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ON THE GLOBAL OFFENSIVE ALLIANCE NUMBER OF A TREE

Abstract. For a graph $G = (V,E)$, a set $S \subseteq V$ is a dominating set if every vertex in $V - S$ has at least a neighbor in $S$. A dominating set $S$ is a global offensive alliance if for every vertex $v$ in $V - S$, at least half of the vertices in its closed neighborhood are in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$, and the global offensive alliance number $\gamma_o(G)$ is the minimum cardinality of a global offensive alliance of $G$. We first show that every tree of order at least three with $\ell$ leaves and $s$ support vertices satisfies $\gamma_o(T) \geq (n - \ell + s + 1)/3$ and we characterize extremal trees attaining this lower bound. Then we give a constructive characterization of trees with equal domination and global offensive alliance numbers.

Keywords: global offensive alliance number, domination number, trees.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Let $G = (V,E)$ be a finite and simple graph of order $n$. The open neighborhood of a vertex $v \in V$ is $N_G(v) = N(v) = \{ u \in V \mid uv \in E \}$ and the closed neighborhoods of $v$ is $N_G[v] = N[v] = N(v) \cup \{ v \}$. The degree of $v$, denoted by $\text{deg}_G(v)$, is the size of its open neighborhood. A vertex of degree one is called a pendant vertex or a leaf and its neighbor is called a support vertex. If $v$ is a support vertex, then $L_v$ will denote the set of the leaves attached at $v$. We also denote the set of leaves of a graph $G$ by $L(G)$, the set of support vertices by $S(G)$, and let $|L(G)| = \ell$, $|S(G)| = s$. A tree $T$ is a double star if it contains exactly two vertices that are not leaves. A subdivided star $SS_q$ is obtained from a star $K_{1,q}$ by subdividing each edge by exactly one vertex.

For a graph $G = (V,E)$, a set $S$ is a dominating set if every vertex in $V - S$ has at least a neighbor in $S$. A dominating set $S$ is called a global offensive alliance if for every $v \in V - S$, $|N[v] \cap S| \geq |N[v] - S|$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$, and the global offensive alliance number $\gamma_o(G)$ is the minimum cardinality of a global offensive alliance of $G$. Clearly for every graph $G,$
\(\gamma_o(G) \geq \gamma(G)\). Every graph has a global offensive alliance, since \(S = V\) is such a set. We abbreviate global offensive alliance as \(\text{goa}\). If \(S\) is a goa of \(G\) and \(|S| = \gamma_o(G)\), then we say that \(S\) is a \(\gamma_o(G)\)-set. Alliances in graphs were introduced by Hedetniemi, Hedetniemi, and Kristiansen in [5]. For the study of offensive alliances we cite for example [1] and [2]. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [3, 4]. In this paper, we show that every tree of order at least three with \(\ell\) leaves and \(s\) support vertices satisfies \(\gamma_o(T) \geq (n - \ell + s + 1)/3\) and we characterize extremal trees attaining this lower bound. We also give a constructive characterization of trees with equal domination and global offensive alliance numbers.

2. LOWER BOUND

We begin with a couple of observations.

**Observation 2.1.** If \(G\) is a connected graph of order at least three, then there is a \(\gamma_o(G)\)-set that contains all the support vertices.

**Observation 2.2.** Let \(T\) be a tree obtained from a nontrivial tree \(T'\) by attaching a star \(K_{1,t}\) of center \(x\) with an edge \(xz\) at a support vertex \(z\) of \(T'\). Then \(\gamma_o(T) = \gamma_o(T') + 1\) and \(\gamma(T) = \gamma(T') + 1\).

**Proof.** By Observation 2.1 there is a \(\gamma_o(T')\)-set \(D\) that contains all the support vertices. Hence \(x, z \in D\); so \(D - \{x\}\) is a goa of \(T'\) and \(\gamma_o(T') \leq \gamma_o(T) - 1\). Since every \(\gamma_o(T')\)-set can be extended to a goa of \(T\) by adding \(x\), \(\gamma_o(T) \leq \gamma_o(T') + 1\). It follows that \(\gamma_o(T) = \gamma_o(T') + 1\). If \(D'\) is any \(\gamma_o(T')\)-set, then \(D' \cup \{x\}\) is a dominating set of \(T\), implying that \(\gamma(T) \leq \gamma(T') + 1\). The equality comes by the fact that \(x, z\) belong to some \(\gamma(T')\)-set, and such a set minus \(x\) is a dominating set of \(T'\).

Let \(F\) be the family of trees of order at least three that can be obtained from \(r\) disjoint stars by first adding \(r - 1\) edges so that they are incident only with centers of the stars and the resulting graph is connected, and then subdividing each new edge exactly once.

**Theorem 2.3.** Let \(T\) be a tree of order \(n \geq 3\) with \(\ell\) leaves and \(s\) support vertices. Then \(\gamma_o(T) \geq (n - \ell + s + 1)/3\) with equality if and only if \(T \in F\).

**Proof.** Let \(T \in F\). Then \(T\) contains \(|S(T)| - 1\) vertices of degree two and the remaining vertices are leaves and support vertices. It follows that \(n = \ell + 2s - 1\) and so \(s = (n - \ell + s + 1)/3\). Now it is clear that every \(\gamma_o(T)\)-set contains at least \(|S(T)|\) vertices and so \(\gamma_o(T) \geq |S(T)|\). The equality follows from the fact that \(S(T)\) is a global offensive alliance of \(T\), implying that \(\gamma_o(T) = |S(T)| = (n - \ell + s + 1)/3\).

To prove that if \(T\) is a tree of order \(n \geq 3\), then \(\gamma_o(T) \geq (n - \ell + s + 1)/3\) with equality only if \(T \in F\), we use an induction on the order \(n\). If \(\text{diam}(T) = 2\), then \(T\) is a star with \(\gamma_o(T) = 1 = (n - \ell + s + 1)/3\) and so \(T \in F\). If \(\text{diam}(T) = 3\), then \(\gamma_o(T) = 2 > (n - \ell + s + 1)/3\). Assume that every tree \(T'\) of order \(n'\), \(3 \leq n' < n\), with \(\ell'\) leaves and \(s'\) support vertices satisfies \(\gamma_o(T') \geq (n' - \ell' + s' + 1)/3\) with equality
On the global offensive alliance number of a tree

if and only if \( T \in \mathcal{F} \). Let \( T \) be a tree of order \( n \) and diameter at least four having \( \ell \) leaves and \( s \) support vertices.

We now root \( T \) at a vertex \( r \) of maximum eccentricity \( \text{diam}(T) \geq 4 \). Let \( u \) be a support vertex at maximum distance from \( r \), \( v \) be the parent of \( u \), and \( w \) be the parent of \( v \) in the rooted tree. Note that \( \text{deg}_\mathcal{F}(w) \geq 2 \) and let \( D \) be a \( \gamma_o(T) \)-set that contains no leaves. Denote by \( T_u \) the subtree induced by a vertex \( x \) and its descendants in the rooted tree \( T \). We distinguish between three cases.

**Case 1.** \( v \) is a support vertex. Let \( T' = T - L_u \cup \{u\} \). Then \( n' = n - 1 - |L_u| \geq 3 \), \( \ell' = \ell - |L_u| \) and \( s' = s - 1 \). By Observation 2.2, \( \gamma_o(T') = \gamma_o(T) + 1 \) and by induction on \( T' \) we obtain \( \gamma_o(T) > (n - \ell + s + 1)/3 \).

**Case 2.** \( \text{deg}_\mathcal{F}(v) \geq 3 \) and \( v \) is not a support vertex. Thus every child of \( v \) is a support vertex. Let \( k \) be the number of children of \( v \) and \( B \) the set of leaves in \( T_v \). We first assume that \( \text{deg}_\mathcal{F}(w) \geq 3 \) and let \( T' = T - T_v \). Then \( n' = n - |B| - k - 1 \geq 3 \), \( \ell' = \ell - |B| \) and \( s' = s - k \). Since \( D \) contains all children of \( v \) and does not contain \( v \) (else replace it by \( w \)), \( D \cap V(T') \) is a goa of \( T' \). It follows \( \gamma_o(T') \leq \gamma_o(T) - k \) and by induction on \( T' \) we have

\[
\gamma_o(T) \geq (n' - \ell' + s' + 1)/3 + k \geq (n - \ell + s + 1 + k - 1)/3
\]

and therefore \( \gamma_o(T) > (n - \ell + s + 1)/3 \) since \( k \geq 2 \).

Now assume that \( \text{deg}_\mathcal{F}(w) = 2 \). Let \( T' = T - (T_v - \{v\}) \). Then \( n' = n - |B| - k \geq 3 \), \( \ell' = \ell - |B| + 1 \) and \( s' = s - k + 1 \). Clearly \( D \) contains all children of \( v \) and does not contain \( v \) (else replace it by \( w \)), and \( D \) must contain \( w \) for otherwise \( w \) would have one neighbor in \( D \) and itself and \( v \) not in \( D \). Thus \( D \cap V(T') \) is goa of \( T' \) and hence \( \gamma_o(T') \leq \gamma_o(T) - k \). By induction on \( T' \) we have

\[
\gamma_o(T) \geq (n' - \ell' + s' + 1)/3 + k \geq (n - \ell + s + 1 + k)/3
\]

and therefore \( \gamma_o(T) > (n - \ell + s + 1)/3 \).

**Case 3.** \( \text{deg}_\mathcal{F}(v) = 2 \). Then \( u, w \in D \) and \( v \notin D \). Assume that \( \text{deg}_\mathcal{F}(w) = 2 \) or \( \text{deg}_\mathcal{F}(v) \geq 3 \) and \( v \) is not a support vertex. Let \( T' = T - L_u \cup \{u\} \). Then \( D \cap V(T') \) is a goa of \( T' \) and so \( \gamma_o(T') \leq \gamma_o(T) - 1 \). Using the induction on \( T' \) and since \( n' = n - 1 - |L_u| \geq 3 \), \( \ell' = \ell - |L_u| + 1 \) and \( s' = s \), we obtain

\[
\gamma_o(T) \geq (n' - \ell' + s' + 1)/3 + 1 \geq (n - \ell + s + 1)/3.
\]

We finally assume that \( \text{deg}_\mathcal{F}(w) \geq 3 \) and \( w \) is a support vertex. Let \( T' = T - L_u \cup \{u, v\} \). Then \( D \cap V(T') \) is a goa of \( T' \), \( n' = n - 2 - |L_u| \geq 3 \), \( \ell' = \ell - |L_u| \) and \( s' = s - 1 \). Hence by induction on \( T' \), we have

\[
\gamma_o(T) \geq \gamma_o(T') + 1 \geq (n' - \ell' + s' + 1)/3 + 1 \geq (n - \ell + s + 1)/3.
\]

Further, if \( \gamma_o(T) \geq (n - \ell + s + 1)/3 \), then we have equality throughout this inequality chain. In particular, \( \gamma_o(T') = (n' - \ell' + s' + 1)/3 \). Thus by the inductive hypothesis on \( T' \), \( T' \in \mathcal{F} \). It follows that \( T \in \mathcal{F} \). \( \square \)
3. TREES T WITH $\gamma_0(T) = \gamma(T)$

**Observation 3.1.** Let $T$ be a tree obtained from a nontrivial tree $T'$ by attaching a subdivided star $SS_k$, $k \geq 2$, of center $x$ with an edge edge $xy$ at a vertex $y$ of $T'$. Then:

(a) $\gamma_0(T') \leq \gamma_0(T) - k$, with equality if $y$ belongs to some $\gamma_0(T')$-set or a strict majority of its closed neighborhood belong to some $\gamma_0(T')$-set.

(b) $\gamma(T) = \gamma(T') + k$.

**Proof.** (a) By Observation 2.1 there is a $\gamma_0(T)$-set $S$ that contains all support vertices of the added subdivided star. Also we may assume that $x \notin S$ (else replace it by $y$). Thus $S \cap V(T')$ is a goa of $T'$, and so $\gamma_0(T') \leq \gamma_0(T) - k$. Now if $y$ belongs to some $\gamma_0(T')$-set or a strict majority of its closed neighborhood belong to some $\gamma_0(T')$-set, then such sets can be extended to a goa of $T$ by adding the set of support vertices of $SS_k$. It follows that $\gamma_0(T) \leq \gamma_0(T') + k$ and the equality holds.

Item (b) is easy to show. □

In order to characterize trees with equal domination and global offensive alliance numbers we define the family $F$ of all trees $T$ that can be obtained from a sequence $T_1, T_2, \ldots, T_k (k \geq 1)$ of trees, where $T_1 = P_2$, $T = T_k$, and, if $k \geq 2$, $T_{i+1}$ is obtained recursively from $T_i$ by one of the four operations defined below. Let one the vertices of $T_1$ be considered a support and the other a leaf.

— **Operation $O_1$:** Attach a vertex by joining it to any support vertex of $T_1$.

— **Operation $O_2$:** Attach a path $P_3 = xy$ by joining $x$ to any support vertex $z$ of $T_1$.

— **Operation $O_3$:** Attach a subdivided star $SS_k$, $k \geq 2$, of center $u$ by joining $u$ to vertex $v$ of $T_1$ with the condition that if $v$ does not belong to a $\gamma_0(T_1)$-set $D$, then a strict majority of $N_{T_1}[v]$ are in $D$.

— **Operation $O_4$:** Attach a path $P_3 = xyz$ by joining $x$ to any vertex of $T_1$ that belongs to a $\gamma_0(T_1)$-set.

**Lemma 3.2.** If $T \in F$, then $\gamma_0(T) = \gamma(T)$.

**Proof.** We use induction on the number of operations $k$ performed to construct $T$. The property is true for $T_1 = P_2$. Suppose the property is true for all trees of $F$ constructed with $k - 1 \geq 0$ operations. Let $T = T_k$ with $k \geq 2$, $T' = T_{k-1}$, and let $D$ be a $\gamma_0(T)$-set that contains no leaf of $T$. We examine the following cases.

Clearly if $T$ was obtained from $T'$ by Operation $O_1$, then $\gamma_0(T') = \gamma_0(T)$, $\gamma(T') = \gamma(T)$ and so $\gamma_0(T) = \gamma(T)$.

If $T$ was obtained from $T'$ by Operation $O_2$, then by Observation 2.2, $\gamma_0(T) = \gamma_0(T') + 1$ and $\gamma(T) = \gamma(T') + 1$. Using the induction on $T'$ it follows that $\gamma_0(T) = \gamma(T)$.

If $T$ was obtained from $T'$ by Operation $O_3$, then by Observation 3.1 $\gamma_0(T) = \gamma_0(T') + k$ and $\gamma(T) = \gamma(T') + k$. By induction on $T'$, we obtain $\gamma_0(T) = \gamma(T)$.

Finally assume that $T$ was obtained from $T'$ by Operation $O_4$. Let $w \in V(T')$ be the neighbor of $x$. Then $y \in D$, and $x \notin D$ (else replace it by $w$). Thus $D \cap V(T')$ is a goa of $T'$ and we have $\gamma_0(T') \leq \gamma_0(T) - 1$. Now since $w$ belongs to a $\gamma_0(T')$-set,
such a set can be extended to goa of $T$ by adding $y$; so $\gamma_o(T) \leq \gamma_o(T') + 1$ and the equality follows. Also it can be seen easily that, $\gamma(T) = \gamma(T') + 1$. By induction on $T'$, we obtain the desired result.

**Theorem 3.3.** Let $T$ be a tree. Then $\gamma_o(T) = \gamma(T)$ if and only if $T = K_1$ or $T \in \mathcal{F}$.

**Proof.** Clearly if $T = K_1$, then $\gamma_o(T) = \gamma(T)$. If $T \in \mathcal{F}$, then by Lemma 3.2, $\gamma_o(T) = \gamma(T)$. Now to prove the converse we use an induction on the order $n$ of $T$.

It is obvious that $\gamma_o(K_1) = \gamma(K_1)$. Let us assume that $n \geq 2$. If $n = 2$, then $T = P_2$ and $T$ belongs to $\mathcal{F}$. If $n = 3$, then $T = P_3$ that belongs to $\mathcal{F}$ since it is obtained from $P_2$ by using Operation $O_1$. Assume that every tree $T'$ of order $n' \geq 2$ satisfying $\gamma_o(T') = \gamma(T')$ is in $\mathcal{F}$.

Let $T$ be a tree of order $n > n'$ such that $\gamma_o(T) = \gamma(T)$. If $T$ is a star $K_{1,t}$, then $\gamma_o(T) = \gamma(T)$ and $T \in \mathcal{F}$ because it is obtained from $P_2$ by using Operation $O_1$. If $T$ is a double star, then $\gamma(T) = \gamma(T')$ and $T \in \mathcal{F}$ because it is obtained from $P_2$ by using Operations $O_2$ and $O_1$. Thus we may assume that $T$ has diameter at least four.

If any support vertex, say $x$, of $T$ is adjacent to two or more leaves, then let $T'$ be the tree obtained from $T$ by removing a leaf adjacent to $x$. Then $\gamma(T') = \gamma_o(T)$, $\gamma(T') = \gamma(T)$ and so $\gamma_o(T') = \gamma(T')$. By induction on $T'$, we have $T' \in \mathcal{F}$. Thus $T \in \mathcal{F}$ because it is obtained from $T'$ by using Operation $O_1$. Henceforth, we can assume that every support vertex of $T$ is adjacent to exactly one leaf.

We now root $T$ at a vertex $r$ of maximum eccentricity $\text{diam}(T) \geq 4$. Let $v$ be a support vertex at maximum distance from $r$, $u$ be the parent of $v$, and $w$ be the parent of $u$ in the rooted tree. Let $v'$ be the unique leaf adjacent to $v$. Note that $\text{deg}_T(w) \geq 2$. Let $D$ be a $\gamma_o(T)$-set that contains no leaves. We distinguish between three cases.

**Case 1.** $u$ is a support vertex. Let $T' = T - \{v, v'\}$. Then by Observation 2.2, $\gamma_o(T) = \gamma_o(T') + 1$ and $\gamma(T) = \gamma(T') + 1$. Thus $\gamma(T') = \gamma(T')$ and by induction on $T'$, we have $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$ and is obtained from $T'$ by using Operation $O_2$.

**Case 2.** $u$ is not a support vertex but has at least one child besides $v$ as a support vertex. Thus $T_u$ is a subdivided star. Let $T' = T - T_u$. Then by Observation 3.1, $\gamma_o(T') \leq \gamma_o(T) - k$ and $\gamma(T) = \gamma(T') + k$, where $k$ is the number of children of $v$. Assume now that $\gamma_o(T') < \gamma_o(T) - k$, then

$$\gamma_o(T') < \gamma_o(T) - k = \gamma(T) - k = (\gamma(T') + k) - k = \gamma(T')$$

and so $\gamma_o(T') < \gamma(T')$, a contradiction. Hence $\gamma_o(T') = \gamma_o(T) - k$ and $D' = D \cap V(T')$ is a $\gamma_o(T')$-set. It follows that $\gamma_o(T') = \gamma(T')$. Note that if $w \notin D'$, then since $D$ is a $\gamma_o(T)$-set, $|N_{T'}[w] \cap D'| > |N_{T'}[w] - D'|$. Applying the inductive hypothesis $T'$ belongs to $\mathcal{F}$, and so $T \in \mathcal{F}$ because it is obtained from $T'$ by using Operation $O_3$.

**Case 3.** $u$ has degree two. Let $T' = T - \{v', v, u\}$. It can be seen that $\gamma(T') = \gamma(T) - 1$. Also $v \in D$, $u \notin D$ (else replace it by $w$), and so $w \in D$. Thus $D \cap V(T')$ is a goa of $T'$ and $\gamma_o(T') \leq \gamma_o(T) - 1$. Now if $\gamma_o(T') < \gamma_o(T) - 1$, then

$$\gamma_o(T') < \gamma_o(T) - 1 = \gamma(T) - 1 = (\gamma(T') + 1) - 1 = \gamma(T')$$
and hence \( \gamma_o(T') < \gamma(T') \), a contradiction. Therefore \( \gamma_o(T') = \gamma_o(T) - 1 \) and so \( D \cap V(T') \) is a \( \gamma_o(T') \)-set containing \( w \). It follows that \( \gamma_o(T') = \gamma(T') \), and by the inductive hypothesis \( T' \in \mathcal{F} \). Thus \( T \in \mathcal{F} \) and is obtained from \( T' \) by using Operation \( O_4 \).

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