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ON THE PROPERTY OF THE BBGKY HIERARCHY SOLUTION IN A CUMULANT REPRESENTATION

Abstract. We consider a one-dimensional nonsymmetric system of particles interacting via the hard-core potential. For this system, we prove that the BBGKY hierarchy solution in a cumulant representation is an equilibrium in the case of equilibrium initial data.

Keywords: BBGKY hierarchy of equations, nonsymmetric system, cumulant (semi--invariant), equilibrium distribution functions.

Mathematics Subject Classification: 35Q72.

1. INTRODUCTION

States of systems of infinitely many particles are described by infinite sequences of distribution functions that are solutions to the BBGKY hierarchy [1]. BBGKY hierarchy solutions are determined by expansions over particle groups (clusters) of increasing size in the form of iteration series or some function series [2, 5, 12].

In this article we consider solution of the initial value problem for the BBGKY hierarchy represented in the form of an expansion over particle clusters whose evolution is governed by the cumulant (semi-invariant) of the evolution operator of the corresponding particle cluster for nonsymmetric system of particles with interaction between the nearest neighbors [4]. Such a representation of solutions enables us to describe the cluster nature of the evolution of infinite particle systems in detail.

Existence and uniqueness results for such BBGKY hierarchy solutions in the space E_ξ of double sequences of nonsymmetric bounded functions were given in articles [13, 14].

Formulation of the problem is given in Section 2. In Section 3 it is proved that the BBGKY hierarchy solution in a cumulant representation is an equilibrium in the case of equilibrium initial data.

2. FORMULATION OF THE PROBLEM

We consider a one-dimensional nonsymmetric system of particles of diameter σ with mass $m = 1$ which interact with their nearest neighbors via the potential $\Phi(q)$ and occupy admissible configurations only: $q_{i+1} \geq q_i + \sigma$, $i \in \mathbb{Z}^1 \setminus \{0\}$. The configurations $W_n = \{(q_{-n_2}, \dots, q_{n_1}) \in \mathbb{R}^{n_1+n_2} | q_{i+1} < q_i + \sigma$ for at least a single pair $(i, i+1) \in ((-n_2, -n_2+1), \dots, (-1, 1), \dots, (n_1-1, n_1))\}$ are said to be forbidden. The potential $\Phi(q)$ satisfies the conditions:

$$1.\ \Phi \in C^2[\sigma, R], \quad 0 < \sigma < R < \infty,$$

$$2.\ \Phi(|q|) = \begin{cases} +\infty, & |q| \in [0, \sigma), \\ 0, & |q| \in (R, +\infty), \end{cases}$$

$$3.\ \Phi'(\sigma + 0) = 0,$$

where R is the range of the interaction.

Notice that the third condition on the potential $\Phi(q)$ means that at the instants of collision t, i.e., $Q_{i+1}(t) - Q_i(t) = \sigma$, the hard core interaction only is effective. In this case completely elastic scattering occurs: the phase states of particles change instantaneously according to

$$\begin{aligned} (Q_i(t), P_i(t)) &= (Q_i(t), P_{i+1}(t)), \\ (Q_{i+1}(t), P_{i+1}(t)) &= (Q_{i+1}(t), P_i(t)). \end{aligned}$$

The interaction potential satisfies the stability condition:

$$\sum_{i=-n_2}^{n_1-1} \Phi(q_i - q_{i+1}) \geq -B(n_1 + n_2), \quad B > 0. \tag{2.1}$$

A solution $F(t) = \{F_{|Y|}(t, Y)\}_{|Y|=s=s_1+s_2 \geq 0}$ of the initial value problem for the BBGKY hierarchy is represented as the expansion in the space E_ξ [4, 13, 14]:

$$F_{|Y|}(t, Y) = \sum_{n=0}^{\infty} \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \geq 0}} \int_{(\mathbb{R}^1 \times \mathbb{R}^1)^{n_1+n_2}} d(X \setminus Y)\, \mathfrak{A}_{(n_2, n_1)}(t, X_Y) F_{|X|}(0, X), \tag{2.2}$$

where

$$Y = (x_{-s_2}, \dots, x_{s_1}), \quad X = (x_{-(n_2+s_2)}, \dots, x_{s_1+n_1}),$$

$$X_Y = (x_{-(n_2+s_2)}, \dots, x_{-(s_2+1)}, Y, x_{s_1+1}, \dots, x_{s_1+n_1}),$$

$$d(X \setminus Y) = dx_{-(n_2+s_2)} \dots dx_{-(s_2+1)} dx_{s_1+1} \dots dx_{s_1+n_1},$$

$$\mathfrak{A}_{(n_2, n_1)}(t, X_Y) = \sum_{P: X_Y = \bigcup_i X_i} (-1)^{|P|-1} \prod_{X_i \subset P} S_{|X_i|}(-t, X_i),$$

$\sum\limits_{P}$ is the sum over all ordered partitions of the partially ordered set X_Y into $|P|$ nonempty pairwise disjoint partially ordered subsets and the set Y lies in one of the subsets X_i.

The evolution operator $S_{s+n}(-t)$ of the Liouville equation is defined by the formula

$$S_{s+n}(-t, x_{-(n_2+s_2)}, \ldots, x_{s_1+n_1}) f_{s+n}(x_{-(n_2+s_2)}, \ldots, x_{s_1+n_1}) =$$
$$\begin{cases} f_{s+n}(X_{-(n_2+s_2)}(-t, x_{-(n_2+s_2)}, \ldots, x_{s_1+n_1}), \ldots, X_{s_1+n_1}(-t, x_{-(n_2+s_2)}, \ldots, x_{s_1+n_1})), \\ \text{ if } (x_{-(n_2+s_2)}, \ldots, x_{s_1+n_1}) \in \mathbb{R}^{n_1+s_1+n_2+s_2} \times (\mathbb{R}^{n_1+s_1+n_2+s_2} \setminus W_{n+s}); \\ 0, \text{ if } (q_{-(n_2+s_2)}, \ldots, q_{s_1+n_1}) \in W_{n+s}; \end{cases}$$

where $X_i(-t, x_{-(n_2+s_2)}, \ldots, x_{s_1+n_1}), i = -(n_2+s_2), \ldots, s_1+n_1$ is a solution of the initial value problem for the Hamilton equations of $(n+s)$-particle system with initial data $X_i(0, x_{-(n_2+s_2)}, \ldots, x_{s_1+n_1}) = x_i, i = -(n_2+s_2), \ldots, s_1+n_1$.

We consider this system of particles situated in a bounded connected domain $\Lambda \subset \mathbb{R}^1$.

3. ON THE PROPERTY OF THE BBGKY HIERARCHY SOLUTION

Introduce the sequence of equilibrium distribution functions for this system of particles situated in a bounded connected domain $\Lambda \subset \mathbb{R}^1$

$$F_\Lambda^{eq} = \big((I - \mathfrak{a}_{(+)})^{-1}(I - \mathfrak{a}_{(-)})^{-1}\Psi_\Lambda\big)_0^{-1} (I - \mathfrak{a}_{(+)})^{-1}(I - \mathfrak{a}_{(-)})^{-1}\Psi_\Lambda, \tag{3.1}$$

where $\mathfrak{a}_{(+)}$ and $\mathfrak{a}_{(-)}$ are the operators of integration with respect to the last arguments with positive and negative numbers, respectively, i.e.,

$$(\mathfrak{a}_{(+)}f)_s(x_{-s_2}, \ldots, x_{s_1}) = \int dx_{s_1+1}\, f_{s_1+1+s_2}(x_{-s_2}, \ldots, x_{s_1+1}),$$
$$(\mathfrak{a}_{(-)}f)_s(x_{-s_2}, \ldots, x_{s_1}) = \int dx_{-(s_2+1)}\, f_{s_1+s_2+1}(x_{-(s_2+1)}, \ldots, x_{s_1}),$$

while $\Psi_\Lambda = \{\Psi_{\Lambda,s}(x_{-s_2}, \ldots, x_{s_1})\}_{s \geq 0}$ is the sequence of the Boltzmann equilibrium functions.

Due to stability condition (2.1) of the regular potential, the distribution functions (3.1) exist [15].

The following theorem holds:

Theorem 3.1. *If the initial data is given in the form of equilibrium distribution functions, then the solution of the initial value problem for the BBGKY hierarchy is an equilibrium.*

Proof. If the initial data takes the form of equilibrium distribution functions (3.1), then the solution (2.2) in a bounded connected domain $\Lambda \subset \mathbb{R}^1$ takes on the form

$$
\begin{aligned}
F_{\Lambda,|Y|}(t,Y) &= \sum_{n=0}^{\infty} \sum_{\substack{n_1+n_2=n\\ n_1,n_2\geq 0}} \int\limits_{(\Lambda\times\mathbb{R}^1)^{n_1+n_2}} d(X\setminus Y)\,\mathfrak{A}_{\Lambda,(n_2,n_1)}(t,X_Y)F^{eq}_{\Lambda,n+s}(0,X) = \\
&= \big((I-\mathfrak{a}_{(+)})^{-1}(I-\mathfrak{a}_{(-)})^{-1}\Psi_\Lambda\big)_0^{-1} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \int\limits_{(\Lambda\times\mathbb{R}^1)^{n_1+n_2}} d(X\setminus Y)\times \\
&\quad\times \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \int\limits_{(\Lambda\times\mathbb{R}^1)^{m_1+m_2}} d(Z\setminus X)\,\mathfrak{A}_{\Lambda,(n_2,n_1)}(t,X_Y)\Psi_{\Lambda,m+n+s}(0,Z) = \\
&= \big((I-\mathfrak{a}_{(+)})^{-1}(I-\mathfrak{a}_{(-)})^{-1}\Psi_\Lambda\big)_0^{-1}\times \\
&\quad\times \sum_{l_1=0}^{\infty}\sum_{n_1=0}^{l_1}\sum_{l_2=0}^{\infty}\sum_{n_2=0}^{l_2} \int\limits_{(\Lambda\times\mathbb{R}^1)^{l_1+l_2}} d(Z\setminus Y)\,\mathfrak{A}_{\Lambda,(n_2,n_1)}(t,X_Y)\Psi_{\Lambda,l+s}(0,Z),
\end{aligned}
\tag{3.2}
$$

where $Z = (x_{-(l_2+s_2)},\ldots,x_{s_1+l_1})$.

Lemma 3.2. *The equality*

$$
\int\limits_{(\Lambda\times\mathbb{R}^1)^{l_1+l_2}} d(Z\setminus Y)\,\mathfrak{A}_{\Lambda,(n_2,n_1)}(t,X_Y)\Psi_{\Lambda,l+s}(0,Z) = 0, \qquad n_1+n_2=n\geq 1,
$$

$$
n_1\leq l_1,\; n_2\leq l_2,
$$

holds.

Proof. The cumulant $\mathfrak{A}_{\Lambda,(n_2,n_1)}(t,X_Y)$ can be represented as a sum of 2^{n-1} pairs of summands, each of them of the form

$$
\begin{aligned}
&S_{\Lambda,|X_1|}(-t,X_1)S_{\Lambda,|X_2|}(-t,X_2)\ldots S_{\Lambda,|X_{k-1}|+|X_k|}(-t,X_{k-1},X_k)- \\
&\quad - S_{\Lambda,|X_1|}(-t,X_1)S_{\Lambda,|X_2|}(-t,X_2)\ldots S_{\Lambda,|X_{k-1}|}(-t,X_{k-1})S_{\Lambda,|X_k|}(-t,X_k),
\end{aligned}
$$

where $X_k = \{x_{s_1+n_1}\}$.

Let us prove that

$$
\begin{aligned}
\int\limits_{(\Lambda\times\mathbb{R}^1)^{l_1+l_2}} d(Z\setminus Y) \prod_{i=1}^{k-2} S_{\Lambda,|X_i|}(-t,X_i)\,\big(S_{\Lambda,|X_{k-1}|+1}(-t,X_{k-1},x_{s_1+n_1})- \\
-S_{\Lambda,|X_{k-1}|}(-t,X_{k-1})S_{\Lambda,1}(-t,x_{s_1+n_1})\big)\,\Psi_{\Lambda,l+s}(0,Z) = 0, \qquad n\geq 1.
\end{aligned}
$$

Indeed,

$$\int\limits_{(\Lambda\times\mathbb{R}^1)^{l_1+l_2}} d(Z\setminus Y)\prod_{i=1}^{k-2} S_{\Lambda,|X_i|}(-t,X_i)\left(S_{\Lambda,|X_{k-1}|+1}(-t,X_{k-1},x_{s_1+n_1})-\right.$$
$$\left.-S_{\Lambda,|X_{k-1}|}(-t,X_{k-1})S_{\Lambda,1}(-t,x_{s_1+n_1})\right)\Psi_{\Lambda,l+s}(0,Z)=$$
$$=\int\limits_{(\Lambda\times\mathbb{R}^1)^{m_1+m_2}} d(Z\setminus X)\int\limits_{(\Lambda\times\mathbb{R}^1)^{n_1+n_2-|X_{k-1}|-1}} dM_1\prod_{i=1}^{k-2} S_{\Lambda,|X_i|}(-t,X_i)\times$$
$$\times\int\limits_{(\Lambda\times\mathbb{R}^1)^{|X_{k-1}|+1}} dM_2\left(S_{\Lambda,|X_{k-1}|+1}(-t,X_{k-1},x_{s_1+n_1})-\right.$$
$$\left.-S_{\Lambda,|X_{k-1}|}(-t,X_{k-1})S_{\Lambda,1}(-t,x_{s_1+n_1})\right)\Psi_{\Lambda,l+s}(0,Z), \tag{3.3}$$

where

$$M_1=(X_1\cup\cdots\cup X_{k-2})\setminus Y,\quad M_2=X_{k-1}\cup\{x_{s_1+n_1}\}.$$

If $Y\subset X_i$, $i\leq k-2$ then, according to the Liouville theorem [2],

$$\int\limits_{(\Lambda\times\mathbb{R}^1)^{|X_{k-1}|+1}} dM_2\left(S_{\Lambda,|X_{k-1}|+1}(-t,X_{k-1},x_{s_1+n_1})-\right.$$
$$\left.-S_{\Lambda,|X_{k-1}|}(-t,X_{k-1})S_{\Lambda,1}(-t,x_{s_1+n_1})\right)\Psi_{\Lambda,l+s}(0,Z)=0.$$

If $Y\subset X_{k-1}$ then, according to the Liouville theorem,

$$\int\limits_{(\Lambda\times\mathbb{R}^1)^{n_1+n_2-|X_{k-1}|-1}} dM_1\prod_{i=1}^{k-2} S_{\Lambda,|X_i|}(-t,X_i)\times$$
$$\times\int\limits_{(\Lambda\times\mathbb{R}^1)^{|X_{k-1}|+1}} dM_2\left(S_{\Lambda,|X_{k-1}|+1}(-t,X_{k-1},x_{s_1+n_1})-\right.$$
$$\left.-S_{\Lambda,|X_{k-1}|}(-t,X_{k-1})S_{\Lambda,1}(-t,x_{s_1+n_1})\right)\Psi_{\Lambda,l+s}(0,Z)=$$
$$=\int\limits_{(\Lambda\times\mathbb{R}^1)^{n_1+n_2-|X_{k-1}|-1}} dM_1\int\limits_{(\Lambda\times\mathbb{R}^1)^{|X_{k-1}|+1}} dM_2\left(S_{\Lambda,|X_{k-1}|+1}(-t,X_{k-1},x_{s_1+n_1})-\right.$$
$$\left.-S_{\Lambda,|X_{k-1}|}(-t,X_{k-1})S_{\Lambda,1}(-t,x_{s_1+n_1})\right)\Psi_{\Lambda,l+s}(0,Z). \tag{3.4}$$

The following equality is true

$$\int\limits_{(\Lambda\times\mathbb{R}^1)^{l_1+l_2}} d(Z\setminus Y)\mathfrak{A}_{\Lambda,(0,1)}(t,X_Y)\Psi_{\Lambda,l+s}(0,Z) =$$

$$= \int\limits_{(\Lambda\times\mathbb{R}^1)^{l_1+l_2}} d(Z\setminus Y)e^{\beta\Phi(q_{s_1+1}-q_{s_1+2})}\Big(e^{-\beta\Phi\left(Q_{s_1+1}(-t,Y,x_{s_1+1})-q_{s_1+2}\right)}-$$
$$-e^{-\beta\Phi\left(Q_{s_1}(-t,Y)-Q_{s_1+1}(-t,x_{s_1+1})\right)-\beta\Phi\left(Q_{s_1+1}(-t,x_{s_1+1})-q_{s_1+2}\right)}\times$$
$$\times e^{\beta\Phi(q_{s_1}-q_{s_1+1})}\Big)\Psi_{\Lambda,l+s}(0,Z) = 0.$$

Note that an analogous result is valid for arguments with negative numbers. Hence,

$$\int\limits_{\Lambda\times\mathbb{R}^1} dx_{s_1+n_1}\left(S_{\Lambda,|X_{k-1}|+1}(-t,X_{k-1},x_{s_1+n_1})-\right.$$
$$\left.-S_{\Lambda,|X_{k-1}|}(-t,X_{k-1})S_{\Lambda,1}(-t,x_{s_1+n_1})\right)\Psi_{\Lambda,l+s}(0,Z) = 0.$$

Therefore, the expression in (3.4) and then the expression in (3.3) are equal to zero, and the statement of the lemma is true. □

Finally, expression (3.2) takes the form

$$F_{\Lambda,|Y|}(t,Y) = \left((I-\mathfrak{a}_{(+)})^{-1}(I-\mathfrak{a}_{(-)})^{-1}\Psi_\Lambda\right)_0^{-1}\times$$
$$\times\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\int\limits_{(\Lambda\times\mathbb{R}^1)^{l_1+l_2}} d(Z\setminus Y)S_{\Lambda,s}(-t,Y)\Psi_{\Lambda,l+s}(0,Z) =$$
$$= \left((I-\mathfrak{a}_{(+)})^{-1}(I-\mathfrak{a}_{(-)})^{-1}\Psi_\Lambda\right)_0^{-1}\left((I-\mathfrak{a}_{(+)})^{-1}(I-\mathfrak{a}_{(-)})^{-1}\Psi_\Lambda\right)_s = F^{eq}_{\Lambda,s},$$

which proves the theorem. □

4. CONCLUSION

While investigating the BBGKY hierarchy, complicated mathematical problems arise [3, 6–11]. One of the problems for a cumulant representation of the solution of the BBGKY hierarchy of infinite particle system was considered in [14].

In the present paper it is proved that the BBGKY hierarchy solution in a cumulant representation is an equilibrium in the case of equilibrium initial data.

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