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CYCLABILITY IN BIPARTITE GRAPHS

Abstract. Let $G = (X, Y; E)$ be a balanced 2-connected bipartite graph and $S \subset V(G)$. We will say that S is *cyclable* in G if all vertices of S belong to a common cycle in G . We give sufficient degree conditions in a balanced bipartite graph G and a subset $S \subset V(G)$ for the cyclability of the set S .

Keywords: graphs, cycles, bipartite graphs.

Mathematics Subject Classification: 05C20, 05C35, 05C38, 05C45.

1. INTRODUCTION

We shall consider only finite graphs without loops and multiple edges.

Several authors have given results about cycles containing specific subsets of vertices, see for example [7] or [9].

The set S of vertices is called *cyclable* in G if all vertices of S belong to a common cycle in G . We also speak about *cyclability* or *noncyclability* of the vertex set S .

In a bipartite graph $G = (X, Y; E)$ we will call the independent sets of vertices X and Y *the partite sets*.

Let $G = (X, Y; E)$ be a bipartite graph and let $S \subset V(G)$, then $S_X = S \cap X$ and $S_Y = S \cap Y$. We will say that S is *balanced* iff $|S_X| = |S_Y|$.

In 1992 Shi Ronghua [8] obtained the following result:

Theorem 1.1. *Let G be a 2-connected graph of order n and S a subset of $V(G)$ with $|S| \geq 3$. If for every pair of nonadjacent vertices x and y in S we have*

$$d(x) + d(y) \geq n,$$

then S is cyclable in G .

Note that the assumption of 2-connectivity may be omitted in Theorem 1.1. It is an easy corollary of a result of K. Ota [7].

Recently R. Čada, E. Flandrin and Z. Ryjáček [3] proved the following generalization of Theorem 1.1:

Theorem 1.2. *Let G be a 2-connected graph of order n and S a subset of $V(G)$. If for every pair of nonadjacent vertices x and y in S we have*

$$d(x) + d(y) \geq n - 1,$$

then either S is cyclable in G , or n is odd and G contains an independent set $S_1 \subseteq S$ such that $|S_1| = \frac{n}{2}$ and every vertex of S_1 is adjacent to all vertices in $G \setminus S_1$.

In 2002 E. Flandrin, H. Li, A. Marczyk and M. Woźniak [4] obtained the following generalization of Theorem 1.1:

Theorem 1.3. *Let G be a k -connected graph, $k \geq 2$ of order n . Denote S_1, \dots, S_k subsets of the vertex set $V(G)$ and let $S = S_1 \cup S_2 \cup \dots \cup S_k$. If for any $x, y \in S_i$, $xy \notin E$ we have*

$$d(x) + d(y) \geq n,$$

then S is cyclable in G .

The notion of cyclability is a generalization of the term of hamiltonicity. If we consider $S = V(G)$ then S is cyclable iff G is hamiltonian. In fact Theorem 1.1 is a generalization of the following result of O. Ore [6]:

Theorem 1.4. *Let G be a graph on $n \geq 3$ vertices. If for all nonadjacent vertices $x, y \in V(G)$ we have*

$$d(x) + d(y) \geq n,$$

then G is hamiltonian.

A similar result for bipartite graphs was proved by J. Moon and M. Moser [5] in 1963:

Theorem 1.5. *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$. If for all nonadjacent vertices $x \in X$ and $y \in Y$ we have*

$$d(x) + d(y) \geq n + 1,$$

then G is hamiltonian.

Given a balanced bipartite graph and a selected subset of vertices, we are interested in properties that imply cyclability.

In 2000 D. Amar, M. El Kadi Abderrezzak, E. Flandrin [2] proved the following generalization of Theorem 1.1 for bipartite graphs:

Theorem 1.6. *Let $G = (X, Y; E)$ be a balanced 2-connected bipartite graph of order $2n$, $S \subset X$. If for every $x \in S$, $y \in Y$, $xy \notin E$ we have*

$$d(x) + d(y) \geq n + 1,$$

then S is cyclable in G .

Note that in this case $S = S_X$ and Theorem 1.6 is also a generalization of Theorem 1.5.

The main result of the present paper, given in Section 3, is Theorem 3.1, which improves upon Theorem 1.6.

2. DEFINITIONS

Let G be a graph and H a subgraph of G .

Definition 2.1. $N_G(H)$ denotes the set of all vertices of the graph G which are adjacent to a vertex of the subgraph H , i.e. $N_G(H) = \{u \in V(G) : \exists v \in V(H) \text{ such that } uv \in E(G)\}$.

Consider an arbitrary vertex $x \in V(G)$. $N(x)$ denotes the set of all neighbors of the vertex x in G , i.e. $N(x) = \{u \in V(G) : xu \in E(G)\}$. $N_H(x)$ denotes the set of all neighbors of the vertex x in the subgraph H , i.e. $N_H(x) = \{u \in V(H) : xu \in E(G)\}$.

$d_H(x)$ denotes the number of neighbors of x in the subgraph H i.e. $d_H(x) = |N_H(x)|$, and $d_H(x)$ denotes the degree of the vertex x in the subgraph H .

In the proof we will only use cycles and paths with a given orientation. For a cycle $C : c_1 \dots c_k$ or a path $P : p_1 \dots p_l$ we will use implicit orientation.

Thus it makes sense to speak of a successor c_{i+1} and a predecessor c_{i-1} of a vertex c_i (addition modulo $l+1$). Denote the successor of a vertex x by x^+ and its predecessor by x^- . This notation can be extended to $A^+ = \{x^+ : x \in A\}$, and similarly, to A^- when $A \subseteq V(G)$.

Let P be a path $p_1 \dots p_k$ and $u, v \in V(G)$ such that $up_1, vp_k \in E(G)$, then: uPv is the path $up_1 \dots p_kv$ and vPu is the path $vp_k \dots p_1u$.

Definition 2.2. We shall call a path $P : p_1 \dots p_l$ a C -path of the cycle C iff $V(P) \cap V(C) = \{p_1, p_l\}$. Note that a C -path is a generalized chord of the cycle.

Definition 2.3. Let $C : c_1 \dots c_l$ be a cycle in G with the orientation, the indices $1, \dots, l$ are considered modulo l . For any pair of vertices $c_i, c_j \in V(C)$ ($i \neq j$) we define four intervals:

- $]c_i, c_j[$ is the path $c_{i+1} \dots c_{j-1}$.
- $[c_i, c_j[$ is the path $c_i \dots c_{j-1}$.
- $]c_i, c_j]$ is the path $c_{i+1} \dots c_j$.
- $[c_i, c_j]$ is the path $c_i \dots c_j$.

Note that these four intervals are subsets of the cycle C .

For notation and terminology not defined above a good reference is [1].

3. THEOREM

Theorem 3.1. If $G = (X, Y; E)$ is a balanced 2-connected bipartite graph of order $2n$ and $S \subset V(G)$ satisfying conditions:

$$\text{For every } x \in S_X, \quad y \in Y, \quad xy \notin E \quad \text{we have} \quad d(x) + d(y) \geq n + 1 \quad (3.1)$$

$$\text{For every } x \in X, \quad y \in S_Y, \quad xy \notin E \quad \text{we have} \quad d(x) + d(y) \geq n + 1 \quad (3.2)$$

then S is cyclable in G .

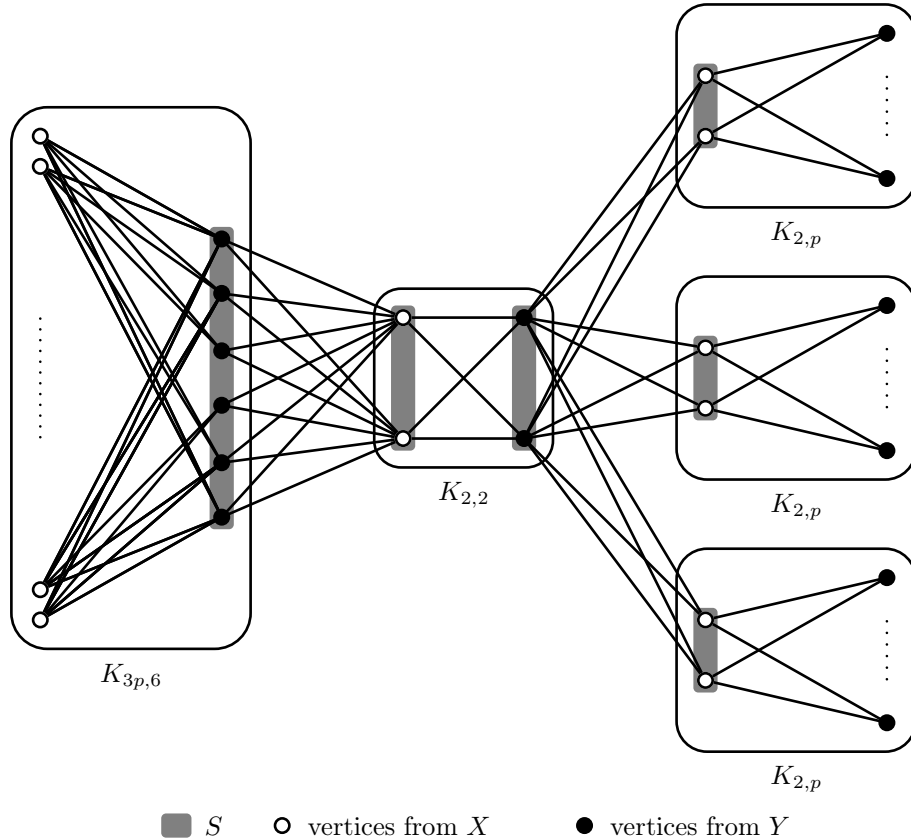


Fig. 1. $G_{p,3}$

Theorem 3.1 is obviously a generalization of Theorem 1.6.

We first tried to find a generalization satisfying two conditions:

- The vertices of S are in both partite sets X and Y .
- The degree sum condition holds only for vertices from S .

However, even if we assume that S is balanced (i.e. $|S_X| = |S_Y|$), such a result is not true.

For every $k \geq 1$ we will give an example of a 2-connected, balanced bipartite graph $G = (X, Y; E)$ and a balanced set $S \subset V(G)$, satisfying the following condition:

$$\text{For every } x \in S_X, y \in S_Y \text{ if } xy \notin E \text{ then } d(x) + d(y) \geq n + k, \quad (3.3)$$

such that S is not cyclable in G .

$$\text{Let } k \geq 1, p \geq k + 2 \text{ and } 2 \leq r \leq 1 + \frac{p - k}{2}.$$

First consider bipartite graphs $K_{pr,2r}$, $K_{2,2}$ and r copies of $K_{2,p}$. In $K_{2,2}$ we have two partite sets say X_2 and Y_2 . The graph $G = (X, Y; E)$ is obtained out of $K_{pr,2r}$, $K_{2,2}$ and the r copies of $K_{2,p}$ by joining every vertex of degree pr from $K_{pr,2r}$ with all vertices from X_2 and every vertex of degree p from the r copies of $K_{2,p}$ with all vertices from Y_2 .

Let S_1 be the set of all vertices from $K_{pr,2r}$ of degree pr in $K_{pr,2r}$.

In each copy of $K_{2,p}$ we take the two vertices of degree p in $K_{2,p}$. In this way we will get $2r$ vertices and we define the set S_2 as the set containing these $2r$ vertices.

We can define now the set S . Let $S = S_1 \cup S_2 \cup V(K_{2,2})$.

For $k \geq 1$, $p \geq k + 2$ and $2 \leq r \leq 1 + \frac{p-k}{2}$ we have obtained a balanced, 2-connected bipartite graph $G_{p,r} = (X, Y; E)$ of order $2n$ with $n = pr + 2r + 2$ and a balanced set S which is not cyclable, but satisfies (3.3).

We can find an example of the graph $G_{p,3}$ on the Figure 1.

This example shows that it is not enough to assume that the degree sum condition holds only for the vertices from S in a bipartite graph. Even increasing the connectivity will not be sufficient, as we can see in the following example.

For every $k \geq 1$ and $l \geq 2$ we will give an example of an l -connected, balanced bipartite graph $G' = (X, Y; E)$ and a balanced set $S' \subset V(G')$, satisfying (3.3), such that S' is not cyclable in G' .

Let $k \geq 1$, $l \geq 2$, $p \geq l^2 - l + k$ and $l \leq r < 1 + \frac{p-k}{l}$.

First consider bipartite graphs $K_{pr,lr}$, $K_{l,l}$ and r copies of $K_{l,p}$. In $K_{l,l}$ we have two partite sets say X_l and Y_l . The graph $G' = (X, Y; E)$ is obtained out of $K_{pr,lr}$, $K_{l,l}$ and the r copies of $K_{l,p}$ by joining every vertex of degree pr from $K_{pr,lr}$ with all vertices from X_l and every vertex of degree p from the r copies of $K_{l,p}$ with all vertices from Y_l .

Let S'_1 be the set of all vertices from $K_{pr,lr}$ of degree pr in $K_{pr,lr}$.

In each copy of $K_{l,p}$ we take the l vertices of degree p in $K_{l,p}$. In this way we will get lr vertices and we define the set S'_2 as the set containing these lr vertices.

We can define now the set S' . Let $S' = S'_1 \cup S'_2 \cup V(K_{l,l})$.

For $k \geq 1$, $l \geq 2$, $p \geq l^2 - l + k$ and $l \leq r < 1 + \frac{p-k}{l}$ we have obtained a balanced, l -connected bipartite graph $G'_{p,r,l} = (X, Y; E)$ of order $2n$ with $n = pr + lr + l$ and a balanced set S' which is not cyclable in G' , but satisfies (3.3).

4. PROOF OF THEOREM 3.1

4.1. PRELIMINARY NOTATIONS

Let $G = (X, Y; E)$ be a bipartite graph and let C be a cycle in G .

In this chapter for a given cycle C and a vertex $x \in V(G \setminus C)$, a C -path Q through x will be denoted $Q : uQ_1xQ_2u'$, where Q_1 and Q_2 are two vertex disjoint paths. The end vertices of the C -path $Q : u$ and u' and the vertex x do not belong to Q_1 nor Q_2 .

Note that the path Q_1 may be empty or in other words $V(Q_1) = \emptyset$ and in this case $xu \in E$. Similarly for Q_2 .

An example of a C -path $P : uP_1xP_2u'$ through a vertex x can be found on Figure 2.

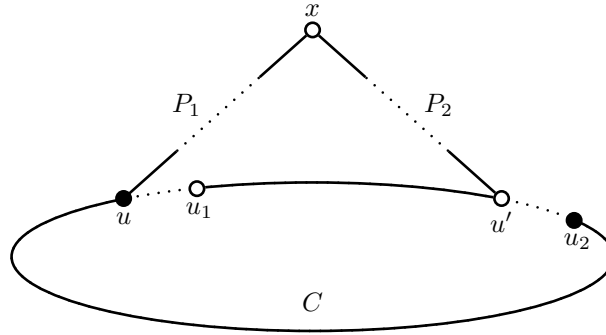


Fig. 2. An example of a cycle C and a C -path P with $x, u', u_1 \in X$ and $u, u_2 \in Y$

Remark 4.1. Given a 2-connected graph G , a nonhamiltonian cycle C and a vertex $x \in V(G \setminus C)$, G contains necessarily a C -path through x .

In the remaining part of Section 4 we will always consider a 2-connected bipartite graph G and a subset $S \subset V(G)$ not cyclable in G . Given a cycle C , a vertex $x \in V(G \setminus C) \cap S$ such that C contains $S \setminus \{x\}$ but does not contain S , we will denote by P a C -path through x . We will always assume that the cycle C and the C -path P are chosen such that P is shortest possible among all C -paths through x for all cycles C containing $S \setminus \{x\}$, i.e. for any cycle C' containing $S \setminus \{x\}$, and for any C' -path P' containing the vertex x we have $|V(P)| \leq |V(P')|$. We will denote this C -path $P : uP_1xP_2u'$ (note that P_1 and/or P_2 may be empty).

We will denote by u_1 the first vertex on the cycle C from S after u (u_1 exists since S is not cyclable). Similarly u_2 is the first vertex on the cycle C from S after u' .

R is the subgraph induced in G by $V(G) \setminus V(C)$.

All the intervals of type $[a, b]$, $[a, b[$, $]a, b]$ and $]a, b[$ are intervals on the cycle C and we sometimes identify the vertex set of an interval with the corresponding interval.

Remark 4.2. The C -path $P : uP_1xP_2u'$ has the following properties:

- If $V(P_1) \neq \emptyset$ or $V(P_2) \neq \emptyset$ then $d_C(x) \leq 1$. (4.1)

- If $V(P_1) \neq \emptyset$ and $V(P_2) \neq \emptyset$ then $d_C(x) = 0$. (4.2)

Remark 4.2 is an immediate consequence of the choice of the cycle C and the C -path P .

4.2. FORMULATION AND PROOF OF LEMMA 4.3

In the proof of Theorem 3.1 we shall use the following lemma. Notations G, S, C, R, x and C -path $P : uP_1xP_2u'$, u_1 and u_2 are defined in Section 4.1 but we recall them for completeness. We denote by C a cycle containing $S \setminus \{x\}$. Let P be a C -path through x . The cycle C and the C -path P are chosen such that P is shortest possible among all C -paths through x for all cycles C containing $S \setminus \{x\}$. Let u_1 be the first vertex from S after u on the cycle C , and let u_2 be the first vertex from S after u' on the cycle C . The subgraph of G induced by $V(G) \setminus V(C)$, will be denoted R .

Lemma 4.3. *Let $G = (X, Y; E)$ be a 2-connected bipartite graph and let C, P, R and S be as above. Then we have:*

$$\begin{aligned} & - \text{For every } C\text{-path } Q : aQ_1xQ_2a' \text{ through } x \text{ we have:} \\ & \quad V(\lceil a, a' \rceil) \cap S \neq \emptyset \text{ and } V(\lceil a', a \rceil) \cap S \neq \emptyset. \end{aligned} \quad (4.3)$$

$$\begin{aligned} & - \text{For any } b \in V(\lceil u, u_1 \rceil) \text{ and } c \in V(\lceil u', u_2 \rceil) \text{ we have:} \\ & \quad N_{P_1xP_2}(b) = N_{P_1xP_2}(c) = \emptyset. \end{aligned} \quad (4.4)$$

$$- \text{If } z \in V(P) \setminus \{u, u'\}, \text{ then } N_R(z) \cap (N(\lceil u, u_1 \rceil) \cup N(\lceil u', u_2 \rceil)) = \emptyset. \quad (4.5)$$

$$- N_R(\lceil u, u_1 \rceil) \cap N_R(\lceil u', u_2 \rceil) = \emptyset. \quad (4.6)$$

$$- \text{For any } y \in N_R(x) \text{ we have } N_{\lceil u, u_1 \rceil}(y) = N_{\lceil u', u_2 \rceil}(y) = \emptyset. \quad (4.7)$$

Proof of Lemma 4.3. Suppose that $V(\lceil a, a' \rceil) \cap S = \emptyset$, then the cycle:

$$C' : aQ_1xQ_2a'a'^+ \dots a, \quad (4.8)$$

is a cycle containing S , a contradiction. If $V(\lceil a', a \rceil) \cap S = \emptyset$, then using similar arguments we get a contradiction and hence (4.3) is proved.

In order to prove (4.4) suppose that there is a vertex $b \in V(\lceil u, u_1 \rceil)$ such that $N_{P_1xP_2}(b) \neq \emptyset$. We have a vertex $z \in N_{P_1xP_2}(b)$ and we assume that the vertices on the path P_1xP_2 are labeled as follows: $P_1xP_2 : p_1^1 \dots p_1^l x p_2^k \dots p_2^1$.

We shall consider three cases.

1. When $z = x$, then the following cycle:

$$C' : uP_1x b b^+ \dots u' \dots u_2 \dots u, \quad (4.9)$$

contains S , a contradiction.

2. If $z \in V(P_1)$, then the following cycle:

$$C' : u p_1^1 \dots p_1^i z b b^+ \dots u_1 \dots u' \dots u_2 \dots u, \quad (4.10)$$

contains $S \setminus \{x\}$ and has a C' -path

$$P' : z p_1^{i+2} \dots x P_2 u', \quad (4.11)$$

shorter than P , a contradiction with the choice of C and P .

3. If $z \in V(P_2)$, then the following cycle:

$$C' : uP_1xp_2^k \dots p_2^j zbb^+ \dots u' \dots u_2 \dots u, \tag{4.12}$$

contains S , a contradiction.

So we have $N_{P_1xP_2}(b) = \emptyset$. Using similar arguments we can prove that for any $c \in V([u', u_2])$ $N_{P_1xP_2}(c) = \emptyset$, and hence (4.4) is true.

We will prove now (4.5).

Suppose that $N_R(z) \cap (N([u, u_1]) \cup N([u', u_2])) \neq \emptyset$.

So we have a vertex $w \in N_R(z) \cap (N([u, u_1]) \cup N([u', u_2]))$. Without loss of generality we can assume that $w \in N_R(z) \cap N([u, u_1])$. Let $a \in V([u, u_1])$, such that $aw \in E$. From (4.4) we know that $w \notin V(P_1xP_2)$.

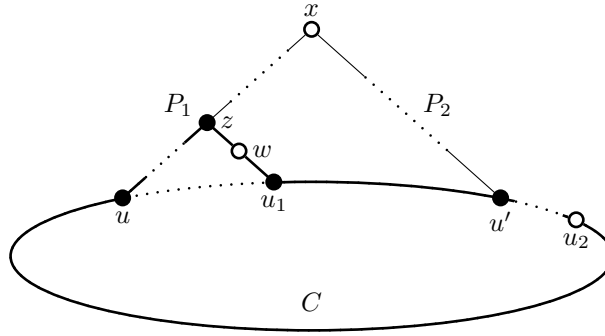


Fig. 3. $u_2, w, x \in X$ and $u, u_1, u', z \in Y$

As in the proof of (4.4), we shall consider three cases:

1. $z \in V(P_1)$.
2. $z \in V(P_2)$.
3. $z = x$.

Using similar arguments we get contradiction.

In any case we obtain a contradiction by replacing in (4.10), (4.12) and (4.9), the edge za by the path $zwua$. Hence (4.5) is true.

For $a = u_1$, you can find the illustrations of Cases 1 and 2 on Figures 3 and 4 respectively.

In order to prove (4.6), suppose that $b \in V([u, u_1])$, $c \in V([u', u_2])$ and $z \in N_R(\{b, c\})$.

From (4.4) we know that $N_{P_1xP_2}(b) = N_c(P_1xP_2) = \emptyset$, and so $z \notin V(P)$. In this case the cycle:

$$C' : uP_1xP_2u'u'^- \dots u_1 \dots bzc \dots u_2u_2^+ \dots u$$

contains S , a contradiction.

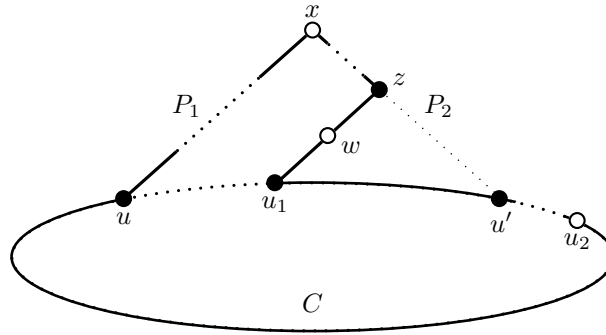


Fig. 4. $u_2, w, x \in X$ and $u, u_1, u', z \in Y$

Hence (4.6) is true.

In order to prove (4.7), suppose that there is a vertex $y \in N_R(x)$ such that $N_{]u, u_1]}(y) \neq \emptyset$. From (4.4) we know that $y \notin V(P_1xP_2)$. We have a vertex $b \in V(]u, u_1])$ such that $yb \in E$ and the following cycle:

$$C' : uP_1xybb^+ \dots u_1 \dots u' \dots u_2 \dots u$$

contains S , a contradiction and so $N_{]u, u_1]}(y) = \emptyset$.

Using the same arguments we can prove $N_{]u, u_2]}(y) = \emptyset$. Hence (4.7) is true and the proof of Lemma 4.3 is finished. □

4.3. PROOF OF THEOREM 3.1

We may assume that $S_Y \neq \emptyset$ and $|S_Y| \geq |S_X|$. We will proceed by induction over the number of vertices in S_X .

If $|S_X| = 0$, then $S = S_Y$ and from Theorem 1.6 we know that S is cyclable in G . So the first step of the induction is finished.

Suppose now that S satisfies the assumptions of Theorem 3.1 and $|S_X| \geq 1$.

From the induction hypothesis, we assume that for any $x \in S_X$ the set $S \setminus \{x\}$ is cyclable in G , while S itself is not cyclable. Let us choose a vertex $x \in S_X$.

We have a cycle C containing $S \setminus \{x\}$ such that $x \notin V(C)$. We recall that the cycle C and the C -path P are chosen such that P is shortest possible among all C -paths containing x for all cycles C containing $S \setminus \{x\}$. As in Section 4.1, u_1 is the first vertex from S on the cycle C after u and u_2 is the first vertex from S on the cycle C after u' , R is the subgraph induced in G by $V(G) \setminus V(C)$.

It is clear that in this case R is a balanced bipartite graph.

Note that if $c = |V(C)|$, $r = |V(R)|$, then c and r are even and $n = \frac{c+r}{2}$.

From Remark 4.2 and Lemma 4.3 C and P satisfy (4.3) — (4.7).

We shall consider four cases:

1. $N_R(x) = \emptyset$.
2. $N_R(x) \neq \emptyset$ and $u_1, u_2 \in S_Y$.
3. $N_R(x) \neq \emptyset$ and $u_1, u_2 \in S_X$.
4. $N_R(x) \neq \emptyset$ and u_1 and u_2 are in different partite sets.

Case 1. $N_R(x) = \emptyset$

In this case P_1 and P_2 are empty and $xu, xu' \in E$.

Since R is balanced there is an $y \in Y \cap V(R)$. Since $xy \notin E$ then from (3.1) we have:

$$d(x) + d(y) \geq n + 1. \tag{4.13}$$

Since $N_R(x) = \emptyset$ we have:

$$d_R(x) = 0 \quad \text{and} \quad d_R(y) \leq \frac{r}{2} - 1. \tag{4.14}$$

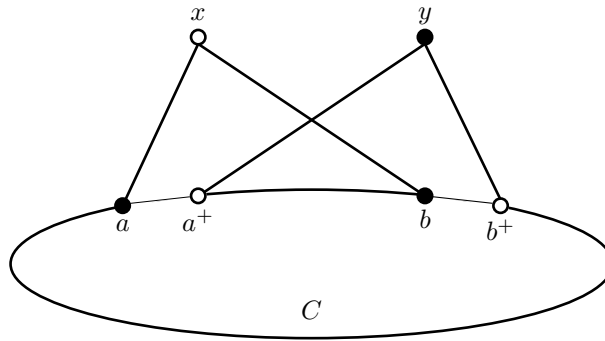


Fig. 5. $a^+, b^+, x \in X$ and $a, b, y \in Y$

Suppose that y has two neighbors a^+, b^+ in $N_C(x)^+$, then $xa, xb \in E$ and the cycle C' (see Fig. 5):

$$C' : \quad xbb^- \dots a^+yb^+b^{++} \dots ax$$

contains S , a contradiction with noncyclability of S . So y has at most one neighbor in $N_C(x)^+$ and thus:

$$d_C(x) + d_C(y) \leq \frac{c}{2} + 1. \tag{4.15}$$

From (4.14) and (4.15) we have:

$$d(x) + d(y) \leq \frac{r}{2} - 1 + \frac{c}{2} + 1 \leq n,$$

a contradiction with (3.1).

Case 2. $N_R(x) \neq \emptyset$ and $u_1, u_2 \in S_Y$

Subcase 2.1. There is a vertex $x_0 \in V(]u, u_1]) \cap X$ or $x_0 \in V(]u', u_2]) \cap X$

We can assume that $x_0 \in V(]u, u_1]) \cap X$.

Since $V(]u, u_1]) \cap S = \emptyset$, then from Lemma 4.3 (4.4) $xu_1 \notin E$.

Note that if $x_0u_2 \in E$, then the cycle:

$$C' : uP_1xP_2u'u'^- \dots u_1 \dots x_0u_2u_2^+ \dots u$$

contains S , a contradiction.

Using the same arguments we can show that:

$$N_{]u, u_1[}(u_2) = \emptyset \text{ and } N_{]u', u_2[}(x_0) = \emptyset. \quad (4.16)$$

So we have $u_1, u_2 \in Y$, $xu_1 \notin E$, $x_0u_2 \notin E$ and from (3.1), (3.2) we have:

$$d(x_0) + d(x) + d(u_1) + d(u_2) \geq 2n + 2. \quad (4.17)$$

Consider the interval $]u_2, u_1[$.

If $a \in V(]u_2, u_1])$ and $x_0a \in E$ then $a^+u_2 \notin E$.

Suppose that there is a vertex $a \in V(]u_2, u_1])$ such that $x_0a, a^+u_2 \in E$. From (4.16) we have $a \in V(]u_2, u])$ and in this case the cycle:

$$C' : x_0aa^- \dots u_2^+u_2a^+ \dots uP_1xP_2u'u'^- \dots u_1 \dots x_0 \quad (4.18)$$

contains S , a contradiction.

Using similar arguments we can show that if $a \in V(]u_1, u_2])$ and $u_2a \in E$ then $a^+x_0 \notin E$.

Since also $u_1u_2 \notin E$ we have:

$$d_C(x_0) + d_C(u_2) \leq \frac{c}{2}. \quad (4.19)$$

If $a \in V(C) \setminus \{u\}$ and $xa \in E$ then $u_1a^+ \notin E$.

Suppose that there is a vertex $a \in V(C) \setminus \{u\}$ such that $xa, u_1a^+ \in E$. From (4.4) we know that $a \notin V(]u, u_1])$ and thus the cycle:

$$C' : xaa^- \dots u_1a^+a^{++} \dots uP_1x$$

contains S , a contradiction.

So we have:

$$d_C(x) + d_C(u_1) \leq \frac{c}{2} + 1. \quad (4.20)$$

From Lemma 4.3 (4.5) we have:

$$d_R(x) + d_R(x_0) \leq \frac{r}{2}. \quad (4.21)$$

From Lemma 4.3 (4.6) we have:

$$d_R(u_1) + d_R(u_2) \leq \frac{r}{2}. \quad (4.22)$$

From (4.19) — (4.22) we have:

$$d(x_0) + d(x) + d(u_1) + d(u_2) \leq \frac{c}{2} + \frac{c}{2} + 1 + \frac{r}{2} + \frac{r}{2} = 2n + 1,$$

a contradiction with (4.17).

Subcase 2.2. $u^+ = u_1$ and $u'^+ = u_2$

If $u^+ = u_1$ and $u'^+ = u_2$ then $u, u' \in X$ and so $V(P_1) \neq \emptyset$ and $V(P_2) \neq \emptyset$ and from Remark 4.2 (4.2) we have $d_C(x) = 0$.

Subcase 2.2.1. $N_R(u_1) = \emptyset$ or $N_R(u_2) = \emptyset$

We can assume that $N_R(u_1) = \emptyset$. From Lemma 4.3 (4.4) $xu_1 \notin E$ and so from (3.1) we have:

$$d(x) + d(u_1) \geq n + 1. \quad (4.23)$$

From Remark 4.2 (4.2) and the assumption that $N_R(u_1) = \emptyset$ we have:

$$d(x) + d(u_1) = d_C(x) + d_R(x) + d_C(u_1) + d_R(u_1) \leq 0 + \frac{r}{2} + \frac{c}{2} + 0 = n,$$

a contradiction with (4.23).

Subcase 2.2.2. $N_R(u_1) \neq \emptyset$ and $N_R(u_2) \neq \emptyset$

Take an $a \in N_R(u_1)$. From Lemma 4.3 (4.4) $a \notin V(P)$ and $u_1x \notin E$. From (4.6) $u_2a \notin E$ and so from (3.1), (3.2) we have:

$$d(u_1) + d(u_2) + d(x) + d(a) \geq 2n + 2. \quad (4.24)$$

Note that for any $b \in V(\lceil u_2, u \rceil)$ if $ab \in E$ then $u_2b^+ \notin E$. Suppose that $ab, u_2b^+ \in E$, then the cycle:

$$C : abb^- \dots u_2b^+ \dots uP_1xP_2u'u'^- \dots u_1a$$

contains S , a contradiction.

Note that for any $b \in V(\lceil u_1, u' \rceil)$ if $ab \in E$ then $u_2b^- \notin E$. Suppose that $ab, u_2b^- \in E$, then the cycle:

$$C : abb^+ \dots u'P_2xP_1uu^- \dots u_2b^- \dots u_1a$$

contains S , a contradiction.

Since S is not cyclable $N_{u_2}(\lceil u, u_1 \rceil) = \emptyset$.

Note that in this case $u_2 = u'^+$ and so it is impossible that $au', u'^+u_2 \in E$.

From the above we have:

$$d_C(u_2) + d_C(a) \leq \frac{c}{2}. \tag{4.25}$$

Since from Remark 4.2 (4.2) we have $d_C(x) = 0$ then:

$$d_C(x) + d_C(u_1) \leq \frac{c}{2}$$

and

$$d_C(x) + d_C(a) + d_C(u_1) + d_C(u_2) \leq c. \tag{4.26}$$

Suppose now that we have a vertex $b \in V(R)$ such that $ab, xb \in E$. Then the cycle:

$$C : u_1abxP_1uu^- \dots u_2u'u'^- \dots u_1$$

contains S , a contradiction and so we have:

$$d_R(x) + d_R(a) \leq \frac{r}{2}. \tag{4.27}$$

From Lemma 4.3 (3.1) for any vertex $b \in V(R)$ if $u_1b \in E$ then $u_2b \notin E$.

Since also $x \notin N(u_1) \cup N(u_2)$ we have:

$$d_R(u_1) + d_R(u_2) \leq \frac{r}{2} - 1. \tag{4.28}$$

From (4.26) — (4.28) we have

$$d(u_1) + d(u_2) + d(x) + d(a) \leq c + r - 1 \leq 2n - 1,$$

a contradiction with (4.24).

Case 3. $N_R(x) \neq \emptyset$ and $u_1, u_2 \in S_X$

Subcase 3.1. $V(\lceil u, u_1 \rceil) \cap Y \neq \emptyset$ or $V(\lceil u', u_2 \rceil) \cap Y \neq \emptyset$

Choose a vertex $y \in V(\lceil u, u_1 \rceil) \cap Y$. From Lemma 4.3 (4.3) we have $xy \notin E$ and so from (3.1) we have:

$$d(x) + d(y) \geq n + 1. \tag{4.29}$$

Subcase 3.1.1. x has a neighbor in $R \setminus P_1$

Let y' be a neighbor of x from $R \setminus P_1$. Note that since S is not cyclable $u_1y' \notin E$ and from (3.1) we have:

$$d(x) + d(y) + d(y') + d(u_1) \geq 2n + 2. \tag{4.30}$$

Note that y and y' cannot have common neighbors in R , because if we have a vertex $b \in V(R) \cap X$ such that $yb, y'b \in E$ then the following cycle:

$$C : uP_1xy'byy^+ \dots u_1 \dots u' \dots u_2 \dots u^-u$$

contains S , a contradiction.

Using the same arguments we can show that u_1 and x don't have common neighbors in R and thus:

$$d_R(x) + d_R(y) + d_R(y') + d_R(u_1) \leq \frac{r}{2} + \frac{r}{2} = r. \quad (4.31)$$

Subcase 3.1.1.1. $V(P_1) \neq \emptyset$

Since $V(P_1) \neq \emptyset$ and P is a shortest C -path containing x , we have $xu \notin E$ and since from Lemma 4.3 (4.4) $d_{]u, u_1]}(x) = 0$, we have $d_{[u, u_1]}(x) = 0$.

Note that by the choice of C and P , for any $a \in V(]u_1, u])$

$V(P_1) \neq \emptyset$

$xa \in E$ then $ya^+ \notin E$. Suppose that $xa, ya^+ \in E$ then the cycle:

$$C : \quad xaa^- \dots ya^+a^{++} \dots uP_1x$$

contains S , a contradiction.

From this:

$$d_C(x) + d_C(y) \leq \frac{c}{2}. \quad (4.32)$$

If $a \in V(]u_1, u])$ and $y'a \in E$ then $u_1a^+ \notin E$. Take a vertex $a \in V(C) \setminus \{u\}$ and suppose that $y'a, u_1a^+ \in E$, then the cycle:

$$C : \quad xy'aa^- \dots u_1a^+a^{++} \dots uP_1x$$

contains S , a contradiction.

From Lemma 4.3 (4.7) $d_{y'}(]u, u_1]) = 0$.

Since it is possible that $y'u \in E$, from the above we get:

$$d_C(y') + d_C(u_1) \leq \frac{c}{2} + 1. \quad (4.33)$$

From (4.32) and (4.33) we have:

$$d_C(x) + d_C(y) + d_C(y') + d_C(u_1) \leq c + 1. \quad (4.34)$$

Subcase 3.1.1.2. $V(P_1) = \emptyset$

For any $a \in V(C) \setminus \{u\}$ if $xa \in E$ then $ya^+ \notin E$, except xu and yu^+ . Hence:

$$d_C(x) + d_C(y) \leq \frac{c}{2} + 1. \quad (4.35)$$

As in Subcase 3.1.1.1 for any $a \in V(]u_1, u])$ if $y'a \in E$ then $u_1a^+ \notin E$.

From Lemma 4.3 (4.7) $d_{]u, u_1]}(y') = 0$.

Since in this case $xu \in E$, we know that $u \in Y$ and $y'u \notin E$, thus $d_{y'}([u, u_1]) = 0$.

From the above:

$$d_C(y') + d_C(u_1) \leq \frac{c}{2}. \quad (4.36)$$

From (4.35) and (4.36) we have:

$$d_C(x) + d_C(y) + d_C(y') + d_C(u_1) \leq c + 1 \quad (4.37)$$

Conclusion from Subcases 3.1.1.1 and 3.1.1.2

Independently of the fact if $V(P_1)$ is empty or not, when x has a neighbor in $R \setminus P_1$ from (4.37) and (4.34) we have:

$$d_C(x) + d_C(y) + d_C(y') + d_C(u_1) \leq c + 1$$

and from (4.31):

$$d_R(x) + d_R(y) + d_R(y') + d_R(u_1) \leq r.$$

Hence from (4.31) and (4.37) we have:

$$d(x) + d(y) + d(y') + d(u_1) \leq r + c + 1 = 2n + 1,$$

a contradiction with (4.30). This ends the proof of Subcase 3.1.1.

Subcase 3.1.2. $N_R(x) \subset P_1$

We recall that $N_R(x) \neq \emptyset$. In this case $N_R(x) = \{y'\}$ and x has no other neighbors in R . We will get a contradiction by calculating the degree sum of the vertices x, y, y' , and u_2 .

We recall that $xy \notin E$ and we have the inequality (4.29):

$$d(x) + d(y) \geq n + 1.$$

Let y' be a neighbor of x in $R \cap P_1$. From (4.4) we have $u_2y' \notin E$ and from (3.1):

$$d(y') + d(u_2) \geq n + 1. \quad (4.38)$$

Hence from (3.1) we have:

$$d(x) + d(y) + d(y') + d(u_2) \geq 2n + 2. \quad (4.39)$$

From Lemma 4.3 (4.4) and (4.7) we have:

$$d_R(y) + d_R(y') \leq \frac{r}{2}. \quad (4.40)$$

Since S is not cyclable x and u_2 cannot have common neighbors in R we have:

$$d_R(x) + d_R(u_2) \leq \frac{r}{2}. \quad (4.41)$$

From (4.40) and (4.41) we get:

$$d_R(x) + d_R(y) + d_R(y') + d_R(u_2) \leq r. \quad (4.42)$$

Using the same arguments as those used to show (4.32) in Subcase 3.1.1.1 we can show that:

$$d_C(x) + d_C(y) \leq \frac{c}{2}. \quad (4.43)$$

Since $N_R(x) \subset P_1$ we have $P_2 = \emptyset$ and $xu' \in ED$. Hence $u' \in Y$. Since $y', u' \in Y$ we know that $y'u' \notin E$ and since also S is not cyclable we have:

- $N_{[u', u_2]}(y') = \emptyset$.
- If $a \in V([u_2, u'])$ and $y'a \in E$ then $a^+u_2 \notin E$.

From the above we have:

$$d_C(u_2) + d_C(y') \leq \frac{c}{2}. \quad (4.44)$$

From (4.42) — (4.44) we have:

$$d(x) + d(y) + d(y') + d(u_2) \leq r + c \leq 2n,$$

a contradiction with (3.1).

Subcase 3.2. $V(]u, u_1]) \cap Y = V(]u', u_2]) \cap Y = \emptyset$

From the main assumption in Case 1 we know that $u_1, u_2 \in X$. Since also $V(]u, u_1]) \cap Y = V(]u', u_2]) \cap Y = \emptyset$ we have:

$$]u, u_1[=]u', u_2[= \emptyset. \quad (4.45)$$

From (4.45): $u_1 = u^+$ and $u_2 = u'^+$ and thus $u, u' \in Y$.

Subcase 3.2.1. $N_R(u_1) = \emptyset$ or $N_R(u_2) = \emptyset$

We may assume that $N_R(u_1) = \emptyset$. In this case $u, u' \in Y$ and since $N_R(x) \neq \emptyset$, x has a neighbor in $R \setminus P_1$ or $R \setminus P_2$. We can assume that there is a vertex $y \in V(R \setminus P_1)$, such that $xy \in E$. From Lemma 4.3 (4.7) we know that $u_1y \notin E$ and from (3.1) we have:

$$d(y) + d(u_1) \geq n + 1. \quad (4.46)$$

Note that for any $a \in V(]u_1, u^-])$ if $ya \in E$ then $u_1a^+ \notin E$, because if $ya, u_1a^+ \in E$ then the cycle:

$$C: \quad xyaa^- \dots u_1a^+a^{++} \dots u^-uP_1x$$

contains S , a contradiction.

Note that since S is not cyclable $yu^- \notin E$ and since $u, y \in Y$ we have $yu \notin E$ and so

$$d_C(y) + d_C(u_1) \leq \frac{c}{2} \quad (4.47)$$

Since $u_1y \notin E$ and $N_R(u_1) = \emptyset$ we have:

$$d_R(y) + d_R(u_1) \leq \frac{r}{2} - 1 \quad (4.48)$$

and from (4.47), (4.48) we have:

$$d(y) + d(u_1) \leq \frac{c}{2} + \frac{r}{2} - 1 \leq n - 1,$$

a contradiction with (3.1).

Subcase 3.2.2. $N_R(u_1) \neq \emptyset$ and $N_R(u_2) \neq \emptyset$

From the noncyclability of S we know that u_1 and x cannot have common neighbors in R . We choose a vertex $y_1 \in N_R(u_1)$. Note that $y_1 \notin V(P)$. We chose also a $y \in N_R(x) \setminus \{y_1\}$.

Since $u_1y \notin E$ and $xy_1 \notin E$, from (3.1) we have:

$$d(x) + d(y) + d(u_1) + d(y_1) \geq 2n + 2. \quad (4.49)$$

From the noncyclability of S we know that y and y_1 cannot have common neighbors in R and so:

$$d_R(y) + d_R(y_1) \leq \frac{r}{2}. \quad (4.50)$$

For the same reasons x and u_1 cannot have common neighbors in R and so:

$$d_R(x) + d_R(u_1) \leq \frac{r}{2}. \quad (4.51)$$

We recall that in this case $N_R(x) \subset P_1$ and $N_R(x) \neq \emptyset$, hence $xu \notin E$. Note that for any $a \in V(C) \setminus \{u\}$, if $ax \in E$ then $a^+y_1 \notin E$. Suppose that $ax, a^+y_1 \in E$, then the cycle:

$$C : \quad xaa^- \dots u_1y_1a^+a^{++} \dots u^-uP_1x$$

contains S , a contradiction. From this:

$$d_C(x) + d_C(y_1) \leq \frac{c}{2}. \quad (4.52)$$

Using the same arguments we can show that for any $a \in V(C)$, if $ay \in E$ then $a^+u_1 \notin E$ and thus:

$$d_C(y) + d_C(u_1) \leq \frac{c}{2}. \quad (4.53)$$

From (4.50) — (4.53) we have

$$d(x) + d(y) + d(y_1) + d(u_1) \leq \frac{r}{2} + \frac{r}{2} + \frac{c}{2} + \frac{c}{2} = 2n,$$

a contradiction with (4.49).

Case 4. $N_R(x) \neq \emptyset$ and u_1 and u_2 are in different partite sets

We can assume that $u_1 \in X$ and $u_2 \in Y$.

Subcase 4.1. $N_R(x) \cap (V(R \setminus P_1)) \neq \emptyset$

We choose a vertex $y \in N_R(x) \cap (V(R \setminus P_1))$. From Lemma 4.3 (4.4) we know that $xu_2 \notin E$ and so from (3.1) we have:

$$d(x) + d(u_2) \geq n + 1. \quad (4.54)$$

From Lemma 4.3 (4.7) we know that $yu_1 \notin E$ and so from (3.2) we have:

$$d(u_1) + d(y) \geq n + 1. \quad (4.55)$$

Thus from (4.54) and (4.55) we have:

$$d(x) + d(y) + d(u_1) + d(u_2) \geq 2n + 2. \quad (4.56)$$

For any $a \in V(\lceil u_2, u' \rceil)$ if $xa \in E$ then $u_2a^+ \notin E$. Suppose that $xa, u_2a^+ \in E$ then the cycle:

$$C : \quad xaa^- \dots u_2^+u_2a^+a^{++} \dots uu^+ \dots u_1 \dots u'P_2x$$

contains S , a contradiction.

Note that however from Lemma 4.3 (4.4) we know that $N_{]u', u_2]}(x) = \emptyset$, but if $a = u'$ it may happen that $xa, u_2a^+ \in E$ and so:

$$\text{if } xu' \notin E \text{ then } d_C(x) + d_C(u_2) \leq \frac{c}{2}, \quad (4.57)$$

$$\text{if } xu' \in E \text{ then } d_C(x) + d_C(u_2) \leq \frac{c}{2} + 1. \quad (4.58)$$

Using the same arguments we can show that for any $a \in V(]u_1, u[)$ if $ya \in E$ then $u_1a^+ \notin E$. Note that however from Lemma 4.3 (4.7) we know that $N_{]u, u_1]}(y) = \emptyset$, but if $a = u$ it may happen that $ya, u_1a^+ \in E$ and so:

$$\text{if } yu \notin E \text{ then } d_C(y) + d_C(u_1) \leq \frac{c}{2}, \quad (4.59)$$

$$\text{if } yu \in E \text{ then } d_C(y) + d_C(u_1) \leq \frac{c}{2} + 1. \quad (4.60)$$

Suppose now that $d_C(x) + d_C(u_2) = \frac{c}{2} + 1$ and $d_C(y) + d_C(u_1) = \frac{c}{2} + 1$. In this case we have $xu', yu \in E$. Since $xu' \in E$ we have $y \notin V(P_2)$ and since $yu \in E$ we have $u \in X$. Since $u \in X$ we have $V(P_1) \neq \emptyset$. Hence from Remark 4.2 (4.1) $d_C(x) = 1$.

From the noncyclability of S we have $u_1u_2 \notin E$ and so $d_C(u_1) \leq \frac{c}{2} - 1$ and $d_C(u_2) \leq \frac{c}{2} - 1$.

From the above if $xu', yu \in E$ then:

$$d_C(x) + d_C(u_2) \leq \frac{c}{2} \quad (4.61)$$

and this improves upon the inequality (4.58). In fact we cannot have $d_C(y) + d_C(u_1) = d_C(x) + d_C(u_2) = \frac{c}{2} + 1$, and so from (4.57) — (4.61) we know that in any case:

$$d_C(x) + d_C(u_2) + d_C(y) + d_C(u_1) \leq c + 1. \quad (4.62)$$

From Lemma 4.3 (4.7) vertices u_2 and y cannot have common neighbors in R and so:

$$d_R(u_2) + d_R(y) \leq \frac{r}{2}. \quad (4.63)$$

For the same reasons we have:

$$d_R(u_1) + d_R(x) \leq \frac{r}{2}. \quad (4.64)$$

From (4.62) — (4.64) we have:

$$d(x) + d(y) + d(u_1) + d(u_2) \leq c + 1 + r = 2n + 1,$$

a contradiction with (4.56).

Subcase 4.2. $N_R(x) \subset P_1$

Since $N_R(x) \subset P_1$ we have: $xu' \in E$ and there is a vertex $y \in V(P_1)$ such that $xy \in E$ and so from Lemma 4.3 we have $d_C(x) \leq 1$.

Note that from noncyclability of S $xu_2 \notin E$. From Lemma 4.3 (4.4) we know that $yu_1 \notin E$ and from (3.1) we have:

$$d(x) + d(y) + d(u_1) + d(u_2) \geq 2n + 2. \quad (4.65)$$

Using the same arguments as in Subcase 4.1 to show (4.57) and (4.58) we have:

$$d_C(x) + d_C(u_2) \leq \frac{c}{2} + 1. \quad (4.66)$$

From Lemma 4.3 (4.4) we have $V(\lceil u, u_1 \rceil) \cap N(y) = \emptyset$. For any $a \in V(\lceil u_1, u \rceil)$ if $ya \in E$ then $u_1a^+ \notin E$. In this case $xy \in E$, $y \in V(P_1)$ and the vertices of the path $P = P_1xP_2$ are labelled in the following way: $p_1^1 \dots p_l^1xp_2^k \dots p_2^1$. Since P is the shortest C -path $y = p_l^1$.

Suppose that $ya, u_1a^+ \in E$, then the cycle:

$$C' : yaa^- \dots u_1a^+ \dots up_1^1 \dots p_{l-1}^1y$$

contains $S \setminus \{x\}$ and has a C -path $P' : yxu'$, shorter than P , a contradiction with the choice of C and P . However it is possible that if $a = u$ then $ya, u_1a^+ \in E$. So we have shown that:

$$\text{if } yu \notin E \text{ then } d_C(y) + d_C(u_1) \leq \frac{c}{2}. \quad (4.67)$$

Suppose now that $yu \in E$. Since $xu' \in E$ we have $u' \in Y$ and $u_2 \in Y$. From this $V(\lceil u', u_2 \rceil) \cap X \neq \emptyset$. We choose a vertex $v \in V(\lceil u', u_2 \rceil) \cap X$. From the noncyclability of S $u_1v^+ \notin E$ and $yv \notin E$.

Since if $yu \in E$, then we have a vertex $v \in V(C) \cap X$ such that $u_1v^+ \notin E$ and $yv \notin E$, so we have:

$$d_C(y) + d_C(u_1) \leq \frac{c}{2}. \quad (4.68)$$

From (4.67) and (4.68) in any case:

$$d_C(y) + d_C(u_1) \leq \frac{c}{2}. \quad (4.69)$$

From Lemma 4.3 (4.7) vertices u_2 and y cannot have common neighbors in R and so:

$$d_R(u_2) + d_R(y) \leq \frac{r}{2}. \quad (4.70)$$

For the same reasons we have:

$$d_R(u_1) + d_R(x) \leq \frac{r}{2}. \quad (4.71)$$

From (4.66) and (4.69) — (4.71) we have:

$$d(x) + d(y) + d(u_1) + d(u_2) \leq \frac{c}{2} + 1 + \frac{c}{2} + \frac{r}{2} + \frac{r}{2} = 2n + 1,$$

a contradiction with (4.65).

We have shown that in any case we get a contradiction with the hypothesis that S is not cyclable, so the proof of Theorem 3.1 is finished. \square

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