Opuscula Mathematica • Vol. 29 • No. 1 • 2009

http://dx.doi.org/10.7494/OpMath.2009.29.1.27

Jolanta Golenia, Yarema A. Prykarpatsky, Eugeniusz Wachnicki

THE CARTAN-MONGE GEOMETRIC APPROACH TO THE GENERALIZED CHARACTERISTICS METHOD AND ITS APPLICATION TO THE HEAT EQUATION $u_t - u_{xx} = 0$

Abstract. The generalized Cartan-Monge type approach to the characteristics method is discussed from the geometric point of view. Its application to the classical one-dimensional linear heat equation $u_t - u_{xx} = 0$ is presented. It is shown that the corresponding exact solution of the Cauchy problem can be represented in a classical functional-analytic Gauss type form.

Keywords: characteristics method, heat equation.

Mathematics Subject Classification: 35A30, 35K05.

1. INTRODUCTION. BACKGROUNDS OF THE GEOMETRIC CARTAN-MONGE APPROACH TO THE CHARACTERISTICS METHOD

Basic principles of the characteristics method [12, 13, 16, 17] were proposed in the 19th century by A. Cauchy. Later it was substantially developed by G. Monge, who introduced the geometric notion of characteristic surface, related to partial differential equations of the first order. The notion of characteristic surface together with related characteristic fields appeared to be fundamental [3,6,7,10,11,16] for the characteristics method, whose main essence consists in bringing the problem of studying solutions of our partial differential equation to the equivalent one of studying some set of ordinary differential equations. This way of reasoning later succeeded in the development of the Hamilton-Jacobi theory, making it possible to describe a wide class of solutions of partial differential equations of the first order in the form

$$H(x; u, u_x) = 0, \tag{1.1}$$

where $H \in C^2(\mathbb{R}^{n+1} \times \mathbb{R}^n; \mathbb{R})$, $|H_x| \neq 0$, $|\cdot|$ is the standard norm in $\mathbb{R}^n$, called a Hamiltonian function and $u \in C^2(\mathbb{R}^n; \mathbb{R})$ is an unknown function under analysis. The Hamilton-Jacobi equation (1.1) is coupled with a boundary value condition

$$u|_{\Gamma_\varphi} = u_0, \tag{1.2}$$

with $u_0 \in C^1(\Gamma_\varphi; \mathbb{R})$, defined on some smooth (almost everywhere) hypersurface

$$\Gamma_\varphi := \{x \in \mathbb{R}^n : \varphi(x) = 0, \quad |\varphi_x| \neq 0\}, \tag{1.3}$$

where $\varphi \in C^1(\mathbb{R}^n; \mathbb{R})$ is some smooth function on $\mathbb{R}^n$.

Following G. Monge's ideas, let us introduce the characteristic surface $S_H \subset M$ on the jet-manifold $M := J^{(1)}(\mathbb{R}^n; \mathbb{R}) \simeq \mathbb{R}^{n+1} \times \mathbb{R}^n$ as

$$S_H := \{(x; u, p) \in M : H(x; u, p) = 0\}, \tag{1.4}$$

where, by definition, $p := u_x \in \mathbb{R}^n$ for all $x \in \mathbb{R}^n$. Characteristic surface (1.4) was described in detail by G. Monge within his geometric approach based on the so-called G. Monge cones $K \subset T(\mathbb{R}^{n+1})$ of hypersurfaces satisfying condition (1.4) and their duals $K^* \subset T^*(\mathbb{R}^{n+1})$ [4,7,16].

The corresponding differential-geometric analysis of G. Monge's scenario was later conducted by E. Cartan, who reformulated [1,4,14,16] the geometric picture drown by G. Monge by means of the related compatibility conditions for dual G. Monge cones and the notion of an integral submanifold $\Sigma_H \subset S_H$, naturally assigned to special vector fields on the characteristic surface S_H. In particular, E. Cartan introduced the differential 1-form on S_H

$$\alpha^{(1)} := du - \langle p, dx \rangle, \tag{1.5}$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$, and demanded its vanishing along the dual G. Monge cones $K^* \subset T^*(\mathbb{R}^{n+1})$, concerning the corresponding integral submanifold embedding mapping

$$\pi : \Sigma_H \to S_H. \tag{1.6}$$

This means that for the 1-form $\pi^* \alpha_1^{(1)}$:

$$\pi^* \alpha_1^{(1)} := du - \langle p, dx \rangle |_{\Sigma_H} = 0 \tag{1.7}$$

for all points $(x; u, p) \in \Sigma_H \subset M$ of a solution surface Σ_H, defined in such a way that $K^* = T^*(\Sigma_H)$. An obvious corollary from condition (1.7) is the second Cartan condition:

$$d\pi^* \alpha_1^{(1)} = \pi^* d\alpha_1^{(1)} = \langle dp, \wedge dx \rangle |_{\Sigma_H} = 0. \tag{1.8}$$

These two Cartan's conditions (1.7) and (1.8) should still be augmented with the characteristic surface S_H invariance condition for the differential 1-form $\alpha_2^{(1)} \in \Lambda^1(S_H)$, as

$$\alpha_2^{(1)} := dH|_{S_H} = 0. \tag{1.9}$$

Conditions (1.7), (1.8) and (1.9), when imposed on the characteristic surface $S_H \subset M$, make it possible to construct the proper characteristic vector fields on S_H, whose suitable characteristic strips [7, 16] generate the solution surface Σ_H searched for.

The above reasoning can naturally be embedded into the classical Cartan theory [2, 6, 14] of integrable ideals in the Grassmann algebra on differentiable manifolds. Within this theory, the solution surface $\Sigma_H \subset S_H$ is exactly the maximal integral submanifold of the integrable ideal $I(S_H) \subset \Lambda(S_H)$, generated by the corresponding one-forms (1.5), (1.9) and two-forms $d\alpha_1^{(1)} \in \Lambda(S_H)$. By construction, this ideal is closed, that is $dI(S_H) \subset I(S_H)$, owing to the Cartan-Frobenius integrability [4,9,12,14] criterion.

Below, based on the results of [3, 6, 16], we will construct the proper characteristic vector fields vanishing the ideal $I(S_H)$ related to partial differential equations of the first order (1.1) and generating the solution surface Σ_H as suitable characteristic strips related to boundary conditions (1.2) and (1.3), and next make use of the generalized Cartan-Monge geometric approach [3,6] for partial differential equations of the second and higher orders.

2. THE CHARACTERISTIC VECTOR FIELDS METHOD: DIFFERENTIAL-GEOMETRIC ASPECTS FOR THE FIRST ORDER DIFFERENTIAL EQUATIONS

Let us first consider first order partial differential equation (1.1) on the surface $S_H \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$ and construct a related characteristic vector field $K_H : S_H \to T(S_H)$, given by general expressions

$$\left.\begin{array}{r} dx/d\tau = a_H(x;u,p) \\ dp/d\tau = b_H(x;u,p) \\ du/d\tau = c_H(x;u,p) \end{array}\right\} := K_H(x;u,p), \tag{2.1}$$

where $\tau \in \mathbb{R}$ is a suitable evolution parameter and $(x;u,p) \in S_H$. Since, owing to the Cartan-Monge geometric approach, there hold conditions (1.7), (1.8) and (1.9) along the solution surface Σ_H, we can satisfy them applying the interior differentiation $i_{K_H} : \Lambda(S_H) \to \Lambda(S_H)$ [2, 4, 9] in the Grassmann agebra $\Lambda(S_H)$ to the corresponding differential forms $\alpha_1^{(1)}$ and $d\alpha_1^{(1)}$:

$$i_{K_H}\alpha_1^{(1)} = 0, \qquad i_{K_H} d\alpha_1^{(1)} = 0. \tag{2.2}$$

After simple computation one finds that

$$\begin{aligned} c_H &= \langle p, a_H \rangle, \\ \beta^{(1)} &:= \langle b_H, dx \rangle - \langle a_H, dp \rangle |_{S_H} = 0 \end{aligned} \tag{2.3}$$

for all points $(x;u,p) \in S_H$. The obtained 1-form $\beta^{(1)} \in \Lambda^1(S_H)$ must evidently be compatible with defining invariance condition (1.9) on S_H. This means that there exists a scalar function $\mu \in C^1(S_H;\mathbb{R})$ such that the condition

$$\mu\, \alpha_2^{(1)} = \beta^{(1)} \tag{2.4}$$

holds on S_H. This gives rise to the following final relationships:

$$a_H = \mu\, \partial H/\partial p, \quad b_H = -\mu(\partial H/\partial x + p\, \partial H/\partial u), \tag{2.5}$$

which, together with the first equality in (2.3), completes the search for the structure of the characteristic vector fields $K_H : S_H \to T(S_H)$:

$$K_H = (\mu\ \partial H/\partial p;\ -\mu(\partial H/\partial x + p\ \partial H/\partial u),\ \langle p, \mu\, \partial H/\partial p\rangle)^{\intercal}. \tag{2.6}$$

Now we can pose a suitable Cauchy problem for the equivalent set of ordinary differential equations (2.1) on $S_H \subset M$ as follows:

$$\begin{aligned}
&dx/d\tau = \mu\, \partial H/\partial p,\ x|_{\tau=0} \stackrel{?}{=} x_0(x) \in \Gamma_\varphi, \quad x|_{\tau=t(x)} = x \in \mathbb{R}^n \backslash \Gamma_\varphi,\\
&du/d\tau = \langle p, \mu\, \partial H/\partial p\rangle\,, \quad u|_{\tau=0} = u_0(x_0(x)),\ \ u|_{\tau=t(x)} \stackrel{?}{=} u(x),\\
&dp/d\tau = -\mu(\partial H/\partial x + p\ \partial H/\partial u),\ p|_{\tau=0} = \partial u_0(x_0(x))/\partial x_0,
\end{aligned} \tag{2.7}$$

where the sign " $\stackrel{?}{=}$ " denotes that the function $u : \mathbb{R}^n \to \mathbb{R}$ as a suitable solution of (1.1) remains to be found and the value $x_0(x) \in \Gamma_\varphi$ is still unknown as the intersection point of the corresponding vector field orbit, starting at a fixed point $x \in \mathbb{R}^n \backslash \Gamma_\varphi$, with the boundary hypersurface $\Gamma_\varphi \subset \mathbb{R}^n$ at the moment of "time" $\tau = t(x) \in \mathbb{R}$. Thus, one can formulate the following proposition.

Proposition 2.1. *The characteristic surface S_H is generated by orbits of suitable characteristic fields* (2.6). *The corresponding solutions of inverse Cauchy problem* (2.7) *give rise to the exact solutions of Cauchy problem* (1.2) *and* (1.3) *for Hamilton-Jacobi equation* (1.1).

As a result, by means of solving corresponding "inverse" Cauchy problem (2.7), one finds the following exact functional-analytic expression for a solution $u \in C^2(\mathbb{R}^n; \mathbb{R})$ to boundary value problem (1.2) and (1.3):

$$u(x) = u_0(x_0(x)) + \int_0^{t(x)} \bar{\mathcal{L}}(x; u, p) d\tau, \tag{2.8}$$

where, by definition,

$$\bar{\mathcal{L}}(x; u, p) := \langle p, \mu\, \partial H/\partial p\rangle \tag{2.9}$$

for all $(x; u, p) \in S_H$. If the Hamiltonian function $H : M \to \mathbb{R}$ is nondegenerate, that is the Hessian $Hess\ H := \det(\partial^2 H/\partial p^2) \neq 0$ for all $(x; u, p) \in S_H$, then the first equation in (2.7) can be solved with respect to the variables

$$p = \psi(x, \dot{x}; u) \tag{2.10}$$

for all $(x, \dot{x}) \in T(\mathbb{R}^n)$, where $\dot{x} := dx/dt$, and $\psi : T(\mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^n$ is some smooth mapping. By means of the following classical Lagrangian function expression

$$\mathcal{L}(x, \dot{x}; u) := \bar{\mathcal{L}}(x; u, p)|_{p=\psi(x,\dot{x};u)} \tag{2.11}$$

solution (2.8) takes the form

$$u(x) = u_0(x_0(x)) + \int_0^{t(x)} \mathcal{L}(x, \dot{x}; u) d\tau, \tag{2.12}$$

which can be rewritten [14, 15] equivalently as

$$u(x) = \inf_{x_0 \in \Gamma\varphi} \left\{ u_0(x_0) + \int_0^{t(x)} \mathcal{L}(x(\tau; x_0), \dot{x}(\tau; x_0); u(\tau; x_0)) d\tau \right\}. \tag{2.13}$$

Thereby, the following proposition holds.

Proposition 2.2. *The solutions of inverse Cauchy problem* (2.7) *are exactly the critical points of corresponding infimum problem* (2.13).

The functional-analytic form (2.13) is called the inf-type Hopf-Lax representation and appears to be very important in finding so-called generalized solutions [11, 15, 17] of Hamilton-Jacobi equation (1.1).

3. THE CHARACTERISTIC VECTOR FIELDS METHOD: THE SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Let us now consider a boundary problem for the second order partial differential equation

$$H(x; u, u_x, u_{xx}) = 0, \ u|_{\Gamma_\varphi} = u_0. \tag{3.1}$$

Here the solution $u \in C^2(\mathbb{R}^n; \mathbb{R})$, a generalized "Hamiltonian" function $H \in J^{(2)}(\mathbb{R}^n; \mathbb{R})$ and a boundary condition $u_0 \in C^1(\Gamma_\varphi; \mathbb{R})$ is defined on some smooth almost everywhere hypersurface

$$\Gamma_\varphi := \{x \in \mathbb{R}^n : \varphi(x) = 0, \quad |\varphi_x| \neq 0\},$$

where $\varphi \in C^1(\mathbb{R}^n; \mathbb{R})$ is a given smooth function on $\mathbb{R}^n$. Putting $p^{(1)} := u_x$, $p^{(2)} := u_{xx}$, $x \in \mathbb{R}^n$, one can similarly construct the related characteristic surface

$$S_H := \{(x; u, p^{(1)}, p^{(2)}) \in M : H(x; u, p^{(1)}, p^{(2)}) = 0\} \tag{3.2}$$

on the suitable jet manifold $M := J^{(2)}(\mathbb{R}^n; \mathbb{R}) \simeq \mathbb{R}^{n+1} \times \mathbb{R}^n \times (\mathbb{R}^n \otimes \mathbb{R}^n)$ within the Cartan-Monge generalized geometric approach, and a suitable Cartan set of differential one- and two-forms:

$$\begin{aligned}
\alpha_1^{(1)} &:= du - \left\langle p^{(1)}, dx \right\rangle |_{\Sigma_H} = 0, \\
d\alpha_1^{(1)} &:= \left\langle dx, \wedge dp^{(1)} \right\rangle |_{\Sigma_H} = 0, \\
\alpha_2^{(1)} &:= dp^{(1)} - \left\langle p^{(2)}, dx \right\rangle |_{\Sigma_H} = 0, \\
d\alpha_2^{(1)} &:= \left\langle dx, \wedge dp^{(2)} \right\rangle |_{\Sigma_H} = 0,
\end{aligned} \tag{3.3}$$

vanishing upon the corresponding solution submanifold $\Sigma_H \subset S_H$. The set of differential forms (3.3) should be augmented with the characteristic surface $S_H \subset M$ invariance differential 1-form

$$\alpha_3^{(1)} := dH|_{S_H} = 0, \tag{3.4}$$

vanishing, respectively, upon the characteristic surface S_H. The solution space $\Sigma_H \subset S_H$ is the maximal integral submanifold of the suitably integrable ideal $I(S_H) \subset \Lambda(S_H)$, generated by the one-forms $\alpha_j^{(1)} \in \Lambda(S_H)$, and two-forms $d\alpha_j^{(1)} \in \Lambda(S_H)$, $j = \overline{1,3}$ (see, [1,2,4,14]). This ideal is, by construction, closed and, thereby, integrable, owing to the well known Cartan criterion [1,4,7,14].

Let the characteristic vector field $K_H : S_H \to T(S_H)$ on S_H be given by the expressions

$$\left.\begin{array}{r} dx/d\tau = a_H(x;u,p^{(1)},p^{(2)}) \\ du/d\tau = c_H(x;u,p^{(1)},p^{(2)}) \\ dp^{(1)}/d\tau = b_H^{(1)}(x;u,p^{(1)},p^{(2)}) \\ dp^{(2)}/d\tau = b_H^{(2)}(x;u,p^{(1)},p^{(2)}) \end{array}\right\} := K_H(x;u,p^{(1)},p^{(2)}) \tag{3.5}$$

for all $(x;u,p^{(1)},p^{(2)}) \in S_H$. To find vector field (3.5) it is necessary that the Cartan compatibility conditions in the following geometric form are satisfied

$$i_{K_H}\alpha_j^{(1)}|_{\Sigma_H} = 0, \quad i_{K_H}d\alpha_j^{(1)}|_{\Sigma_H} = 0 \tag{3.6}$$

for $j = \overline{1,3}$, where, as above, $i_{K_H} : \Lambda(S_H) \to \Lambda(S_H)$ is the internal derivative of the Grassmann algebra of differential forms along the vector field $K_H : S_H \to T(S_H)$. As a result of conditions (3.6), one finds that relationships

$$\begin{aligned} &c_H = \left\langle p^{(1)}, a_H \right\rangle, \quad b_H^{(1)} = \left\langle p^{(2)}, a_H \right\rangle, \\ &\left.\langle a_H, \partial H/\partial x\rangle + \langle c_H, \partial H/\partial u\rangle + \left\langle b_H^{(1)}, \partial H/\partial p^{(1)} \right\rangle + \left\langle b_H^{(2)}, \partial H/\partial p^{(2)} \right\rangle\right|_{S_H} = 0, \\ &\beta_1^{(1)} := \left.\left\langle a_H, dp^{(1)} \right\rangle - \left\langle b_H^{(1)}, dx \right\rangle\right|_{S_H} = 0, \\ &\beta_2^{(1)} := \left.\left\langle a_H, dp^{(2)} \right\rangle - \left\langle b_H^{(2)}, dx \right\rangle\right|_{S_H} = 0 \end{aligned} \tag{3.7}$$

should be satisfied upon S_H identically. It is necessary to mention here that, in contrast to the first-order equation case discussed in the previous section, we actually deal with vector field (3.5) defined on the characteristic surface $S_H \subset M$, reduced to some invariant submanifold $M_c \subset M$. It arises naturally from the fifth condition in (3.7), which cannot be satisfied identically by any suitably chosen vector field $K_H : S_H \to T(S_H)$ defined on the whole surface $S_H \subset M$. In particular, we need to take into account that expressions

$$\begin{aligned} &a_H = \mu^{(1|2)}\partial H/\partial p^{(2)}, \quad c_H = \left\langle p^{(1)}, a_H \right\rangle, \quad b_H^{(1)} = \left\langle p^{(2)}, a_H \right\rangle, \\ &b_H^{(2)} = -\mu^{(1|2),*}(\partial H/\partial x + p^{(1)}\partial H/\partial u + \left\langle p^{(2)}, \partial H/\partial p^{(1)} \right\rangle) \end{aligned} \tag{3.8}$$

identically satisfy for an arbitrary smooth tensor field $\mu^{(1|2)}: S_H \to \mathbb{R}^n \otimes \mathbb{R}^{n(n+1)/2}$, the first four conditions in (3.7), while the fifth one can be satisfied on some naturally defined invariant submanifold $M_c \subset M$ only, serving as a description of suitable boundary data to problem (3.1).

The following proposition characterizes the second order partial differential equation case and the way of constructing its exact solutions.

Proposition 3.1. *The characteristic surface S_H defined by* (3.3) *is generated by orbits of suitable characteristic fields* (3.5). *The corresponding exact solutions of second order equation* (3.1) *are given by solutions of the suitable inverse Cauchy problem similar to* (2.7) *posed for the set of ordinary differential equations* (3.5).

A tensor field $\mu^{(1|2)}: S_H \to \mathbb{R}^n \otimes \mathbb{R}^{n(n+1)/2}$ can be both degenerate or not with respect to the following definition.

Definition 3.2. *It is said that a tensor $\mu^{(1|k)} \in \mathbb{R}^n \otimes (\mathbb{R}^m)^k$, $m, n, k \in \mathbb{Z}_+$, is nondegenerate with respect to the problem $H(x,t;u,p) = 0$ if rank $\left(\frac{\partial}{\partial p^{(2)}}\mu^{(1|k)}\frac{\partial H}{\partial p^{(2)}}\right) = \min(km, n)$. In the other case, we say that it is degenerate.*

Having now satisfied relationships (3.7) on the submanifold $M_c \subset M$, we can determine a suitable vector field $K_H : S_H^{(c)} \to T(S_H^{(c)})$ on the surface $S_H^{(c)} := S_H \cap M_c$ and, thereby, construct functional-analytic solutions of our partial differential equation (3.1) of the second order via solving the equivalent boundary problem for the set of ordinary differential equations (3.5) on the characteristic surface $S_H^{(c)} \subset S_H$. This means that to exactly solve the second order partial differential equation (3.1) we need to solve the suitable Cauchy inverse problem similar to (2.7) for the set of ordinary differential equations (3.6).

A quite simple but not trivial example of second order problem (3.1) will be analyzed in detail in the following section.

4. EXAMPLE: THE HEAT EQUATION $u_t - u_{xx} = 0$

Let us now consider a Cauchy problem for the heat equation in $\mathbb{R} \times \mathbb{R}_+$, that is the equation

$$u_t - u_{xx} = 0, \qquad u|_{t=0} = u_0, \tag{4.1}$$

where $(x,t) \in \mathbb{R}\times\mathbb{R}_+$ and u_0 is a smooth function in the Sobolev space $W_2^{(2)}(\mathbb{R};\mathbb{R})$. To apply the generalized characteristics method devised in [5,7,8,17] and described above, let us rewrite equation (4.1) in the jet-manifold coordinates $M := J^2(\mathbb{R}\times\mathbb{R}_+;\mathbb{R})$, that is in the form

$$p_{0,1} - p_{2,0} = 0, \tag{4.2}$$

where $p_{i,j} := \partial^{i+j}u/\partial x^i \partial t^j$, $1 \leq i + j \leq 2$. Now we can define the generalized "Hamiltonian" function:

$$H(x,t;u,p) := p_{0,1} - p_{2,0}, \quad p := (p^{(1)}, p^{(2)}) := (p_{0,1}, p_{1,0}, p_{2,0}, p_{1,1}, p_{0,2}) \in \mathbb{R}^5, \tag{4.3}$$

that generates [5, 7, 18, 20, 21] the characteristic vector field

$$
\begin{aligned}
&\begin{pmatrix} dx/d\tau \\ dt/d\tau \end{pmatrix} = \mu^{(1|2)} \frac{\partial H}{\partial p^{(2)}}, \\
&dp^{(2)}/d\tau = -\mu^{(1|2),*} \begin{pmatrix} \frac{\partial H}{\partial x} + p_{1,0}\frac{\partial H}{\partial u} + p_{1,1}\frac{\partial H}{\partial p_{0,1}} \\ \frac{\partial H}{\partial t} + p_{0,1}\frac{\partial H}{\partial u} + p_{1,1}\frac{\partial H}{\partial p_{1,0}} \end{pmatrix}, \\
&dp_{0,1}/d\tau = p_{1,1}dx/d\tau + p_{0,2}dt/d\tau, \\
&dp_{1,0}/d\tau = p_{2,0}dx/d\tau + p_{1,1}dt/d\tau, \\
&du/d\tau = p_{1,0}dx/d\tau + p_{0,1}dt/d\tau
\end{aligned}
\tag{4.4}
$$

on the manifold M for all $\tau \in \mathbb{R}_+$. Here $\mu^{(1|2)} \in C^1(M; \mathbb{R}^2 \otimes \mathbb{R}^3)$ denotes the proper smooth tensor field on M which is at our disposal during the construction of characteristic vector field (4.4). Vector field (4.4) *a priori* satisfies the invariance condition for the hypersurface:

$$
S_H := \{(x, t; u, p^{(1)}, p^{(2)}) \in M : H(x, t; u, p) = 0\}, \tag{4.5}
$$

that is, for all $\tau \in \mathbb{R}_+$ the equality

$$
\frac{d}{d\tau} H(x, t; u, p) = 0 \tag{4.6}
$$

holds. The last three equalities in (4.4) are the Cartan ordinary compatibility conditions. Below we apply the generalized characteristics method, developed in [5, 7], to construct solutions of Cauchy problem (4.1).

Let us rewrite equations (4.4) in a more extensive form:

$$
\begin{pmatrix} dx/d\tau \\ dt/d\tau \\ dp_{0,1}/d\tau \\ dp_{1,0}/d\tau \\ du/d\tau \end{pmatrix} = \begin{pmatrix} \mu_{1|2,0}\frac{\partial H}{\partial p_{2,0}} + \quad \mu_{1|1,1}\frac{\partial H}{\partial p_{1,1}} + \quad \mu_{1|0,2}\frac{\partial H}{\partial p_{0,2}} \\ \mu_{2|2,0}\frac{\partial H}{\partial p_{2,0}} + \quad \mu_{2}|_{1,1}\frac{\partial H}{\partial p_{1,1}} + \quad \mu_{2|0,2}\frac{\partial H}{\partial p_{0,2}} \\ p_{1,1}dx/d\tau + \quad p_{0,2}dt/d\tau \\ p_{2,0}dx/d\tau + \quad p_{1,1}dt/d\tau \\ p_{1,0}dx/d\tau + \quad p_{0,1}dt/d\tau \end{pmatrix}, \tag{4.7}
$$

and the compatibility condition

$$
\frac{dp_{1,1}}{d\tau}dx - \frac{dx}{d\tau}dp_{1,1} + \frac{dp_{0,2}}{d\tau}dt - \frac{dt}{d\tau}dp_{0,2} = 0. \tag{4.8}
$$

The last condition defines, in the unique way, the functional submanifold $M_c \subset M$ of the Cauchy conditions for (4.1), *a priori* invariant for vector field (4.7). Now from expression (4.3) we obtain the following system of ordinary differential equations on M_c:

$$
\begin{aligned}
\frac{dx}{d\tau} &= -\mu_{1|2,0}, \quad \frac{dt}{d\tau} = -\mu_{2|2,0},\\
\frac{dp_{2,0}}{d\tau} &= -\mu_{1|2,0}p_{1,1} - \mu_{2|2,0}p_{0,2},\\
\frac{dp_{1,1}}{d\tau} &= -\mu_{1|1,1}p_{1,1} - \mu_{2|1,1}p_{0,2},\\
\frac{dp_{0,2}}{d\tau} &= -\mu_{1|0,2}p_{1,1} - \mu_{2|0,2}p_{0,2},\\
\frac{dp_{0,1}}{d\tau} &= -\mu_{1|2,0}p_{1,1} - \mu_{2|2,0}p_{0,2},\\
\frac{dp_{1,0}}{d\tau} &= -\mu_{1|2,0}p_{2,0} - \mu_{2|2,0}p_{1,1},\\
\frac{du}{d\tau} &= -\mu_{1|2,0}p_{1,0} - \mu_{2|2,0}p_{0,1}.
\end{aligned}
\tag{4.9}
$$

For convenience and some analytical simplifications, put now $\mu_{1|1,1} = 0 = \mu_{2|1,1}$, $\mu_{2|2,0} = -1$, $\mu_{1|0,2} = 0 = \mu_{2|0,2}$, and $\mu_{1|2,0} = -c \in \mathbb{R}$. Then we obtain

$$
\begin{aligned}
&\frac{dx}{d\tau} = c, \quad \frac{dt}{d\tau} = 1, \quad \frac{dp_{2,0}}{d\tau} = cp_{1,1} + p_{0,2},\\
&\frac{dp_{1,1}}{d\tau} = 0, \quad \frac{dp_{0,2}}{d\tau} = 0, \quad \frac{dp_{0,1}}{d\tau} = cp_{1,1} + p_{0,2},\\
&\frac{dp_{1,0}}{d\tau} = cp_{2,0} + p_{1,1}, \quad \frac{du}{d\tau} = cp_{1,0} + p_{0,1}, \quad dp_{1,1} = 0,
\end{aligned}
\tag{4.10}
$$

hence the values $p_{1,1} = \bar{p}_{1,1} \in \mathbb{R}$, $p_{0,2} = \bar{p}_{0,2} \in \mathbb{R}$ are constant for all $\tau \in \mathbb{R}$, and the value $p_{1,1}$ is constant for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, which is compatible with others conditions. It means that vector field (4.10) is defined on the submanifold $M_c := \{p_{1,1} = const\} \subset M$. Solving now system (4.10) on M_c, we find:

$$
\begin{aligned}
&x = y + c\tau, \quad x|_{\tau=0} = y(x,t) \in \mathbb{R}, \quad x|_{\tau=t} = x \in \mathbb{R}, \quad \bar{p}_{0,1} = \bar{p}_{2,0},\\
&p_{0,1} = (c\bar{p}_{1,1} + \bar{p}_{0,2})\tau + \bar{p}_{0,1}, \quad p_{2,0} = (c\bar{p}_{1,1} + \bar{p}_{0,2})\tau + \bar{p}_{2,0},
\end{aligned}
\tag{4.11}
$$

and

$$
p_{1,0} = c(c\bar{p}_{1,1} + \bar{p}_{0,2})\tau^2/2 + (c\bar{p}_{2,0} + \bar{p}_{1,1})\tau + \bar{p}_{1,0}. \tag{4.12}
$$

So for the solution $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, from (4.11) and (4.12), we obtain the expression

$$
\begin{aligned}
u(x,t) &= u_0(y(x,t)) + \int_0^t [cp_{1,0}(\tau) + p_{0,1}(\tau)]d\tau =\\
&= u_0(y(x,t)) + (c^3\bar{p}_{1,1} + c^2\bar{p}_{0,2})t^3/6 +\\
&\quad + (c^2\bar{p}_{2,0} + 2c\bar{p}_{1,1} + \bar{p}_{0,2})t^2/2 + (c\bar{p}_{1,0} + \bar{p}_{2,0})t,
\end{aligned}
\tag{4.13}
$$

where $c = (x - y(x,t))/t$ for all $t \in \mathbb{R}_+$. Since solution (4.13) includes the parameters $\bar{p}_{0,1}, \bar{p}_{1,0}, \bar{p}_{1,1}, \bar{p}_{2.0}$ and $\bar{p}_{0,2} \in \mathbb{R}$, we find them from the initial condition and expressions (4.12) and (4.11) taking $\tau = 0$:

$$
\begin{aligned}
&\bar{p}_{0,1} = \bar{p}_{2,0} = u_0^{(2)}(y), \quad \bar{p}_{0,2} = u_0^{(4)}(y),\\
&\bar{p}_{1,1} = u_0^{(3)}(y), \qquad\qquad\quad \bar{p}_{1,0} = u_0^{(1)}(y)
\end{aligned}
\tag{4.14}
$$

in the point $y = y(x,t) \in \mathbb{R}$ for all $t \in \mathbb{R}_+$, $x \in \mathbb{R}$. Now from (4.14) and (4.13) we obtain the following expression for the solution $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$:

$$
\begin{aligned}
u(x,t;c) = u_0(y) &+ \Big[\frac{c^3t^3}{3!}u_0^{(3)}(y) + \frac{c^2t^3}{6}u_0^{(4)}(y)\Big] +\\
&+ \Big[\frac{c^2t^2}{2}u_0^{(2)}(y) + ct^2u_0^{(3)}(y) + \frac{t^2}{2}u_0^{(4)}(y)\Big] + [ctu_0^{(1)}(y) + tu_0^{(2)}(y)],
\end{aligned}
\tag{4.15}
$$

where $c \in \mathbb{R}$ is an arbitrary parameter. It is the solution of equation (4.1) for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. We can now check that expression (4.15) is the solution of equation (4.1) and satisfies, for all $x \in \mathbb{R}$, the initial condition $u_0(x) = \lim_{t \to +0} u(x,t) \in M_c$. Notice that expression (4.15), like the solution of (4.1) on the submanifold M_c, depends, in the general case, on a parameter $c \in \mathbb{R}$, which we can verify directly by differentiation:

$$du(x,t;c)/dc = O(|c|^4), \tag{4.16}$$

where, for convenience, we put $c \to 0$. Result (4.16) is an obvious consequence of the fact that the tensor field $\mu^{(1|2)} = \begin{pmatrix} c & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is strongly degenerate and compability condition (4.8) does not define a sufficiently large submanifold $M_c \subset M$ of the initial conditions for problem (4.1), while the existence of the submanifold is crucial in finding solutions from the last expression in (4.4).

One can also prove that for every fixed $n \in \mathbb{Z}_+$ we may in the proper way choose the nondegenerate tensor $\mu^{(1|2)}$ such that if $c \to 0$ then the equality

$$du(x,t;c)/dc = O(|c|^n), \tag{4.17}$$

holds. This means that the function $u \in C^{(\infty)}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R})$ found above really does not depend on the parameter $c \in \mathbb{R}$. (It also means that vector field (4.4), constructed above, fully served the purpose of extending our solution $u_0 \in C^{(\infty)}(\Gamma_\varphi; \mathbb{R})$ from the initial points ("boundary") of the form $(x,0) \in \mathbb{R} \times \{0\} := \Gamma_\varphi$ onto the points $(x,t) \in (\mathbb{R} \times \mathbb{R}_+) \backslash \Gamma_\varphi$).

Thus, one can easily find the solution of problem (4.1) by calculating the limit of (4.15) as $c \to 0$, that is

$$\begin{aligned} u(x,t) &= \lim_{c \to 0} u(x,t;c) = u_0(x) + \frac{t}{1!} u_0^{(2)}(x) + \frac{t^2}{2!} u_0^{(4)}(x) + \frac{t^3}{3!} u_0^{(6)}(x) + \\ &+ \frac{t^4}{3!} u_0^{(8)}(x) + \frac{t^5}{4!} u_0^{(10)}(x) + \ldots = \\ &= \Big(\sum_{k \in \mathbb{Z}_+} \frac{1}{k!} (t \frac{\partial^2}{\partial x^2})^k \Big) u_0(x) = \exp(t \frac{\partial^2}{\partial x^2}) u_0(x) = \\ &= \frac{1}{\sqrt{2\pi}} \exp(t \frac{\partial^2}{\partial x^2}) \int_{\mathbb{R}} \hat{u}_0(y) \exp(ixy) dy = \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}_0(y) \exp(ixy - y^2 t) dy = \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} u_0(z) dz \int_{\mathbb{R}} \exp(ixy - izy - y^2 t) dy = \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \exp[\frac{-(x-z)^2}{4t}] u_0(z) dz, \end{aligned} \tag{4.18}$$

which is the classical, well-known [17] exact solution to heat equation (1.1) in the standard Gauss form.

5. CONCLUSIONS

We need to make some remarks on the proposed generalized characteristics method in the case of our linear heat equation (4.1). One of preconditions of the effective application of this method to the second order partial differential equation in the general form $H(x, t; p) = 0$ is that the tensor $\mu^{(1|2)}$ should be nondegenerate, but the choice of this tensor is often very important for the construction of vector field (4.4), which leads to the final expression of the solution in an analytical form on the submanifold $M_c \subset M$.

The second important fact featuring the general characteristics method is enabling the representation of solutions in the exact Hopf-Lax type form, which is a simple consequence of the suitable representation of solutions to proper Hamiltonian vector fields in the variational Lagrangian form. For partial differential equations of the first order ($H(x, t; u, p) = 0$) this representation is natural and almost obvious, but in the case of nonlinear partial differential equations of the second (or higher) orders the representation in the Lagrangian form is possible only when the proper general system of Hamiltonian type vector fields can be represented in the analogical variational Lagrange-Ostrogradski type form with higher derivatives. The necessary condition for the existence of this form is the possibility of representation of the solution to our equation in the form of some function with dual quantity of independent variables, but today, the effective criterium is quite complicated.

Thereby, the generalization of the Monge-Cartan characteristics method onto nonlinear partial differential equations of higher orders is a quite natural consequence of the related inverse Cauchy problem analysis of the associated characteristic Hamiltonian type vector fields on the basic invariant submanifold.

Acknowledgements
The authors express their most sincere appreciation to Prof. A.K. Prykarpatsky for posing the problem and fruitful discussions.

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Jolanta Golenia
goljols@tlen.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Cracow, Poland

Yarema A. Prykarpatsky
yarpry@gmail.com

Pedagogical University of Cracow
Institute of Mathematics
ul. Podchorążych 2, 30-084 Cracow, Poland

Institute of Mathematics of NAS of Ukraine
Kyiv, 01601, Ukraine

I. Franko Pedagogical University
Drohobych, Lviv region, 82100, Ukraine

Eugeniusz Wachnicki
euwachni@ap.krakow.pl

Pedagogical University of Cracow
Institute of Mathematics
ul. Podchorążych 2, 30-084 Cracow, Poland

Received: March 18, 2008.
Revised: July 16, 2008.
Accepted: July 22, 2008.