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ON AN EVOLUTION INCLUSION IN NON-SEPARABLE BANACH SPACES

Abstract. We consider a Cauchy problem for a class of nonconvex evolution inclusions in non-separable Banach spaces under Filippov-type assumptions. We prove the existence of solutions.

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1. INTRODUCTION

In this paper we study differential inclusions of the form

\[ x'(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0, \tag{1.1} \]

where \( F : [0,T] \times X \to P(X) \) is a set-valued map, Lipschitzian with respect to the second variable, \( X \) is a Banach space, \( A(t) \) is the infinitesimal generator of a strongly continuous evolution system of a two parameter family \( \{G(t,\tau), t \geq 0, \tau \geq 0\} \) of bounded linear operators of \( X \) into \( X \), \( D = \{(t,s) \in [0,T] \times [0,T]; t \geq s\} \), \( K(.,.) : D \to \mathbb{R} \) is continuous and \( x_0 \in X \).

The existence and qualitative properties of mild solutions of problem (1.1) have been obtained in [1,2–7,13] etc.. Most of the existence results mentioned above are obtained using fixed point techniques. In [9] it is shown that Filippov’s ideas ([11]) can suitably be adapted in order to prove the existence of solutions to problem (1.1). All these approaches are have proved successful the Banach space \( X \) separable.

De Blasi and Pianigiani ([10]) established the existence of mild solutions for semilinear differential inclusions on an arbitrary, not necessarily separable, Banach space \( X \). Even if Filippov’s ideas are still present, the approach in [10] is fundamental different: it consists in the construction of the measurable selections of the multifunction.
This construction does not use classical selection theorems such as Kuratowski and Ryb-Nardzewski’s ([12]) or Bressan and Colombo’s ([8]).

The aim of this paper is to obtain an existence result for problem (1.1) similar to the one in [10]. We will prove the existence of solutions for problem (1.1) in an arbitrary space \( X \) under Filippov-type assumptions on \( F \).

The paper is organized as follows: in Section 2 we present the notations, definitions and preliminary results to be used in the sequel, and in Section 3 we prove the main result.

2. PRELIMINARIES

Consider \( X \), an arbitrary real Banach space with norm \( ||.|| \) and with the corresponding metric \( d(.,.) \). Let \( P(X) \) be the space of all bounded nonempty subsets of \( X \) endowed with the Hausdorff pseudometric

\[
d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup_{a \in A} d(a, B),
\]

where \( d(x, A) = \inf_{a \in A} |x - a|, \ A \subset X, x \in X \).

Let \( \mathcal{L} \) be the \( \sigma \)-algebra of the (Lebesgue) measurable subsets of \( R \) and, for \( A \in \mathcal{L} \), let \( \mu(A) \) be the Lebesgue measure of \( A \).

Let \( X \) be a Banach space and \( Y \) be a metric space. An open (resp., closed) ball in \( Y \) with center \( y \) and radius \( r \) is denoted by \( B_Y(y, r) \) (resp., \( \overline{B}_Y(y, r) \)). In what follows, \( B = B_X(0, 1) \).

A multifunction \( F : Y \to P(X) \) with closed bounded nonempty values is said to be \( d_H \)-continuous at \( y_0 \in Y \) if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( y \in B_Y(y_0, r) \) there is \( d_H(F(y), F(y_0)) \leq \varepsilon \). \( F \) is called \( d_H \)-continuous if it is so at each point \( y_0 \in Y \).

Let \( A \in \mathcal{L} \), with \( \mu(A) < \infty \). A multifunction \( F : Y \to P(X) \) with closed bounded nonempty values is said to be \( \text{Lusin measurable} \) if for every \( \varepsilon > 0 \) there exists a compact set \( K_\varepsilon \subset A \), with \( \mu(A \setminus K_\varepsilon) < \varepsilon \) such that \( F \) restricted to \( K_\varepsilon \) is \( d_H \)-continuous.

It is clear that if \( F, G : A \to P(X) \) and \( f : A \to X \) are Lusin measurable, then so are \( F \) restricted to \( B \) (\( B \subset A \) measurable), \( F + G \) and \( t \to d(f(t), F(t)) \). Moreover, the uniform limit of a sequence of Lusin measurable multifunctions is Lusin measurable, too.

Let \( I \) stand for the interval \([0, T]\), \( T > 0 \).

In what follows, \( \{A(t); t \in I\} \) is the infinitesimal generator of a strongly continuous evolution system \( G(t, s), 0 \leq s \leq t \leq T \).

Recall that a family of bounded linear operators \( G(t, s) \) on \( X \), \( 0 \leq s \leq t \leq T \) depending on two parameters is said to be a strongly continuous evolution system if the following conditions hold: \( G(s, s) = I \), \( G(t, r)G(r, s) = G(t, s) \) for \( 0 \leq s \leq r \leq t \leq T \) and \( (t, s) \to G(t, s) \) is strongly continuous for \( 0 \leq s \leq t \leq T \), i.e., \( \lim_{t \to s, t > s} G(t, s)x = x \) for all \( x \in X \).
In what follows, we are concerned with the evolution inclusion
\[ x'(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0, \tag{2.1} \]

where \( F : I \times X \to \mathcal{P}(X) \) is a set-valued map, \( X \) is a Banach space, \( A(t) \) is the infinitesimal generator of a strongly continuous evolution system of a two parameter family \( \{ G(t,\tau), t \geq 0, \tau \geq 0 \} \) of bounded linear operators of \( X \) into \( X \), \( D = \{(t,s) \in I \times I; t \geq s \} \), \( K(\cdot,\cdot) : D \to \mathbb{R} \) is continuous and \( x_0 \in X \).

A continuous mapping \( x(\cdot) \in C(I,X) \) is called a mild solution of problem (2.1) if there exists a (Bochner) integrable function \( f(\cdot) \in L^1(I,X) \) such that
\[ f(t) \in F(t,x(t)) \quad a.e. \ (I), \tag{2.2} \]
\[ x(t) = G(t,0)x_0 + \int_0^t G(t,\tau) \int_0^\tau K(\tau,s)f(s)dsd\tau, \quad t \in I. \tag{2.3} \]

In this case, we shall call \((x(\cdot),f(\cdot))\) a trajectory-selection pair of (2.1). We note that condition (2.3) can be rewritten as
\[ x(t) = G(t,0)x_0 + \int_0^t U(t,s)f(s)ds, \quad t \in I, \tag{2.4} \]

where \( U(t,s) = \int_s^t G(t,\tau)K(\tau,s)d\tau \).

In what follows, we assume the following hypotheses.

**Hypothesis 2.1.**

(i) \( \{A(t); t \in I\} \) is the infinitesimal generator of the strongly continuous evolution system \( G(t,s), 0 \leq s \leq t \leq T \).

(ii) \( F(\cdot,\cdot) : I \times X \to \mathcal{P}(X) \) has nonempty closed bounded values and, for any \( x \in X \), \( F(\cdot,x) \) is Lusin measurable on \( I \).

(iii) There exists \( l(\cdot) \in L^1(I,(0,\infty)) \) such that for each \( t \in I \):
\[ d_H(F(t,x_1),F(t,x_2)) \leq l(t)|x_1 - x_2|, \quad \forall x_1, x_2 \in X. \]

(iv) There exists \( q(\cdot) \in L^1(I,(0,\infty)) \) such that for each \( t \in I \):
\[ F(t,0) \subset q(t)B. \]

(v) \( D = \{(t,s) \in I \times I; t \geq s \} \), \( K(\cdot,\cdot) : D \to \mathbb{R} \) is continuous.

Set \( n(t) = \int_0^t l(u)du, t \in I, M := \sup_{t,s \in I} |G(t,s)| \) and \( M_0 := \sup_{(t,s) \in D} |K(t,s)| \) and note that \( |U(t,s)| \leq MM_0(t-s) \leq MM_0T. \)

The technical results summarized in the following lemma are essential in the proof of our result. For the proof, we refer the reader to [10].
Lemma 2.2 ([10] i)). Let $F_i : I \to \mathcal{P}(X)$, $i=1,2$, be two Lusin measurable multifunctions and let $\varepsilon_i > 0$, $i=1,2$ be such that

\[ H(t) := (F_1(t) + \varepsilon_1 B) \cap (F_2(t) + \varepsilon_2 B) \neq \emptyset, \quad \forall t \in I. \]

Then the multifunction $H : I \to \mathcal{P}(X)$ has a Lusin measurable selection $h : I \to X$.

ii) Assume that Hypothesis 2.1 is satisfied. Then for any continuous $x(.) : I \to X$, $u(.) : I \to X$ measurable and any $\varepsilon > 0$ there is:

a) the multifunction $t \to F(t,x(t))$ is Lusin measurable on $I$,

b) the multifunction $G : I \to \mathcal{P}(X)$ defined by

\[ G(t) := (F(t,x(t)) + \varepsilon B) \cap B_X(u(t), d(u(t), F(t,x(t))) + \varepsilon) \]

has a Lusin measurable selection $g : I \to X$.

3. THE MAIN RESULT

We are now ready to prove our main result.

Theorem 3.1. We assume that Hypothesis 2.1 is satisfied. Then, for every $x_0 \in X$, Cauchy problem (1.1) has a mild solution $x(.) \in C(I,X)$.

Proof. Let us first note that if $z(.) : I \to X$ is continuous, then every Lusin measurable selection $u : I \to X$ of the multifunction $t \to F(t,z(t)) + B$ is Bochner integrable on $I$. More precisely, for any $t \in I$, there holds

\[ |u(t)| \leq d_H(F(t,z(t)) + B,0) \leq d_H(F(t,z(t)), F(t,0)) + d_H(F(t,0),0) + 1 \leq l(t)|z(t)| + q(t) + 1. \]

Let $0 < \varepsilon < 1$, $\varepsilon_n = \frac{\varepsilon}{n+1}$. Consider $f_0(.) : I \to X$, an arbitrary Lusin measurable, Bochner integrable function, and define

\[ x_0(t) = G(t,0)x_0 + \int_0^t U(t,s)f_0(s)ds, \quad t \in I. \]

Since $x_0(.)$ is continuous, by Lemma 2.2 ii) there exists a Lusin measurable function $f_1(.) : I \to X$ which, for $t \in I$, satisfies

\[ f_1(t) \in (F(t,x_0(t)) + \varepsilon_1 B) \cap B(f_0(t), d(f_0(t), F(t,x_0(t))) + \varepsilon_1). \]

Obviously, $f_1(.)$ is Bochner integrable on $I$. Define $x_1(.) : I \to X$ by

\[ x_1(t) = G(t,0)x_0 + \int_0^t U(t,s)f_1(s)ds, \quad t \in I. \]
By induction, we construct a sequence \( x_n : I \to X \), \( n \geq 2 \) given by
\[
x_n(t) = G(t, 0)x_0 + \int_0^t U(t, s)f_n(s)ds, \quad t \in I,
\] (3.1)
where \( f_n(.) : I \to X \) is a Lusin measurable function which, for \( t \in I \), satisfies:
\[
f_n(t) \in (F(t, x_{n-1}(t)) + \varepsilon_n B) \cap B(f_{n-1}(t), d(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n). \quad (3.2)
\]

At the same time, as we saw at the beginning of the proof, \( f_n(.) \) is also Bochner integrable.

From (3.2), for \( n \geq 2 \) and \( t \in I \), we obtain
\[
|f_n(t) - f_{n-1}(t)| \leq d(f_{n-1}(t), F(t, x_{n-1}(t))) + \varepsilon_n \leq d(f_{n-1}(t), F(t, x_{n-2}(t))) + d_H(F(t, x_{n-2}(t)), F(t, x_{n-1}(t))) + \varepsilon_n \leq \varepsilon_n - l(t)|x_{n-1}(t) - x_{n-2}(t)| + \varepsilon_n.
\]
Since \( \varepsilon_n - l < \varepsilon_n - \varepsilon_n + \varepsilon_n \), for \( n \geq 2 \), we deduce that
\[
|f_n(t) - f_{n-1}(t)| \leq \varepsilon_n - l(t)|x_{n-1}(t) - x_{n-2}(t)|. \quad (3.3)
\]

Denote \( p_0(t) := d(f_0(t), F(t, x_0(t))) \), \( t \in I \). We next prove by recurrence, that for \( n \geq 2 \) and \( t \in I \):
\[
|x_n(t) - x_{n-1}(t)| \leq \sum_{k=0}^{n-2} \int_0^t \varepsilon_{n-2-k} \frac{(MM_0T)^{k+1}(n(t) - n(u))^k}{k!} \, du + \varepsilon_0 \int_0^t \frac{(MM_0T)^n(n(t) - n(u))^{n-1}}{(n-1)!} \, du + \int_0^t (MM_0T)^n(n(t) - n(u))^{n-1} \, du.
\]
(3.4)

We start with \( n = 2 \). In view of (3.1), (3.2) and (3.3), for \( t \in I \), there is
\[
|x_2(t) - x_1(t)| \leq \int_0^t |U(t, s)| \cdot |f_2(s) - f_1(s)| \, ds \leq MM_0T[\varepsilon_0 + l(t)|x_1(t) - x_0(s)|] \, ds \leq
\]
\[
\leq \varepsilon_0 MM_0T \cdot t + \int_0^t [MM_0Tl(s)] \cdot [U(s, r)|f_1(r) - f_0(r)|] \, dr \, ds \leq
\leq \varepsilon_0 MM_0T \cdot t + \int_0^t [(MM_0T)^2l(s)] \cdot (p_0(u) + \varepsilon_1) \, du \, ds \leq
\leq \varepsilon_0 MM_0T \cdot t + \int_0^t [(MM_0T)^2(p_0(u) + \varepsilon_1)] \, l(s) \, ds \, du =
\]
\[
= \varepsilon_0 MM_0T \cdot t + \int_0^t (MM_0T)^2(n(t) - n(s)) \, p_0(u) + \varepsilon_0 \, ds,
\]
i.e., (3.4) is verified for \( n = 2 \).
Using again (3.3) and (3.4), we conclude:

\[ |x_{n+1}(t) - x_n(t)| \leq \int_0^t |U(t, s)| \cdot |f_{n+1}(s) - f_n(s)| ds \leq \]

\[ \leq \int_0^t M M_0 T [\varepsilon_{n-1} + L(s)|x_n(s) - x_{n-1}(s)|] ds \leq \varepsilon_{n-1} M M_0 T + \]

\[ + \int_0^t L(s) \sum_{k=0}^{n-2} \int_0^s \varepsilon_{n-2-k} \frac{(M M_0 T)^{k+2}(n(s) - n(u))^k}{k!} du + \]

\[ + \int_0^s \frac{(M M_0 T)^{n+1}(n(s) - n(u))^{n-1}}{(n-1)!} (p_0(u) + \varepsilon_0) du | ds = \]

\[ = \varepsilon_{n-1} M M_0 T + \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \int_0^t \int_0^s \frac{(M M_0 T)^{k+2}(n(s) - n(u))^k}{k!} l(s) du ds + \]

\[ + \int_0^t L(s) \int_0^s \frac{(M M_0 T)^{n+1}(n(s) - n(u))^{n-1}}{(n-1)!} l(s) |p_0(u) + \varepsilon_0| du ds = \]

\[ = \varepsilon_{n-1} M M_0 T + \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \int_0^t \int_0^u \frac{(M M_0 T)^{k+2}(n(s) - n(u))^k}{k!} l(s) ds du + \]

\[ + \int_0^t (\int_0^u \frac{(M M_0 T)^{n+1}(n(s) - n(u))^{n-1}}{(n-1)!} l(s) ds |p_0(u) + \varepsilon_0| du) = \]

\[ = \varepsilon_{n-1} M M_0 T + \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \int_0^t \frac{(M M_0 T)^{k+2}(n(s) - n(u))^{k+1}}{(k+1)!} du + \]

\[ + \int_0^t \frac{(M M_0 T)^{n+1}(n(s) - n(u))^n}{n!} [p_0(u) + \varepsilon_0] du = \]

\[ = \sum_{k=0}^{n-1} \varepsilon_{n-k} \int_0^t \frac{(M M_0 T)^{k+1}(n(s) - n(u))^k}{k!} du + \]

\[ + \int_0^t \frac{(M M_0 T)^{n+1}(n(s) - n(u))^n}{n!} [p_0(u) + \varepsilon_0] du, \]

and statement (3.4) it is true for \( n + 1 \).
From (3.4) it follows that for $n \geq 2$ and $t \in I$:
\[
|x_n(t) - x_{n-1}(t)| \leq a_n,
\]
where
\[
a_n = \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \frac{(M'M_0T)^{k+1}n(T)^k}{k!} + \frac{(M'M_0T)^n n(T)^{n-1}}{(n-1)!} \int_0^1 p_0(u)du + \varepsilon_0,
\]

Obviously, the series whose $n$-th term is $a_n$ converges. So, from (3.5) we infer that $x_n(.)$ converges to a continuous function, $x(.) : I \to X$, uniformly on $I$.

On the other hand, in view of (3.3) there is
\[
|f_n(t) - f_{n-1}(t)| \leq \varepsilon_{n-2} + l(t)a_{n-1}, \quad t \in I, n \geq 3
\]
which implies that the sequence $f_n(.)$ converges to a Lusin measurable function $f(.) : I \to X$.

Since $x_n(.)$ is bounded and
\[
|f_n(t)| \leq l(t)|x_{n-1}(t)| + q(t) + 1,
\]
we infer that $f(.)$ is also Bochner integrable.

Passing with $n \to \infty$ in (3.1) and using the Lebesgue dominated convergence theorem, we obtain
\[
x(t) = G(t, 0)x_0 + \int_0^t U(t, s)f(s)ds, \quad t \in I.
\]

On the other hand, from (3.2) we get
\[
f_n(t) \in F(t, x_n(t)) + \varepsilon_n B, \quad t \in I, n \geq 1
\]
and letting $n \to \infty$ we obtain
\[
f(t) \in F(t, x(t)), \quad t \in I,
\]
which completes the proof.

**Remark 3.2.** If $A(t) \equiv A$ and $A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{G(t); t \geq 0\}$ from $X$ to $X$, then problem (1.1) reduces to the problem
\[
x'(t) = Ax(t) + \int_0^t K(t, s)F(s, x(s))ds, \quad x(0) = x_0,
\]
well known ([1,2-7,13] etc.) as an integrodifferential inclusion.

Obviously, a result similar to that of Theorem 3.1 may be obtained for problem (3.6).
REFERENCES


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