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BEST APPROXIMATION
IN CHEBYSHEV SUBSPACES OF $L(l_1^n, l_1^n)$

Abstract. Chebyshev subspaces of $L(l_1^n, l_1^n)$ are studied. A construction of a $k$-dimensional Chebyshev (not interpolating) subspace is given.

Keywords: interpolating subspace, Chebyshev subspace, strongly unique best approximation.

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1. INTRODUCTION

Let $\mathbb{K}$ be the field of real or complex numbers and let $(X, \| \cdot \|)$ be a normed space over $\mathbb{K}$. Let $ext S_{X^*}$ denote the set of all extreme points of $S_{X^*}$, where $S_{X^*}$ is the unit sphere in $X^*$.

For any $x \in X$, we put

$$E(x) = \{ f \in ext S_{X^*} : f(x) = \| x \| \}$$ (1)

and with any $Y \subset X$ we associate the set

$$P_Y(x) = \{ y \in Y : \| x - y \| = dist(x, Y) \}.$$

Note that, by the Hahn-Banach and Krein-Milman Theorems, $E(x) \not= \emptyset$.

A linear subspace $Y \subset X$ is called a Chebyshev subspace if for any $x \in X$ the set $P_Y(x)$ contains one element only.

If $Y$ is a linear subspace of $X$, then the following holds

Theorem 1 ([3]). Assume $X$ is a normed space, $Y \subset X$ is its linear subspace, and let $x \in X \setminus Y$. Then $y_0 \in P_Y(x)$ if and only if for every $y \in Y$ there exists $f \in E(x - y_0)$ with $ref(y) \leq 0$.

Let us recall a well-known definition
Definition 1 (see e.g. [7]). An element \(y_0 \in Y\) is called a strongly unique best approximation for \(x \in X\) if and only if there exists \(r > 0\) such that for every \(y \in Y\),
\[
\|x - y\| \geq \|x - y_0\| + r\|y - y_0\|.
\]

The largest constant \(r\) satisfying the above inequality is called the strong unicity constant. There exist two main applications of the strong unicity constant:

1. The error estimate of the Remez algorithm (see e.g. [10]).
2. The Lipschitz continuity of the best approximation mapping at \(x_0\) (assuming that there exists a strongly unique best approximation to \(x_0\)) (see e.g. [5, 8, 9]).

The following holds true:

Theorem 2 ([14]). Let \(x \in X \setminus Y\) and let \(Y\) be a linear subspace of \(X\). Then \(y_0 \in Y\) is a strongly unique best approximation for \(x\) with a constant \(r > 0\) if and only if for every \(y \in Y\) there exists \(f \in E(x - y_0)\) with \(\text{ref}(y) \leq -r\|y\|\).

In this paper, we consider \(X = L(l^n_1, l^n_1)\), \(n > 1\) (the space of all linear and continuous operators from \(l^n_1\) to \(l^n_1\) equipped with the operator norm denoted by \(\| \cdot \|_{op}\)), where
\[
l^n_1 = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \|x\| := \sum_{i=1}^{n} |x_i| \right\}.
\]

It is known [11] that for any operator \(A \in L(l^n_1, l^n_1)\):
\[
\|A\|_{op} = \max_{x \in \text{ext}S^n_1} \|Ax\|.
\]

Since (see [1])
\[
\text{ext}S^n_1 = \{ e_i = (\delta_i, \delta_2, \ldots, \delta_n), i = 1, 2, \ldots, n \},
\]
then for any \(A = [a_{ij}]_{i,j=1,2,\ldots,n} \in L(l^n_1, l^n_1)\), we obtain
\[
\|A\|_{op} = \max \left\{ \sum_{i=1}^{n} |a_{i1}|, \ldots, \sum_{i=1}^{n} |a_{in}| \right\}.
\]

The aim of this paper is to show that, for any \(k \leq n\), there exists a \(k\)-dimensional Chebyshev subspace of \(L(l^n_1, l^n_1)\) which is not an interpolating subspace.

This result is quite different from the result obtained for the space \(L(l^n_1, c_0)\) (see [6]), where any finite dimensional Chebyshev subspace is an interpolating subspace. Additionally, as the space \(L(l^n_1, l^n_1)\) is a finite dimensional space, we get (see [13]) that the unicity of best approximation is equivalent to the strong unicity of best approximation.
2. ONE-DIMENSIONAL CHEBYSHEV SUBSPACES OF $\mathcal{L}(l_1^n, l_1^n)$

Let an operator $A \in \mathcal{L}(l_1^n, l_1^n)$ be represented by a matrix $[a_{ij}]_{i,j=1,2,\ldots,n}$. Since (see [4,12])

$$extS_{\mathcal{L}^*}(l_1^n,l_1^n) = extS_{\mathcal{L}^*} \otimes extS_{\mathcal{L}^*},$$

(2)

where

$$l_1^n = \left\{ x = (x_1,x_2,\ldots,x_n) : \|x\| := \max_{i\in\{1,2,\ldots,n\}} |x_i| \right\},$$

we get

$$extS_{\mathcal{L}^*}(l_1^n,l_1^n) = \{ x \otimes e_j : j = 1,2,\ldots,n \},$$

where $e_j = (\delta_{1j}, \delta_{2j},\ldots,\delta_{nj})$, $j = 1,2,\ldots,n$, and $x = (x_1,x_2,\ldots,x_n)$, $x_i \in \{-1,1\}$, $i = 1,2,\ldots,n$. So, for any operator $A \in \mathcal{L}(l_1^n,l_1^n)$ represented by a matrix $[a_{ij}]_{i,j=1,2,\ldots,n}$, there is

$$(x \otimes e_j)(A) = \sum_{i=1}^{n} x_ia_{ij}.$$ 

Since $E(A) \neq \emptyset$, then by (1) and (2), we get that there exist $x \in l_{1\infty}^n$ and $j \in \{1,2,\ldots,n\}$ such that

$$\|A\|_{\text{op}} = (x \otimes e_j)(A).$$

Let us recall [2] that a $k$-dimensional subspace $\mathcal{V}$ of the normed space $X$ is called an interpolating subspace if and only if for any linearly independent $f_1,f_2,\ldots,f_k \in extS_{X^*}$ and for any $v \in \mathcal{V}$, the following holds: if $f_i(v) = 0$, $i = 1,2,\ldots,k$, then $v = 0$. It is known [2] that any finite dimensional interpolating subspace is a finite dimensional Chebyshev subspace.

**Theorem 3.** Let $\mathcal{V} \subset \mathcal{L}(l_1^n,l_1^n)$ be a $k$-dimensional ($k < n^2$) subspace such that $\mathcal{V} = \text{lin}\{V_1,V_2,\ldots,V_k\}$, $V_m \in \mathcal{L}(l_1^n,l_1^n)$, $m = 1,2,\ldots,k$ and $V_1,V_2,\ldots,V_k$ are linearly independent. For any $m \in \{1,2,\ldots,k\}$, let the operator $V_m$ be represented by the matrix $[(v_{mj})_{j=1,2,\ldots,n}]_{i,j=1,2,\ldots,n}$. Then $\mathcal{V}$ is an interpolating subspace if and only if

$$\begin{vmatrix}
(x^{j_1} \otimes e_{j_1})(V_1) & \cdots & (x^{j_1} \otimes e_{j_1})(V_k) \\
\vdots & \ddots & \vdots \\
(x^{j_k} \otimes e_{j_k})(V_1) & \cdots & (x^{j_k} \otimes e_{j_k})(V_k)
\end{vmatrix} \neq 0,$$

where $(x^{j_l} \otimes e_{j_l})(x^{j_r} \otimes e_{j_r}) \in extS_{\mathcal{L}^*}(l_1^n,l_1^n)$ are linearly independent for $l \neq r$, $l,r \in \{1,2,\ldots,k\}$.

**Proof.** This is a consequence of (2), the definition of a $k$-dimensional interpolating subspace and the theory of linear equations.

**Example 1.** Let $\mathcal{V} \in \mathcal{L}(l_1^n,l_1^n)$ be represented by a matrix $[v_{ij}]_{i,j=1,2,\ldots,n}$, where $v_{1j} = j$, $v_{ij} = 0$, $i = 2,\ldots,n$, $j = 1,2,\ldots,n$. Then $\mathcal{V} = \text{lin}\{V\}$ is a one-dimensional interpolating subspace.
Theorem 4. Let \( \mathcal{V} = \text{lin}\{V\}, \ V \in \mathcal{L}(l_1^n, l_1^n), \) \( n > 1, \ V \neq 0, \ V = [v_{ij}]_{i,j=1,2,\ldots,n}. \) \( \mathcal{V} \) is a Chebyshev subspace of \( \mathcal{L}(l_1^n, l_1^n) \) if and only if \( \mathcal{V} \) is an interpolating subspace.

Proof. Let us assume that \( \mathcal{V} \) is not an interpolating subspace. Hence, there exists \( f = x^{jo} \odot e_{jo}, \) \( j_0 \in \{1, 2, \ldots, n\}, \ x^{jo} = ((x^{jo})_1, (x^{jo})_2, \ldots, (x^{jo})_n), \ (x^{jo})_i \in \{-1, 1\}, \) \( i = 1, 2, \ldots, n, \) such that \( f(V) = 0. \) Let us define \( A = [a_{ij}]_{i,j=1,2,\ldots,n} \) as follows:

\[
a_{ij} = -(x^{jo})_i, \quad a_{ij} = 0, \quad j \neq j_0, \ j \in \{1, 2, \ldots, n\}, \ i = 1, 2, \ldots, n.
\]

Note that \( \|A\| = n. \) Let us consider an operator \( A - \alpha V, \) where \( \alpha \in \mathbb{R}. \) For small enough \( \alpha \) we get \( \|A - \alpha V\| = \|A\|. \) The proof is complete. \( \square \)

3. \( k\)-DIMENSIONAL CHEBYSHEV SUBSPACES OF \( \mathcal{L}(l_1^n, l_1^n) \)

Theorem 5. Let \( \mathcal{V} = \text{lin}\{V_1, V_2, \ldots, V_k\} \subset \mathcal{L}(l_1^n, l_1^n), \) \( k < n^2, \ V_m \in \mathcal{L}(l_1^n, l_1^n) \) (where \( V_m \) are linearly independent for \( m = 1, 2, \ldots, k \)) be a \( k\)-dimensional subspace of \( \mathcal{L}(l_1^n, l_1^n). \) Let \( V_m, \ m \in \{1, 2, \ldots, k\} \) be represented by a matrix \( [(v_m)_{ij}]_{i,j=1,2,\ldots,n}. \) If \( \mathcal{V} \) is a Chebyshev subspace, then vectors \( w_1, w_2, \ldots, w_h \) where

\[
\begin{align*}
  w_1 &= (f_1(V_1), \ldots, f_1(V_k)), \\
  w_2 &= (f_2(V_1), \ldots, f_2(V_k)), \\
  &\vdots \\
  w_h &= (f_h(V_1), \ldots, f_h(V_k))
\end{align*}
\]

are linearly independent for any \( f_1, \ldots, f_h \in \text{ext} S_{\mathcal{L}(l_1^n, l_1^n)} \) such that \( f_m = x^{j_m} \odot e_{j_m}, \) \( m = 1, 2, \ldots, h, \ j_m \neq j_r \) for \( m \neq r, \) where \( h = k \) if \( \dim \mathcal{V} = k \leq n, \ h = n \) if \( n < \dim \mathcal{V} = k < n^2. \)

Proof. Let us assume that (3) does not hold. From this assumption there follows that there exist \( f_1, \ldots, f_h \in \text{ext} S_{\mathcal{L}(l_1^n, l_1^n)} \) such that \( f_m = x^{j_m} \odot e_{j_m}, \) \( m = 1, 2, \ldots, h, \) where \( j_m \neq j_r \) for \( m \neq r \) and \( w_1, w_2, \ldots, w_h \) are linearly dependent. Hence, there exists \( l \in \{1, 2, \ldots, h\} \) and there exist \( \gamma_p \in \mathbb{R}, \ p \in \{1, 2, \ldots, h\}, \) such that

\[
(f_1(V_1), \ldots, f_l(V_k)) = \sum_{p \in \{1, 2, \ldots, h\}, \ p \neq l} \gamma_p(f_p(V_1), \ldots, f_p(V_k)).
\]

From (4) we obtain:

\[
f_l(V_m) = \sum_{p \in \{1, 2, \ldots, h\}, \ p \neq l} \gamma_p f_p(V_m), \ m = 1, 2, \ldots, k.
\]

We shall construct an operator \( A \in \mathcal{L}(l_1^n, l_1^n) \) which has more than one best approximation in \( \mathcal{V}. \) Let \( [a_{ij}]_{i,j=1,2,\ldots,n} \) be a matrix representation for \( A. \) If in (4), \( \gamma_p < 0 \) for some \( p \in \{1, 2, \ldots, h\}, \) \( p \neq l, \) we put

\[
a_{ijp} = (x^{j_p})_i, \quad i = 1, 2, \ldots, n.
\]
If in (4), \( \gamma_p > 0 \) for some \( p \in \{1, 2, \ldots, h\} \), \( p \neq l \), we put
\[
a_{ijp} = -(x^j)_i, \quad i = 1, 2, \ldots, n.
\]
Additionally, we put
\[
a_{ij} = (x^j)_i, \quad i = 1, 2, \ldots, n.
\]
If \( j \neq j_p, \ j \in \{1, 2, \ldots, n\} \), we put \( a_{ij} = 0 \), \( i = 1, 2, \ldots, n \). Let us consider the operator
\[
A(\alpha_1, \alpha_2, \ldots, \alpha_k) := A - (\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_k V_k), \quad \text{where} \quad \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}.
\]
For \( (\alpha_1, \alpha_2, \ldots, \alpha_k) = (0, 0, \ldots, 0) \) there is \( \|A(0, 0, \ldots, 0)\| = \|A\| \). As we have assumed that (3) does not hold, we conclude that there exists \( (\alpha_1, \alpha_2, \ldots, \alpha_k) \neq (0, 0, \ldots, 0) \) such that
\[
\|A(\alpha_1, \alpha_2, \ldots, \alpha_k)\| = \|A\|.
\]
For \( \alpha_i \) small enough for \( i = 1, 2, \ldots, k \), the norm of the operator \( A(\alpha_1, \alpha_2, \ldots, \alpha_k) \) is equal to the largest of the following values:
\[
\|A\| - [\alpha_1 f_p(V_1) + \alpha_2 f_p(V_2) + \ldots + \alpha_k f_p(V_k)],
\]
for some \( p \in \{1, 2, \ldots, h\} \), \( p \neq l \) for which \( \gamma_p < 0 \);
\[
\|A\| + [\alpha_1 f_p(V_1) + \alpha_2 f_p(V_2) + \ldots + \alpha_k f_p(V_k)],
\]
for some \( p \in \{1, 2, \ldots, h\} \), \( p \neq l \) for which \( \gamma_p > 0 \); or
\[
\|A\| - [\alpha_1 f_p(V_1) + \alpha_2 f_p(V_2) + \ldots + \alpha_k f_p(V_k)] =
\|A\| - \left[ \sum_{p \in \{1, 2, \ldots, h\}, \ p \neq l} \gamma_p (\alpha_1 f_p(V_1) + \ldots + \alpha_k f_p(V_k)) \right]. \tag{5}
\]
From the above, if for some \( \alpha_1^0, \alpha_2^0, \ldots, \alpha_k^0 \) we want to obtain
\[
\|A(\alpha_1^0, \alpha_2^0, \ldots, \alpha_k^0)\| < \|A\|,
\]
we need the inequality
\[
[\alpha_1 f_p(V_1) + \alpha_2 f_p(V_2) + \ldots + \alpha_k f_p(V_k)] > 0
\]
to hold for \( p \in \{1, 2, \ldots, h\} \), \( p \neq l \) for which \( \gamma_p < 0 \), and
\[
[\alpha_1 f_p(V_1) + \alpha_2 f_p(V_2) + \ldots + \alpha_k f_p(V_k)] < 0
\]
for \( p \in \{1, 2, \ldots, h\} \), \( p \neq l \) for which \( \gamma_p > 0 \). But then, by (5), we get
\[
\|A(\alpha_1^0, \alpha_2^0, \ldots, \alpha_k^0)\| > \|A\|.
\]
\( \square \)
Note that (3) is satisfied for any \( k \)-dimensional interpolating subspace. But the condition presented in Theorem 5 is not sufficient for a \( k \)-dimensional subspace \( (k \geq 2) \) to be Chebyshev.

**Example 2.** Let \( V = \text{lin}\{V_1, V_2\} \), where
\[
V_1 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}.
\]
Note that (3) is satisfied for \( V = \text{lin}\{V_1, V_2\} \). Let
\[
A = \begin{bmatrix} 0 & 0 \\ 100 & 0 \end{bmatrix}.
\]
Then \( \|A\| = 100 \). Let
\[
A(\alpha_1, \alpha_2) := A - (\alpha_1 V_1 + \alpha_2 V_2) = \begin{bmatrix} -\alpha_1 - 3\alpha_2 & -2\alpha_1 - \alpha_2 \\ 100 & 0 \end{bmatrix}.
\]

Hence, for \( (\alpha_1, \alpha_2) = (0, 0) \), we get
\[
\|A(\alpha_1, \alpha_2)\| = \|A\| = 100 = \inf_{\alpha_1, \alpha_2 \in \mathbb{R}} \|A(\alpha_1, \alpha_2)\|.
\]

But for \( (\alpha_1, \alpha_2) = (3, -1) \) we get \( \|A(\alpha_1, \alpha_2)\| = \|A\| = 100 \).

Now we shall construct a \( k \)-dimensional Chebyshev subspace of \( L(l^n_1, l^n_1) \) which is not an interpolating subspace.

**Theorem 6.** Let \( V_1, V_2, \ldots, V_k \in L(l^n_1, l^n_1) \), \( k \leq n \), \( n > 1 \) be linearly independent and let \( V_m \), \( m \in \{1, 2, \ldots, k\} \) be represented by a matrix
\[
V_m = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
v_{m1} & v_{m2} & \ldots & v_{mn} \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0
\end{bmatrix},
\]
where \( v_{mj} \neq 0 \) for any \( j = 1, 2, \ldots, n \), \( m = 1, 2, \ldots, k \). \( \mathcal{V}(m_1, \ldots, m_r) := \text{lin}\{V_{m_1}, \ldots, V_{m_r}\} \), \( m_1, \ldots, m_r \in \{1, 2, \ldots, k\} \), \( m_p \neq m_q \), \( p \neq q \) is an \( r \)-dimensional Chebyshev subspace of \( L(l^n_1, l^n_1) \) for any \( 1 \leq r \leq k \) if and only if
\[
det((x^m)_{l=1,2,\ldots,r} v_{m,j_l})_{m=1,2,\ldots,r, \ l=1,2,\ldots,r} \neq 0. \tag{6}
\]
Proof. Let us assume that (6) holds. If \( r = 1 \), then \( \mathcal{V}(m_1) = \text{lin}\{V_{m_1}\}, m_1 \in \{1,2,\ldots,k\} \) is a Chebyshev subspace, because it is an interpolating subspace. Let us now assume that for \( 1 < r < k \) the space
\[
\mathcal{V}_r := \mathcal{V}(m_1, \ldots, m_r) = \text{lin}\{V_{m_1}, \ldots, V_{m_r}\},
\]
\[
m_1, \ldots, m_r \in \{1,2,\ldots,k\}, \; m_p \neq m_q, \; p \neq q
\]
is a Chebyshev subspace of \( \mathcal{L}(l_1^n, l_1^n) \) and let
\[
\mathcal{V}_{r+1} := \mathcal{V}(m_1, \ldots, m_r, m_{r+1}) = \text{lin}\{V_{m_1}, \ldots, V_{m_r}, V_{m_{r+1}}\},
\]
\[
m_1, \ldots, m_r, m_{r+1} \in \{1,2,\ldots,k\}, \; m_{r+1} \in \{1,2,\ldots,k\} \setminus \{m_1, \ldots, m_r\}
\]
be not a Chebyshev subspace. From this we conclude that there exists an operator \( A \in \mathcal{L}(l_1^n, l_1^n) \) such that \( \mathcal{P}_{\mathcal{V}_{r+1}}(A) > 1 \). We can assume that 0, \( W \in \mathcal{P}_{\mathcal{V}_{r+1}}(A) \), where \( W \neq 0 \). Let \( \mathcal{U} := \{j \in \{1,2,\ldots,n\} : \|A \circ e_j\| = \|A\|\} \), where \( e_j = (\delta_{ij}, \ldots, \delta_{nj}) \). For any \( j \in \mathcal{U} \), we put
\[
E_j := \{x = (x_1, x_2, \ldots, x_n) : x_i \in \{-1,1\}, \; i = 1,2,\ldots,n : (x \circ A)j = \|A\|\}.
\]
Since 0, \( W \in \mathcal{P}_{\mathcal{V}_{r+1}}(A) \), we conclude that for \( j \in \mathcal{U} \) and \( x \in E_j \) the following holds
\[
(x \otimes e_j)(W) \geq 0. \tag{7}
\]
Let
\[
\mathcal{U}_1 := \{j \in \mathcal{U} : \exists x \in E_j : (x \otimes e_j)(W) = 0\}.
\]
Since 0 \( \in \mathcal{P}_{\mathcal{V}_{r+1}}(A) \), then \( \mathcal{U}_1 \neq \emptyset \). Now we shall show that
\[
\forall j \in \mathcal{U}_1 \; \exists! x \in E_j : (x \otimes e_j)(W) = 0. \tag{8}
\]
Let us assume that (8) does not hold. Let \( x \neq y, \; x, y \in E_j \) be such that
\[
(x \otimes e_j)(W) = 0, \; (y \otimes e_j)(W) = 0.
\]
Without loss of generality, we may assume that
\[
x_i = y_i, \; i = 1,2,\ldots,p, \; p < r+1, \; x_i = -y_i, \; i = p+1, p+2,\ldots,r+1.
\]
Then we get
\[
\sum_{i=1}^{p} x_i(w)_{ij} = 0, \; \sum_{i=p+1}^{r+1} x_i(w)_{ij} = 0. \tag{9}
\]
Since \( x_i = -y_i, \; i = p+1, p+2,\ldots,r+1 \), we obtain \( a_{ij} = 0, \; i = p+1, \ldots, r+1 \). By (7) there follows:
\[
\sum_{i=p+1}^{r} x_i(w)_{ij} - x_{r+1}(w)_{r+1j} \geq 0,
\]
\[
\sum_{i=p+1}^{r} -x_i(w)_{ij} + x_{r+1}(w)_{r+1j} \geq 0,
\]
and then

\[ \sum_{i=p+1}^{r} x_i(w)_{ij} = x_{r+1}(w)_{r+1j}. \]

Applying (9) we conclude that \( x_{r+1}(w)_{r+1j} = 0 \) and then \( (w)_{r+1j} = 0 \). Hence \( W \in V_r := \text{lin}\{V_{m_1}, \ldots, V_{m_r}\} \). Additionally, 0 \( \in V_r := \text{lin}\{V_{m_1}, \ldots, V_{m_r}\} \). But \( V_r \) is a Chebyshev subspace and hence (8) is proved. We shall show that there exists \( \alpha_0 > 0 \) such that for any \( 0 < \alpha \leq \alpha_0 \) the following holds:

\[ E(A - \alpha W) = \{ x \otimes e_j : j \in \mathcal{U}_1, (x \otimes e_j)(W) = 0, (x \otimes e_j)(A) = \|A\| \}. \]  

(10)

Let \( f \notin E(A) \). Then there exist \( \alpha_0 > 0, b > 0 \) such that for any \( 0 < \alpha \leq \alpha_0 \) the following holds:

\[ f(A - \alpha W) \leq b < \|A\| \leq \|A - \alpha W\|. \]

Let \( f \in E(A - \alpha W), \ f(A) = \|A\|. \) If \( f(W) > 0 \), we get

\[ \|A - \alpha W\| = f(A - \alpha W) = \|A\| - \alpha f(W) < \|A\|. \]

From the above we conclude that if \( f \in E(A - \alpha W) \), then \( f \in E(A), \ f(W) = 0 \). Since

\[ \|A - \alpha W\| = \|A\| = \text{dist}(A, V_{r+1}), \]

(10) is proved. Since \( \alpha W \in \mathcal{P}_{V_{r+1}}(A) \) we obtain (see [13]):

\[ \exists 1 \leq q \leq r + 2, \ \exists \lambda_1, \ldots, \lambda_q > 0, \ \sum_{i=1}^{q} \lambda_i = 1, \]

such that

\[ \sum_{i=1}^{q} \lambda_i (x^{j_i} \otimes e_{j_i})|_{V_{r+1}} = 0, \]

(11)

and \( (x^{j_i} \otimes e_{j_i})(A - \alpha W) = \|A - \alpha W\|. \) By (8) we get \( j_i \neq j_l, \ i \neq l, \ i, l \in \{1, 2, \ldots, q\} \). Let us take the least \( q \) such that \( 1 \leq q \leq r + 2 \) and (11) is satisfied. If \( q = r + 2 \), then (see [15]) we get that \( \alpha W \) is a strongly unique best approximation for \( A \) in \( V_{r+1} \). If \( 1 \leq q \leq r + 1 \), we have a contradiction with (6).

Let us assume that \( V_r = \text{lin}\{V_{m_1}, \ldots, V_{m_r}\}, m_1, \ldots, m_r \in \{1, 2, \ldots, k\}, m_p \neq m_q, \ p \neq q \) is a Chebyshev subspace of \( L(l_1^n, l_1^n) \) for any \( 1 \leq r \leq k \) and let (6) does not hold. Hence, there exist

\[ 1 \leq r \leq k, \ i_1, \ldots, i_r, \ j_1, \ldots, j_r \in \{1, 2, \ldots, n\}, \ x^1, x^2, \ldots, x^r \in \{-1, 1\}^n \]

such that

\[ \det([x^m]_{m=1}^{l} |_{i_1, j_1})_{m=1, 2, \ldots, r, l=1, 2, \ldots, r} = 0. \]

From this we conclude that there exist

\[ \lambda_1, \ldots, \lambda_r \in \mathbb{R}, \ \sum_{l=1}^{r} | \lambda_l | > 0. \]
such that
\[ \sum_{i=1}^{r} \lambda_i (x^i \otimes e_{j_i})|_{\mathcal{V}_r} = 0. \] (12)

Without loss of generality, we may assume that \( \lambda_i > 0 \), \( l = 1, 2, \ldots, r \). Let us now define an operator \( B = [b_{ij}]_{i,j=1,2,\ldots,n} \) as follows:
\[ b_{ij} = \text{sgn}(x^i)_1, \quad b_{ij} = 0, \quad j \neq j_i, \quad l \in \{1, 2, \ldots, r\}, \quad i \in \{1, 2, \ldots, n\}. \]
Then \( (x^i \otimes e_{j_i})(B) = \|B\|, \quad l = 1, 2, \ldots, r \). By (12), there follows that \( 0 \in \mathcal{P}_{\mathcal{V}_r}(B) \) and
\[ \dim \text{span}\{x^i \otimes e_{j_i}|_{\mathcal{V}_r}\} < r, \]
where \( \dim \mathcal{V}_r = r \). It means that there exists \( V \in \mathcal{V}_r \setminus \{0\} \) such that
\[ (x^i \otimes e_{j_i})(V) = 0, \quad l = 1, 2, \ldots, r. \]
Note that if \( f \notin E(B) \), then there exist \( \alpha_0 > 0 \), \( b > 0 \) such that for any \( \alpha \in (0, \alpha_0) \):
\[ f(B - \alpha V) < \|B - \alpha V\|. \]

From the above we conclude that \( \|B - \alpha V\| = \|B\| \).

**Corollary 1.** Let \( \mathcal{V} \subset \mathcal{L}(l^1_1, l^1_n) \) be a \( k \)-dimensional subspace from Theorem 6. Any operator \( A \in \mathcal{L}(l^1_n, l^1_n) \) has a unique best approximation in \( \mathcal{V} \) if and only if \( A \) has a strongly unique best approximation in \( \mathcal{V} \).

**Proof.** It is a consequence of [13] and the fact that \( \mathcal{L}(l^1_n, l^1_n) \) is a finite dimensional space.

**Example 3.** We shall construct a \( k \)-dimensional Chebyshev subspace \( \mathcal{V} \subset \mathcal{L}(l^1_1, l^1_n) \), \( k \leq n \). The construction is as follows. Let \( 0 < t_1 < t_2 < \ldots < t_{k-1} \). We put \( V_m = [(v_m)_{ij}]_{i,j=1,2,\ldots,n} , \quad m = 1, 2, \ldots, k - 1 \) as follows:
\[ (v_m)_{mj} = t^m_j, \quad j = 1, 2, \ldots, n, \]
\[ (v_m)_{ij} = 0, \quad i \neq m, \quad j = 1, 2, \ldots, n. \]
Let us assume that the subspace \( \mathcal{V}_{k-1} := \text{lin}\{V_1, V_2, \ldots, V_{k-1}\} \) satisfies formula (6) for any \( 1 \leq r \leq k - 1 \). We shall construct an operator \( V_k \in \mathcal{L}(l^1_n, l^1_n) \) such that \( V_k := \text{lin}\{V_1, V_2, \ldots, V_{k-1}, V_k\} \) satisfies formula (6) for any \( 1 \leq r \leq k \), which means that \( \mathcal{V}_k := \text{lin}\{V_1, V_2, \ldots, V_{k-1}, V_k\} \) is a Chebyshev subspace of \( \mathcal{L}(l^1_1, l^1_n) \). We are looking for such \( x \in \mathbb{R} \) that, for any \( r \in \{1, 2, \ldots, k\} \), there holds:
\[ W(x, y^1, \ldots, y^r, j_1, \ldots, j_r, m_1, \ldots, m_{r-1}) := \begin{vmatrix} y^1_{j_1m_1} & \cdots & y^1_{j_1m_1} \\ \vdots & \ddots & \vdots \\ y^r_{j_rm_{r-1}} & \cdots & y^r_{j_rm_{r-1}} \\ y^r_{r,x^{j_1}} & \cdots & y^r_{r,x^{j_r}} \end{vmatrix} \neq 0, \]
(13)
for any $j_1, j_2, \ldots, j_r \in \{1, 2, \ldots, n\}$, $y_1, \ldots, y_r \in \{-1, 1\}^r$, $m_1, m_2, \ldots, m_{r-1} \in \{1, 2, \ldots, k-1\}$.

By the assumption, $W(x, y_1, \ldots, y_r, j_1, \ldots, j_r, m_1, \ldots, m_{r-1})$ is not identically equal to zero. Hence, the set of roots of $W(x, y_1, \ldots, y_r, j_1, \ldots, j_r, m_1, \ldots, m_{r-1})$ is finite for arbitrary fixed $y_1, \ldots, y_r, j_1, \ldots, j_r, m_1, \ldots, m_{r-1}$. Hence, the set of roots of $W(x, y_1, \ldots, y_r, j_1, \ldots, j_r, m_1, \ldots, m_{r-1})$, $y_1, \ldots, y_r, j_1, \ldots, j_r, m_1, \ldots, m_{r-1}$ is countable. But $\mathbb{R}$ is not countable, so there exists $x \in \mathbb{R}$ which satisfies (13).

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Best approximation in Chebyshev subspaces of $L(l^n_1, l^n_1)$


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