Theorem 3.2. *The assignment* $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}$ *is a *-morphism, i.e. the matrix* $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}(T^*)$ *is the Hermitian adjoint to the matrix* $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}(T)$ *for any bounded linear operator* $T : H \to H$ *if and only if the frame* $\mathcal{G}$ *is tight.*

To prove this result we use Schur Lemma which says that the commutant (the centre) of $\mathcal{B}(H)$ consists exactly of scalar multiples of the identity. This follows easily from calculating the products with rank one operators of the form $Tx = \langle x, u\rangle w$ with $x, u, w \in H$. Then $ATx = \langle x, u\rangle Aw$, while $TAx = \langle Ax, u\rangle w$. Hence $Aw = \alpha w$, where $\alpha = \langle u, u\rangle^{-1}\langle Au, u\rangle$.

Proof of Theorem 3.2. Let $S = S^*$ be the frame operator for $\mathcal{G}$. Its inverse is also self-adjoint. Since the entries of $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}(T^*)$ are

$$\langle T^* g_m, \widetilde{g_n}\rangle = \langle T^* S\widetilde{\widetilde{g_m}}, \widetilde{g_n}\rangle = \langle T^* S\widetilde{\widetilde{g_m}}, S^{-1} g_n\rangle = \langle S^{-1} T^* S\widetilde{\widetilde{g_m}}, g_n\rangle,$$

these are equal to $\overline{\langle T g_n, \widetilde{g}_m\rangle} = \langle T^* \widetilde{g}_m, g_n\rangle$-the entries of the Hermitian conjugate to $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}(T)$ if and only if $S^{-1}T^*S = T^*$. Indeed the values of these (bounded linear) operators coincide on each $\widetilde{\widetilde{g_m}}$, due to the linear density of the g_n's. Multiplying both sides on the left by S we see (as T^* run through all $\mathcal{B}(H)$) that this amounts to S being in the centre of $\mathcal{B}(H)$. But the latter equals $\{\alpha I_H : \alpha \in \mathbb{C}\}$, as Schur Lemma asserts. The frame is tight with bound κ iff $S = \kappa I_H$ (cf. [6]). □

Corollary 3.3. *The matrix* $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}(T)$ *representing bounded linear operator* $T \in \mathcal{B}(H)$ *w.r. to a tight frame is self-adjoint (normal or a projection – respectively) iff the operator* T *has the corresponding property.*

This follows directly from Theorem 3.2 and from the product formula (iii) of Theorem 3.1. Indeed, $\mathrm{Matr}^{(\mathcal{G},\mathcal{G})}$ is then a C^*-algebraic monomorphism and the considered properties involve just involutions and products.

Some care must be taken only with those properties that refer to the unit element, not preserved by our morphism, in general.

Corollary 3.4. *The polynomial functional calculus is consistent with the matrix representation* $\mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}$ *in tight frames, i.e.* $\mathrm{Matr}(p(T)) = p(\mathrm{Matr}(T))$ *only for the polynomials vanishing at zero. The same applies to the polynomials in the (noncommuting) variables* $(z, \bar{z})$*, i.e.* $\mathrm{Matr}(p(T, T^*)) = p(\mathrm{Matr}(T), \mathrm{Matr}(T^*))$.

Proof. Indeed, if a polynomial p satisfies $p(0) = 0$, then $p(z) = c_1 z + \cdots + c_k z^k$, so that p has no free term. Consequently, $p(T) = c_1 T + \cdots + c_k T^k$ has no nonzero summand of the form $c_0 T^0 = c_0 I_H$ and higher degree summands transform as required, by the Product Formula (iii) of Theorem 3.1 and by linearity. Since $\bar{z}(T) = T^*$, the remaining claim follows analogously from Theorem 3.2. □

4. $\mathcal{S}_p$-IDEALS MEMBERSHIP

Again we fix a tight frame $\mathcal{G}$ and let $\mathrm{Matr} := \mathrm{Matr}^{(\mathcal{G},\widetilde{\mathcal{G}})}$. Recall that an operator $T \in \mathcal{B}(H)$ is said to belong to the Schatten–von Neumann ideal $\mathcal{S}_p = \mathcal{S}_p(H)$ if the eigenvalues of $|T|$ (repeated according to their multiplicity) belong to ℓ^p, where $|T| := (T^*T)^{\frac{1}{2}}$. The S_p-norm $\|T\|_p$ is the ℓ_p-norm of the sequence of these eigenvalues. As the square root function is a uniform limit of some polynomials p_n on finite intervals $[0, \|T\|^2]$, the star-homomorphism $\mathrm{Matr}(\cdot)$ sends $|T|$ into $|\mathrm{Matr}(T)|$. Indeed, as $p_n(0) \to \sqrt{0} = 0$, the polynomials $p_n(z) - p_n(0)$ involving no free terms also do the job. Here we are using the (isometric) functional calculus based on the algebra of continuous functions on the spectrum of a self-adjoint element of a C^*-algebra. Now one applies Corollary 3.4 of the previous section.

The same is true of the function $t \mapsto |t|^{\frac{p}{2}}$. Hence to $|T|^{\frac{p}{2}}$ there corresponds the matrix $|\mathrm{Matr}(T)|^{\frac{p}{2}}$. The relation $T \in \mathcal{S}_p$ is equivalent to the membership of $|T|^{\frac{p}{2}}$ in $\mathcal{S}_2$ – the set of all Hilbert – Schmidt operators.

Applying the Parseval Identity for $\mathcal{G}$, or its counterpart (estimates (2.1)) in the case of general (even non-tight) frames, one obtains equivalence between the membership of an operator T in the Hilbert–Schmidt class $\mathcal{S}_2$ with the analogous property for its matrix (in an arbitrary pair of frames), as in [2]. The Hilbert–Schmidt condition for matrices is known as being a Frobenius matrix, ie. having square-summable in modulus entries. The H-S norm of T is estimated in [2] by $\sqrt{K_1K_2}$ times the H-S norm of its matrix $\mathrm{Matr}^{(\mathcal{G}_1,\mathcal{G}_2)}(T)$, where K_j are the Bessel constants of $\mathcal{G}_j$. In our tight frame case $\mathcal{G}_2 = \widetilde{\mathcal{G}_1}$, implying $K_1K_2 = 1$. The H-S norms of T and of its matrix are then equal, since a second estimate of Proposition 3.6 in [2] applies.

In order to extend the above for other values of p we may apply this $p = 2$ – case to the operator $|T|^{\frac{p}{2}}$ and to the corresponding matrix $|\mathrm{Matr}(T)|^{\frac{p}{2}}$.

Corollary 4.1. *$T \in \mathcal{B}(H)$ belongs to the Schatten–von Neumann ideal $\mathcal{S}_p$ if and only if so does its matrix M with respect to a tight frame $\mathcal{G}$. Moreover, the following equality takes place in the normalised tight frame case:*

$$\|T\|_p = \|M\|_p$$

for any finite number $p \geq 1$.

In [3] operators $\mathbf{M}_m = \mathbf{M}_{m,\mathcal{G}_2,\mathcal{G}_1}$ called *Bessel multipliers* (defined by bounded scalar sequences $m \in \ell^\infty$) were considered. In the case of frames, denoting by $\mathrm{diag}(m)$ the diagonal matrix whose main diagonal entries are given by $a_{jj} = m_j$, we note that $\mathbf{M}_m = \mathcal{O}^{(\mathcal{G}_2,\mathcal{G}_1)}(\mathrm{diag}(m))$. The linear mapping that assigns $\mathbf{M}_m$ to a given $m \in \ell^\infty$ is bounded in various pairs of norms. It is shown in [3] (even for Bessel sequences) that $\|\mathbf{M}_m\|$ (operator norm) is estimated by $\sqrt{K_1K_2}\|m\|_\infty$, $\|\mathbf{M}_m\|_p \leq \sqrt{K_1K_2}\|m\|_p$ in the special cases $p = 1, 2$, leaving open the question for other values of $p > 1$.

When trying to argue as in the proof of Corollary 3.3, one meets difficulty: Unlike $\mathrm{Matr}(\cdot)$, the assignment $\mathcal{O}(\cdot)$ is no longer homomorphic. Nonetheless, one may apply the interpolation method, yielding the same estimate by $\sqrt{K_1K_2}\|m\|_p$. Indeed, the scale of Shatten–von Neumann ideals interpolates in the same way as the scale of ℓ^p-spaces (cf. [4], Sect. 7.4). This works in the general case of Bessel sequences, thanks to the mentioned estimates of [3]. Even simpler argument is based on the inequalities $\|ST\|_p \leq \|S\|_p\|T\|$ and the ideal property of $\mathcal{S}_p$ together with (ii) of Theorem 3.1.

The Banach space analogue to frames, called atomic decompositions, may also be considered (cf. [7,9]). It will be interesting to extend the above results to this general case.

5. SPECTRA AND THE MATRIX REPRESENTATION

Now we consider matrices of a bounded linear operator $T : H \to H$ with respect to a given dual pair $(\mathcal{G}, \bar{\mathcal{G}})$ of frames. The bar indicates the possibly non-canonical dual and the shorthand notation $\bar{D}$ will frequently be used for the synthesis operator $D_{\bar{\mathcal{G}}}$. In this case the equality $\bar{D}C = I$ will replace the formula (2.2).

If the frame $\mathcal{G}$ is overcomplete, we have no equality between the spectra $\sigma(T)$ of T and $\sigma(M)$ of its matrix counterpart $M = \mathrm{Matr}^{\mathcal{G},\bar{\mathcal{G}}}(T) = C_{\mathcal{G}} T D_{\bar{\mathcal{G}}}$. Similar problem occurs for basic parts of the spectrum (like the *point spectrum* denoted by $\sigma_p(T)$ and consisting of all eigenvalues of T). Indeed, if only $\bar{\mathcal{G}}$ is overcomplete, $D_{\bar{\mathcal{G}}}$ has nontrivial nullspace $\ker(D_{\bar{\mathcal{G}}})$ and than for any T one has $0 \in \sigma_p(M)$, since

$$\ker(D_{\bar{\mathcal{G}}}) \subset \ker(M).$$

Similarly, non-density of the range of $C_{\mathcal{G}}$ has the "adding zero effect" on the continuous spectrum $\sigma_c(M)$ of M. (We recall its definition later.) Nonetheless, point 0 is the only possible discrepancy between the corresponding spectra:

$$\sigma(M) \setminus \{0\} = \sigma(T) \setminus \{0\},$$

but this follows easily from algebraic fact that nonzero parts of the spectra of AB and BA coincide – here applied to $A = C$, $B = T\bar{D}$. Similar equality holds for the point spectra. (We provide a short proof for the sake of completeness, although this type of result seems known in the perturbation theory setup.)

Theorem 5.1. *For any frame $\mathcal{G}$ and its dual $\bar{\mathcal{G}}$ we have*

$$\sigma_p(M) \setminus \{0\} = \sigma_p(T) \setminus \{0\}.$$

Proof. Assume first that $\lambda \in \sigma_p(T)$. Hence for some $v \in H \setminus \{0\}$

$$\lambda v = Tv = T\bar{D}Cv.$$

Multiplying from the left both sides by C, we get $\lambda Cv = CT\bar{D}Cv = MCv$. By the frame condition (2.1), we have $Cv \neq 0$, hence $\lambda \in \sigma_p(M)$.

Working in the opposite direction, multiply both sides of $\lambda w = Mw$ from the left by $\bar{D}$, taking into account the fact that $\bar{D}C = I_H$, to obtain

$$\lambda \bar{D}w = \bar{D}Mw = \bar{D}CT\bar{D}w = T\bar{D}w.$$

The eigenvector $\bar{D}w$ of T is nonzero, if we assume that $\lambda w \neq 0$, since the latter equals $Mw = CT\bar{D}w$. This yields $\lambda \in \sigma_p(T)$. □

In what follows we adopt the notation

$$T_\lambda = T - \lambda I_H$$

for $T \in \mathcal{B}(\mathcal{H})$, $\lambda \in \mathbb{C}$, useful in dealing with various parts of spectra. Let us first consider *the continuous spectrum* $\sigma_c(\cdot)$ defined by the following two conditions:

$$0 \in \sigma_c(T) \Leftrightarrow 0 \in \sigma(T) \setminus (\sigma_p(T) \cup \sigma_p(T^*)), \quad \lambda \in \sigma_c(T) \Leftrightarrow 0 \in \sigma_c(T_\lambda).$$

Here $0 \notin \sigma_p(T^*)$ is equivalent to the density (in H) of $\mathcal{R}(\mathcal{T})$, the range of T. Since for overcomplete frames any dual frame synthesis operator $\bar{D}$ is non-injective, so is $M = CT\bar{D}$. In this case $0 \notin \sigma_c(M)$ for any operator T.

Let us fix an infinite matrix K which yields a bounded operator on ℓ^2 and satisfies the equalities

$$C\bar{D}K = KC\bar{D} = C\bar{D} \quad \text{(equivalently, } KC = C \text{ and } \bar{D}K = \bar{D}\text{)}. \tag{5.1}$$

For example, one can take either $K = I$, or $K = C\bar{D}$. With any choice of K the continuous spectrum of T has the same description in terms of the related matrix $M = CT\bar{D}$ with respect to $\mathcal{G}$ and any of $\mathcal{G}$'s dual frames $\bar{\mathcal{G}}$:

Theorem 5.2. *A nonzero λ belongs to $\sigma_c(T)$ if and only if $\lambda \notin \sigma_p(M)$, $\mathcal{R}(\mathcal{M}-\lambda\mathcal{K}) \not\supset \mathcal{R}(\mathcal{C})$ and $\overline{\mathcal{R}(\mathcal{M}-\lambda\mathcal{K})} \supset \mathcal{R}(\mathcal{C})$.*

Proof. To show one containment, fix $\lambda \in \sigma_c(T) \setminus \{0\}$. Then $\lambda \notin \sigma_p(T)$. By Theorem 5.1, $\lambda \notin \sigma_p(M)$. Also, the range of T_λ does not contain some vector $y \in H$. If Cy were in $\mathcal{R}(\mathcal{M}-\lambda\mathcal{K})$, for some $x \in \ell^2$ we would have $Cy = CT\bar{D}x - \lambda Kx$ and multiplying by $\bar{D}$ from the left we get $y = T\bar{D}x - \lambda\bar{D}Kx = T_\lambda(\bar{D}x)$, in contradiction to our choice of y outside the range of T_λ. To see the density of $\mathcal{R}(\mathcal{M}-\lambda\mathcal{K})$ in $\mathcal{R}(\mathcal{C})$, consider b of the form $b = Cy$ for some $y \in H$. The range of T_λ is dense, so one finds a sequence of vectors $x_n \in H$ with $T_\lambda x_n \to y$. Then $(M-\lambda K)Cx_n = CT\bar{D}Cx_n - \lambda KCx_n = CT_\lambda x_n$ converge to Cy, since C is continuous, showing the required density.

Conversely, let $\lambda \neq 0$ satisfy these three conditions. As above, we see that $\lambda \notin \sigma_p(T)$. Also T_λ cannot be surjective, otherwise the range of $(M-\lambda K)C = CT\bar{D}C - \lambda KC = CT_\lambda$ would coincide with that of C, which is not the case. Similarly to the previous part, let us assume, that $y \in H$. Then there exists such a sequence $a_n \in \ell^2$, that $(M-\lambda K)a_n \to Cy$. By continuity of $\bar{D}$ we have $T_\lambda\bar{D}a_n = \bar{D}CT\bar{D}a_n - \lambda\bar{D}Ka_n = \bar{D}(M-\lambda K)a_n \to y$. □

Finally, to describe the essential spectrum $\sigma_{\text{ess}}(T)$ (with $\lambda \notin \sigma_{\text{ess}}(T)$ meaning the closedness and finite codimension of $\mathcal{R}(T_\lambda)$ plus finiteness of $\dim\ker(T_\lambda)$) one has to know not only the range of C, as was the case with $\sigma_c(T)$, but $\ker(\bar{D})$ as well.

In the canonical dual frame case

$$\widetilde{D}(\mathbf{a}) = \sum a_n\widetilde{g}_n = \sum a_n S^{-1}g_n = S^{-1}D(\mathbf{a})$$

for any $\mathbf{a} = (a_n) \in \ell^2$, showing $\widetilde{D} = S^{-1}D$ and, consequently,

$$\ker(\widetilde{D}) = \ker D = \mathcal{R}(C)^\perp.$$

Note, that in general the dimension of $\ker(\bar{D})$ (a dual frame synthesis operator) can be infinite, leaving 0 in $\sigma_{\text{ess}}(M)$, hence the discrepancy at 0 depends on the frame behaviour. The nonzero part is again described below under additional assumption that the dual frame used to represent T is the canonical one, i.e. in the $\bar{\mathcal{G}} = \widetilde{\mathcal{G}}$ case. We conjecture that this requirement can be dropped, but at the moment we can only prove the following description:

Theorem 5.3. *Let $M = CT\widetilde{D}$ be the matrix representation of T using a frame $\mathcal{G}$ and its canonical dual frame $\widetilde{\mathcal{G}}$. Then $\lambda \notin \sigma_{\text{ess}}(T) \cup \{0\}$ iff the range of M_λ is closed and the dimensions of $\ker(M_\lambda) \ominus \ker(\widetilde{D})$ and of $\mathcal{R}(\mathcal{C}) \ominus \mathcal{R}(\mathcal{M}_\lambda)$ are both finite.*

Proof. Take first $\lambda \notin \sigma_{\text{ess}}(T) \cup \{0\}$. To see the closedness of $\mathcal{R}(\mathcal{M}_\lambda)$ consider any convergent sequence of its points b_n. We want to show that its limit, say $b = \lim b_n$ also belongs to $\mathcal{R}(\mathcal{M}_\lambda)$, i.e. is of the form $M_\lambda a$ for some $a \in \ell^2$. Since for some points a_n in ℓ^2 we have $b_n = CT\widetilde{D}a_n - \lambda a_n$, applying $\widetilde{D}$ on both sides and using $\widetilde{D}C = I$, it follows that the sequence

$$T_\lambda \widetilde{D}a_n = T\widetilde{D}a_n - \lambda\widetilde{D}a_n$$

converges to $\widetilde{D}b$. In view of the closedness of $\mathcal{R}(\mathcal{T}_\lambda)$, the latter limit is in this range: $\widetilde{D}b = T_\lambda x$ for some $x \in H$. Then

$$M_\lambda Cx = CT\widetilde{D}Cx - \lambda Cx = CT_\lambda x = C\widetilde{D}b$$

and to solve the equation $M_\lambda a = b$ for a it suffices to put $a = Cx + \frac{1}{\lambda}(C\widetilde{D}b - b)$. Indeed, developing $M_\lambda a = (CT\widetilde{D} - \lambda I)(Cx + \frac{1}{\lambda}(C\widetilde{D}b - b))$, we see that it is

$$CT\widetilde{D}Cx - \lambda Cx + \frac{1}{\lambda}(CT\widetilde{D}C\widetilde{D} - CT\widetilde{D})b - (C\widetilde{D} - I)b = M_\lambda Cx - C\widetilde{D}b + b = b.$$

For $a \in \ker M_\lambda$ orthogonal to $\ker \widetilde{D}$, we are going to show that $\widetilde{D}a \in \ker T_\lambda$. Indeed,

$$T_\lambda \widetilde{D}a = \widetilde{D}CT\widetilde{D}a - \lambda\widetilde{D}a = \widetilde{D}(M - \lambda I)a = 0.$$

Hence the inclusion $\widetilde{D}(\ker(M_\lambda) \ominus \ker(\widetilde{D})) \subset \ker T_\lambda$ follows. But $\widetilde{D}$ restricted to $\ker(M_\lambda) \ominus \ker(\widetilde{D})$ is injective, so we deduce

$$\dim(\widetilde{D}(\ker(M_\lambda) \ominus \ker(\widetilde{D}))) = \dim(\ker(M_\lambda) \ominus \ker(\widetilde{D})) \leq \dim(\ker T_\lambda) < \infty.$$

To see the finite dimensionality of $\mathcal{R}(C) \ominus \mathcal{R}(M_\lambda)$ it suffices to note that since $\mathcal{R}(C) \perp \ker D$, D maps this subspace injectively and its dimension equals to that of $D(\mathcal{R}(C) \ominus \mathcal{R}(M_\lambda))$. The latter is contained in $H \ominus \mathcal{R}(T_\lambda)$. Indeed, let $b \in \mathcal{R}(C) \ominus \mathcal{R}(M_\lambda)$. Then for any $x \in H$ we have

$$\langle Db, T_\lambda x\rangle = \langle b, CT\widetilde{D}Cx - \lambda Cx\rangle = \langle b, MCx - \lambda Cx\rangle = 0.$$

The other half of the proof follows similar pattern. Given λ satisfying the three conditions for M, we check that the range of T_λ is closed by considering a convergent (to some limit y) sequence $y_n \in \mathcal{R}(T_\lambda)$. Then $y_n = T_\lambda x_n$ for certain $x_n \in H$ and

$$(M - \lambda I)Cx_n = MCx_n - \lambda Cx_n = CT\widetilde{D}Cx_n - \lambda Cx_n = Cy_n.$$

But $Cy_n \to Cy$ and the range of M_λ, being closed, contains Cy. Consequently, $Cy = M_\lambda a$ for some $a \in \ell^2$. Applying $\widetilde{D}$ from the left, we get

$$y = \widetilde{D}Cy = \widetilde{D}Ma - \lambda\widetilde{D}a = T\widetilde{D}a - \lambda\widetilde{D}a = T_\lambda\widetilde{D}a.$$

To estimate $\dim(\ker T_\lambda)$ we check that $C(\ker T_\lambda) \subset \ker M_\lambda$, by showing

$$(M - \lambda I)Cx = CT\widetilde{D}Cx - \lambda Cx = CT_\lambda x = 0.$$

This suffices, C being injective, to get

$$\dim \ker T_\lambda = \dim C(\ker T_\lambda) \leq \dim \ker M_\lambda < \infty.$$

The cokernel $H \ominus \mathcal{R}(\mathcal{T}_\lambda)$ of T_λ is mapped by the (injective) dual frame analysis operator $\widetilde{C}$ into the (finite dimensional) subspace $\mathcal{R}(C) \ominus \mathcal{R}(M_\lambda)$, which implies the finiteness of its dimension. Indeed, for any $a \in \ell^2$

$$\langle \widetilde{C}y, M_\lambda a\rangle = \langle \widetilde{C}y, CT\widetilde{D}a - \lambda a\rangle = \langle y, T\widetilde{D}a - \lambda \widetilde{D}a\rangle = \langle y, T_\lambda \widetilde{D}a\rangle = 0.$$

Moreover, as $D = S\widetilde{D}$, by taking the adjoints of both sides we obtain $C = \widetilde{C}S^*$. As S is bijective, the ranges of C and $\widetilde{C}$ have to coincide and $\widetilde{C}(H \ominus \mathcal{R}(\mathcal{T}_\lambda)) \subset \mathcal{R}(\mathcal{C})$. □

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