Mostafa Blidia, Rahma Lounes

VERTICES BELONGING TO ALL OR TO NO MINIMUM LOCATING DOMINATING SETS OF TREES

Abstract. A set $D$ of vertices in a graph $G$ is a locating-dominating set if for every two vertices $u, v$ of $G \setminus D$ the sets $N(u) \cap D$ and $N(v) \cap D$ are non-empty and different. In this paper, we characterize vertices that are in all or in no minimum locating dominating sets in trees. The characterization guarantees that the $\gamma_L$-excellent tree can be recognized in a polynomial time.

Keywords: domination, locating domination.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION AND PRELIMINARY RESULTS

For a simple graph $G = (V, E)$, the open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. A set $D \subseteq V$ is a dominating set if for each vertex $v \in V - D$, $N(v) \cap D \neq \emptyset$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$, see [2].

We are interested in a variation of domination in graphs. A set $D \subseteq V$ is a locating-dominating set (LDS) if it is dominating and every two vertices $x, y$ of $V \setminus D$ satisfy $N(x) \cap D \neq N(y) \cap D$. The locating-domination number $\gamma_L(G)$ is the minimum cardinality of a locating-dominating set. Locating-domination was introduced by Slater [9, 10]. Moreover, since every locating-dominating set is a dominating set, then every graph $G$ satisfies the inequality

$$\gamma(G) \leq \gamma_L(G).$$

For any parameter $\mu(G)$ associated with a graph property $\mathcal{P}$, we refer to a set of vertices with Property $\mathcal{P}$ and cardinality $\mu(G)$ as a $\mu(G)$-set. A graph $G$ is called a $\mu(G)$-excellent graph if every vertex of $G$ is contained in a $\mu(G)$-set.

For more details on domination in graphs, see the monographs by Haynes, Hedetniemi and Slater [5, 6] and also [7].
Many researchers have been interested in characterizing the vertices of $G$ that are in all or in no set with the cardinality $\mu(G)$. Indeed, Hammer et al. [4] have characterized those vertices in a graph for independent sets with maximum cardinalities, Mynhardt [8] has characterized the vertices in all or in no minimum dominating sets of trees, Cockayne et al. [3] have characterized the set of vertices contained in all or in no total dominating sets of trees, and Blidia et al. [1] have characterized the set of vertices contained in all or in no minimum double dominating sets of trees.

In this paper, we investigate vertices belonging to all or to no minimum locating dominating sets of a tree and we deduce a polynomial algorithm to recognize a $\gamma_L$-excellent tree.

For this purpose, we introduce the following notation.

For a tree $T$ we define the sets $A_L(T)$ and $N_L(T)$ by

$$A_L(T) = \{v \in V(T) \mid v \text{ is in every } \gamma_L(T)\text{-set}\} \quad \text{and} \quad N_L(T) = \{v \in V(T) \mid v \text{ is in no } \gamma_L(T)\text{-set}\}.$$ 

The degree of a vertex $v$, denoted by $\deg_G(v)$, is the number of vertices adjacent to $v$ and the diameter of $G$ is $\text{diam}(G) = \max\{d(x, y) \mid x, y \in V(G)\}$ where $d(x, y)$ is the length of the shortest path between $x$ and $y$. Specifically, for a vertex $v$ in a rooted tree $T$, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of $v$, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$; it is denoted by $T_v$. A leaf (or pendant vertex) of $T$ is a vertex of degree one, while a support vertex of $T$ is a vertex adjacent to a leaf. We denote the set of leaves and support vertices set of $T$ by $L(T)$ and $S(T)$, respectively. Let $T$ be a rooted tree. We denote by $L(v)$ the set of leaves of $T_v$ distinct from $v$, that is, $L(v) = D(v) \cap L(T)$. A vertex of degree at least three is called a branch vertex. We denote by $B(T)$ the set of all branch vertices of $T$. We also define the sets

$$L'(v) = \{u \in L(v) \mid d(u, v) \equiv j \pmod 5\}, \quad \text{where} \quad j = 0, 1, 2, 3, 4.$$ 

A path on $n$ vertices is denoted by $P_n$.

Below we give some straightforward observations.

**Observation 1.** If $T$ is a tree of diameter at least 2 and $y$ a vertex of $L(T)$, then there is a $\gamma_L(T)$-set that does not contain $y$.

**Observation 2.** In a nontrivial path $P_n$, $L(P_n) \subseteq N_L(P_n)$ if and only if $n \equiv 0 \pmod 5$.

The following lemma will be used in the next section.

**Lemma 1.** Let $T'$ be a tree and $v$ a vertex of $V(T')$. Let $u$ be a vertex of $T'$ such that $u \neq v$. Let $T$ be the tree obtained from $T'$ by adding a path $P_5 = x_1x_2x_3x_4x_5$ and the edge $ux_1$. Then:

1. $\gamma_L(T) = \gamma_L(T') + 2$,
2. $v \in A_L(T')$ if and only if $v \in A_L(T)$,
3. $v \in N_L(T')$ if and only if $v \in N_L(T)$.
Vertices belonging to all or to no minimum locating dominating sets of trees

Proof. Let $T$ be the tree obtained from $T'$ by adding a path $P_5 = x_1x_2x_3x_4x_5$ and the edge $w_1$ where $u \neq v$.

(1) Every $\gamma_L(T')$-set can be extended to an $LDS$ of $T$ by adding the vertices $x_2$ and $x_4$, so $\gamma_L(T) \leq \gamma_L(T') + 2$. On the other hand, let $S$ be a $\gamma_L(T)$-set. If $u \in S$, then clearly $|S \cap P_5| = 2$ and $S' = S \setminus P_5$ is an $LDS$ of $T'$ with $|S'| = \gamma_L(T) - 2 \geq \gamma_L(T')$. Conversely, assume that $u \notin S$, then clearly $|S \cap P_5| = 2$ and $S' = S - S \cap P_5$ is an $LDS$ of $T'$ with $|S'| = \gamma_L(T) - 2 \geq \gamma_L(T')$. If $x_1 \in S$, in this case $|S \cap P_5| = 3$, let $S' = (S - (S \cap P_5)) \cup \{u\}$ then $S'$ is an $LDS$ of $T'$ with $|S'| = \gamma_L(T) - 3 + 1 = \gamma_L(T) - 2$.

So in each case we have $\gamma_L(T) \geq \gamma_L(T') + 2$. Therefore, $\gamma_L(T) = \gamma_L(T') + 2$.

(2) Assume that $v \notin A_L(T')$ and let $S'$ be a $\gamma_L(T')$-set which does not contain $v$. Then $S = S' \cup \{x_2, x_4\}$ is a $\gamma_L(T)$-set that does not contain $v$, and so $v \notin A_L(T)$. Conversely, assume that $v \in A_L(T')$ and let $S$ be any $\gamma_L(T')$-set with $S' = S \cap V(T')$.

If $u \notin S$, then $S'$ is an $LDS$ of $T'$ with $|S'| = |S| - 2$. Hence, $S'$ is a $\gamma_L(T')$-set with $v \in S' \subset S$. If $u \in S$, then, as discussed in (1), $S'$ is an $LDS$ of $T'$ with $|S'| = |S| - 2$. Hence, $S'$ is a $\gamma_L(T')$-set with $v \in S'$, and $v \in S$ since $v \neq u$. Therefore, $v \in A_L(T)$.

(3) Suppose that $v \notin N_L(T')$. Let $S'$ be a $\gamma_L(T')$-set that contains $v$. Clearly, $S' \cup \{x_2, x_4\}$ is a $\gamma_L(T)$-set containing $v$ so $v \notin N_L(T)$. Conversely, suppose that $v \in N_L(T')$ and let $S$ be any $\gamma_L(T)$-set with $S' = S \cap V(T')$. Then, as discussed in (1), $S'$ is an $LDS$ of $T'$ with $|S'| = |S| - 2$. Hence, $S'$ is a $\gamma_L(T')$-set with $v \notin S'$, and $v \notin S$ since $v \neq u$. We deduce that $v \in N_L(T)$. \qed

2. PRUNING OF A TREE

In order to characterize the sets $A_L(T)$ and $N_L(T)$ for any nontrivial tree $T$, we will use a technique called tree pruning, introduced by Mynhardt [8] and later used by Cockayne, Henning and Mynhardt [3].

Let $v$ be a vertex of a nontrivial tree $T$. Using the process described below, with respect to the root $v$, on every branch vertex (vertex of $B(T)$), the tree $T_v$ is transformed into another tree $T_v'$, called the pruning of $T_v$, in which every vertex different from $v$ has degree at most two. As a consequence, if a vertex $v$ is in $A_L(T)$ or $N_L(T)$, then it has the same properties with respect to $T_v'$.

Let $T = T_v$ be a nontrivial tree rooted at a vertex $v$. If every vertex $u \neq v$ has degree at least two, then $T_v = T_v'$. Otherwise, let $w$ be a branch vertex (vertex of $B(T)$ with degree at least 3) at maximum distance from $v$. Then apply the following process:

(a) If $|L^1(w)| \geq 1$, delete $D(w)$ and attach a $P_1$ to $w$.
(b) If $|L^1(w)| = 0$ and $|L^2(w)| \geq 1$, delete $D(w)$ and attach a $P_3$ to $w$.
(c) If $|L^1(w)| \cup L^2(w)| = 0$ and $|L^2(w)| \geq 1$, delete $D(w)$ and attach a $P_4$ to $w$.
(d) If $|L^1(w)| \cup L^2(w)| \cup L^4(w)| = 0$ and $|L^2(w)| \geq 2$, delete $D(w)$ and attach a $P_4$ to $w$.
(e) If $|L^1(w)| \cup L^3(w)| \cup L^4(w)| = 0$ and $|L^2(w)| = 1$, delete $D(w)$ and attach a $P_2$ to $w$.
(f) If $|L^1(w)| \cup L^2(w)| \cup L^3(w)| \cup L^4(w)| = 0$, $|L^2(w)| \geq 2$, delete $D(w)$ and attach a $P_5$ to $w$. 

To illustrate this technique, we consider the tree of Figure 1(a) where \( x, u, s, w, y \) and \( z \) are the branch vertices of \( T \). At this step, \( z \) is the branch vertex at maximum distance from \( v \), since \( |L^1(z)| = 2 \), so we delete \( D(w) \) and attach a path \( P_1 \) at \( z \) (see Figure 1(b)).

Now there remain five branch vertices: \( x, u, s, w, y \). The vertex \( y \) is at distance two from \( v \). Since \( |L^3(y)| = 1 \) and \( |L^1(y)| = 0 \), we delete \( D(y) \) and attach a path \( P_3 \) at \( y \) (see Figure 1(c)). All the remaining branch vertices \( s, u, w, x \) are at distance one from \( v \). Since \( |L^1(x)| = 0, |L^3(x)| = 0 \) and \( |L^1(x)| = 1 \), \( (|L^1(u)| = 0, |L^3(u)| = 2 \), \( (|L^1(u)| = 0, |L^3(u)| = 1) \) and \( (|L^1(w)| = 1) \). So, we delete \( D(x) \) and attach a path \( P_5 \) at \( x \), delete \( D(u) \) and attach a path \( P_4 \) at \( u \), delete \( D(s) \) and attach a path \( P_3 \) at \( s \), and finally delete \( D(w) \) and attach a path \( P_1 \) at \( w \). Now the vertex \( v \) is the unique branch vertex, so we have obtained the pruning \( T_v \) of \( T_v \), where \( deg_{T_v}(u) \leq 2 \) for every \( u \in V(T_v) - \{v\} \) (see Figure 1(d)).

By Lemma 1, we may delete the two \( P_5 \) attached at \( v \) (with \( x \) and \( u \)) and finally we obtain the pruning \( T_v \) of \( T_v \) where \( deg_{T_v}(u) \leq 2 \) for every \( u \in V(T_v) - \{v\} \) and \( d(u,v) \leq 4 \) (see Figure 1(e)).

Since \( |L^1(v)| \cup L^3(v) = 0 \) and \( |L^2(v)| \cup L^3(v) = 3 \), then by Lemma 3, \( v \in N_L(T_v) \), and by Corollary 1, \( T \) is not a \( \gamma_L \)-excellent tree.

---

*Fig. 1*
Lemma 2. Let \( T \) be a tree rooted at a vertex \( v \) and \( w \) a branch vertex at maximum distance from \( v \) (\( w \neq v \)). Set \( k_1 = |L^1(w)| \), \( k_2 = |L^2(w)| \), \( k_3 = |L^3(w)| \), \( k_4 = |L^4(w)| \), and \( k_5 = |L^5(w)| \). If:

(a) \( k_1 \geq 1 \), let \( T' \) be the tree obtained from \( T \) by deleting \( D(w) \) and attaching a \( P_1 \) to \( w \).

(b) \( k_1 = 0 \) and \( k_3 \geq 1 \), let \( T' \) be the tree obtained from \( T \) by deleting \( D(w) \) and attaching a \( P_3 \) to \( w \).

(c) \( k_1 + k_3 = 0 \) and \( k_4 \geq 1 \), let \( T' \) be the tree obtained from \( T \) by deleting \( D(w) \) and attaching a \( P_4 \) to \( w \).

(d) \( k_1 + k_3 + k_4 = 0 \) and \( k_2 \geq 2 \), let \( T' \) be the tree obtained from \( T \) by deleting \( D(w) \) and attaching a \( P_5 \) to \( w \).

Then in each case:

(a) \( v \in A_L(T') \) if and only if \( v \in A_L(T) \).

(b) \( v \in N_L(T') \) if and only if \( v \in N_L(T) \).

Proof. For the sake of simplicity with use of Lemma 1, the tree \( T_0 \) will be simplified by replacing any \( w-x \) path with a \( w-x \) path of length \( j \), where \( j = 1, 2, 3, 4 \) if \( x \in L^1(w) \) for \( i = 1 \), \( 2, 3, 4 \), \( 5 \) respectively.

Let \( a_i, b_j, c_j, d_k, e_k, f_k, g_k, h_k, p, q_i, x_m \in D(w) \cap L(T) \), for \( 0 \leq i \leq k_1 \), \( 0 \leq j \leq k_2 \), \( 0 \leq k \leq k_3 \), \( 0 \leq l \leq k_4 \) and \( 0 \leq m \leq k_5 \).

Case (a). \( k_1 \geq 1 \).

Let \( T' = T - (D(w) - \{ a_1 \}) \).

Every \( \gamma_L(T') \)-set can be extended to an \( LDS \) of \( T \) by adding

\[
X = \{ a_i; i \in \{ 2, \ldots, k_1 \} \} \cup \{ b_j, j \in \{ 1, \ldots, k_2 \} \} \cup \{ e_k; k \in \{ 1, \ldots, k_3 \} \} \cup \{ g_k, k \in \{ 1, \ldots, k_3 \} \} \cup \{ s_m, m \in \{ 1, \ldots, k_5 \} \}.
\]

Let \( D' \) be an arbitrary \( \gamma_L(T') \)-set. We may assume that \( w \in D' \); otherwise, we replace \( a_1 \) with \( w \), then \( D = D' \cup X \) is an \( LDS \) of \( T \); so there is \( \gamma_L(T) \leq |D'| + |X| = \gamma_L(T') + (k_1 - 1) + k_2 + k_3 + 2k_4 + 2k_5 \).

On the other hand; let \( D \) be an arbitrary \( \gamma_L(T) \)-set, then \( D' = D \cap T' \) is an \( LDS \) of \( T' \). There follows \( |D' \cap D(w)| \geq (k_1 - 1) + k_2 + k_3 + 2k_4 + 2k_5 = |X| \) when \( a_1 \notin D \) or \( |D' \cap D(w)| > (k_1 - 1) + k_2 + k_3 + 2k_4 + 2k_5 = |X| \) when \( a_1 \in D \) (i.e., all \( a_i \in D \)), so when \( a_1 \notin D \), \( D' = D \cap D(w) \) and when \( a_1 \in D \), \( D' = (D - D \cap D(w)) \cup \{ a_1 \} \). In each case, \( \gamma_L(T') \leq |D'| \leq \gamma_L(T) - (k_1 - 1) + k_2 - k_3 - 2k_4 - 2k_5 \).

Thus \( \gamma_L(T) = \gamma_L(T') + (k_1 - 1) + k_2 + k_3 + 2k_4 + 2k_5 \).

(1) Suppose that \( v \in A_L(T') \) and let \( D \) be an arbitrary \( \gamma_L(T) \)-set. We have above seen that either \( D' = D - (D \cap D(w)) \) if \( w \notin D \) and \( a_1 \notin D \) or \( D' = (D - (D \cap D(w))) \cup \{ a_1 \} \) if \( w \notin D \) and \( a_1 \in D \) (that is \( D' = D \cap T' \)) is a \( \gamma_L(T') \)-set. Since \( v \in D' \subset D \), then \( v \in A_L(T) \).
Conversely, suppose that $v \in A_L(T)$ and let $S'$ be a $\gamma_L(T')$-set. We know that $S'$ can be extended to a $\gamma_L(T)-$set $S$ by adding the set $X$. So, $S = S' \cup X$ is a $\gamma_L(T)$-set. Since $v \in S$ and $v \notin D[w]$, $v \in A_L(T')$.

(2) Suppose now that $v \in N_L(T')$ and let $D$ be an arbitrary $\gamma_L(T')$-set. As seen above, $D' = D \cap T'$ is a $\gamma_L(T')$-set. Since $v \notin D'$ and $v \notin D[w]$, then $v \notin D$ and thus $v \in N_L(T)$. Conversely, assume that $v \in N_L(T)$ and let $S'$ be a $\gamma_L(T')$-set. Then $S'$ can be extended to a $\gamma_L(T)-$set $S$ by adding the set $X$, and since $v \notin S$ and $v \notin D[w]$, then $v \notin S'$ and so $v \in N_L(T)$.

**Case (b)**, $k_1 = 0$ and $k_3 \geq 1$.

Let $T' = T - (D(w) - \{d_1, e_1, f_1\})$.

Every $\gamma_L(T')$-set can be extended to an $LDS$ of $T$ by adding

$$X = \{b_j, j \in \{1, \ldots, k_2\} \} \cup \{e_k; k \in \{2, \ldots, k_3\} \} \cup$$

$$\cup \{g_l, p_l; \in \{1, \ldots, k_4\} \} \cup \{s_m, u_m; m \in \{1, \ldots, k_5\} \}.$$ 

Let $D'$ be an arbitrary $\gamma_L(T')$-set. Without loss of generality, we may assume that $D'$ contains $e_1$ and $w$; otherwise, we replace $f_1$ with $e_1$ in the first case and $d_1$ (or $f_1$) with $w$ in the second case. So $\gamma_L(T) \leq \gamma_L(T') + k_2 + (k_3 - 1) + 2k_4 + 2k_5$. On the other hand, let $D$ be an arbitrary $\gamma_L(T)$-set. We may assume that $w \in D$; otherwise, we replace $d_1$ or $f_1$ with $w$ and take $e_1$ in $D$; then $D' = D \cap T'$ is an $LDS$ of $T'$ and clearly $|D \cap D(w) - \{d_1, e_1, f_1\}| \geq k_2 + (k_3 - 1) + 2k_4 + 2k_5 = |X|$, then $\gamma_L(T') \leq |D'| = |D| - |D \cap D(w) - \{d_1, e_1, f_1\}| \leq \gamma_L(T) - k_2 - (k_3 - 1) - 2k_4 - 2k_5$. Thus $\gamma_L(T) = \gamma_L(T') + k_2 + (k_3 - 1) + 2k_4 + 2k_5$.

In this case and also in cases (c), (d), (e) and (f), the proofs of part (1) and (2) are similar to the proof of “case (a) (part (1) and part (2))”. So the similar proofs are omitted.

**Case (c)**, $k_1 + k_3 = 0$ and $k_4 \geq 1$.

Let $T' = T - (D(w) - \{g_1, h_1, p_1, q_1\})$.

Every $\gamma_L(T')$-set can be extended to an $LDS$ of $T$ by adding

$$X = \{b_j, j \in \{1, \ldots, k_2\} \} \cup \{g_l, p_l; \in \{2, \ldots, k_4\} \} \cup \{s_m, u_m; m \in \{1, \ldots, k_5\} \}.$$ 

Let $D'$ be an arbitrary $\gamma_L(T')$-set. Then $\gamma_L(T) \leq \gamma_L(T') + k_2 + 2(k_4 - 1) + 2k_5$. On the other hand, let $D$ be an arbitrary $\gamma_L(T)$-set. Without loss of generality, we may assume that $D$ contains $g_1$ and $p_1$; otherwise, we replace $h_1$ with $g_1$ and $q_1$ with $p_1$, then $D' = D \cap T'$ is an $LDS$ of $T'$ and clearly $|D \cap D(w) - \{g_1, h_1, p_1, q_1\}| \geq k_2 + 2(k_4 - 1) + 2k_5 = |X|$, so $\gamma_L(T) \leq |D'| = |D| - |D \cap D(w) - \{g_1, h_1, p_1, q_1\}| \leq \gamma_L(T) - k_2 - 2(k_4 - 1) - 2k_5$. Thus $\gamma_L(T) = \gamma_L(T') + k_2 + 2(k_4 - 1) + 2k_5$.

**Case (d)**, $k_1 + k_3 + k_4 = 0$ and $k_2 \geq 2$.

Let $T' = T - (D(w) - \{b_1, c_1, b_2, c_2\})$ and let $T'' = T' - \{b_1, c_1, b_2, c_2\} + P_4$, that is we replace $\{b_1, c_1, b_2, c_2\}$ by attaching $P_4$ to $w$, where $P_4 = b_1c_1b_2c_2$.

Clearly, every $\gamma_L(T')$-set of $T'$ is a $\gamma_L(T'')$-set of $T''$ and can be extended to an $LDS$ of $T$ by adding

$$X = \{b_j, j \in \{3, \ldots, k_2\} \} \cup \{s_m, u_m; m \in \{1, \ldots, k_5\} \}.$$
Let $D'$ be an arbitrary $\gamma_L(T')$-set. So $\gamma_L(T) \leq \gamma_L(T') + (k_2 - 2) + 2k_5$. On the other hand, let $D$ be an arbitrary $\gamma_L(T)$-set. Without loss of generality, we may assume that $D$ contains $s_1$ and $u_1$; otherwise, we replace $r_1$ with $w$ and take $s_1$ and $u_1$ in $D$. Then $D' = D \cap T'$ is an LDS of $T'$ and clearly $|D \cap (D(w) - \{b_1, c_1, c_2\})| \geq (k_2 - 2) + 2k_5 = |X|$. So $\gamma_L(T') \leq |D'| = |D| - |D \cap (D(w) - \{b_1, c_1, c_2\})| \leq \gamma_L(T) - (k_2 - 2) - 2k_5$. Thus $\gamma_L(T) = \gamma_L(T') + (k_2 - 2) + 2k_5$.

**Case (e).** $k_1 + k_3 + k_4 = 0$ and $k_2 = 1$.

Let $T' = T - (D(w) - \{b_1, c_1\})$.

Every $\gamma_L(T')$-set can be extended to an LDS of $T$ by adding

$$X = \{s_m, u_m; m \in \{1, \ldots, k_5\}\}.$$ 

Let $D'$ be an arbitrary $\gamma_L(T')$-set. So $\gamma_L(T) \leq \gamma_L(T') + |X| = \gamma_L(T') + 2k_5$. On the other hand, let $D$ be an arbitrary $\gamma_L(T)$-set. Without loss of generality, we may replace $r_j$ with $w$ and take $s_j, u_j$ in $D$ if $w \notin D$ and $r_j \in D$, then $D' = D \cap T'$ is an LDS of $T'$ and clearly $|D \cap (D(w) - \{b_1, c_1\})| \geq |X| = 2k_5$. So $\gamma_L(T') \leq |D'| = |D| - |D \cap (D(w) - \{b_1, c_1\})| \leq \gamma_L(T) - |X| = \gamma_L(T) - 2k_5$. Thus $\gamma_L(T) = \gamma_L(T') + |X| = \gamma_L(T') + 2k_5$.

**Case (f).** $k_1 + k_3 + k_4 + k_5 = 0$ and $k_3 \geq 2$.

Let $T' = T - (D(w) - \{r_1, s_1, t_1, u_1, x_1\})$.

Every $\gamma_L(T')$-set can be extended to an LDS of $T$ by adding

$$X = \{s_m, u_m; m \in \{2, \ldots, k_5\}\}.$$ 

Let $D'$ be an arbitrary $\gamma_L(T')$-set. So we have $\gamma_L(T) \leq \gamma_L(T') + |X| = \gamma_L(T') + 2(k_3 - 1)$. On the other hand, let $D$ be an arbitrary $\gamma_L(T)$-set. Without loss of generality, we may replace $r_j$ with $w$ and take $s_j, u_j$ in $D$ if $w \notin D$ and $r_j \in D$, then $D' = D \cap T'$ is an LDS of $T'$ and clearly $|D \cap (D(w) - \{r_1, s_1, t_1, u_1, x_1\})| \geq |X| = 2(k_3 - 1)$, then $\gamma_L(T') \leq |D'| = |D| - |D \cap (D(w) - \{r_1, s_1, t_1, u_1, x_1\})| \leq \gamma_L(T) - 2(k_3 - 1)$. Thus $\gamma_L(T) = \gamma_L(T') + |X| = \gamma_L(T') + 2(k_3 - 1)$.

3. CHARACTERIZATIONS

The following lemma gives a necessary and sufficient condition for the special vertex $v$ of a nontrivial tree $T_v$ to be in $A_L(T)$ (resp. in $N_L(T)$).

**Lemma 3.** Let $T$ be a nontrivial tree rooted at a vertex $v$ such that $\text{deg}_T(v) \leq 2$ for every vertex $u \in V(T) - \{v\}$. Then:

1) $v \in A_L(T)$ if and only if either $(|L^3(v)| \geq 2)$, or $(|L^3(v)| = 1$ and $|L^1(v)| \geq 1)$.
2) $v \in N_L(T)$ if and only if $(|L^3(v) \cup L^1(v)| = 0$ and $|L^2(v) \cup L^1(v)| \geq 2$) or $(|L^3(v) \cup L^2(v) \cup L^1(v)| = 0$ and $|L^1(v)| = 1$).

**Proof.** Clearly if $L(v) = L^0(v)$, that is all the vertices of $T$ are at distance $j$ from $v$, then $j \equiv 0 \pmod{5}$. In this case we may obtain $T$ from $T_v^w = P_5$ by applying Lemma 1 and then $v \notin A_L(T_v^w) \cup N_L(T_v^w)$, therefore $v \notin A_L(T) \cup N_L(T)$. 

So now we suppose that \( L(v) \neq L^0(v) \). According to Lemma 1, it will be sufficient to prove the lemma by considering the tree \( T_v^- \) in which every vertex distinct from \( v \) has degree at most 2 and every leaf of \( T_v^- \) is at distance at most 4 from \( v \) (that is if \( T_v^- \) contains leaves at distance 5 from \( v \), we just consider the remaining tree obtained by removing the paths \( P_5 \) attached to \( v \)). So we may assume that no path \( P_3 \) is attached to \( v \) in \( T_v^- \).

Let \( k_i \) denote the number of leaves in \( T_v^- \) at distance \( i \) from \( v \), where \( i = 1, 2, 3, 4 \). So \( v \in A_L(T) \) (resp. \( N_L(T) \)) if and only if \( v \in A_L(T_v^-) \) (resp. \( N_L(T_v^-) \)).

If \( v \) is a pendant vertex, then \( T_v^- \) is a path \( P_n \) with \( 5 \geq n \geq 2 \) and \( |L_1(v) \cup L_2(v) \cup L_3(v) \cup L_4(v)| = 1 \). Then, by Observations 1 and 2, \( v \notin A_L(T_v^-) \cup N_L(T_v^-) \) if and only if \( n = 2, 3, 4 \), and \( v \notin N_L(T_v^-) \) if and only if \( n = 5 \), which yields the result. We will from now assume that \( v \) is not a pendant vertex. Thus \( v \) has degree at least 2.

Let \( D \) be a \( \gamma_L(T_v^-) \)-set. For a leaf \( t_i \) at distance \( i \) from \( v \), we denote the \( v-t_i \) path by \( v, t_1, \ldots, t_i \). It remains now to examine the following cases:

**Case 1.** \( k_3 \geq 2 \).

Let \( x_3 \) and \( y_3 \) be two leaves at distance 3 from \( v \). Assume that \( v \notin D \). Then \( D \) must contain two vertices from each of \( \{x_1, \ldots, x_3\} \) and \( \{y_1, \ldots, y_3\} \). Without loss of generality, suppose that \( x_i, y_i \in D \) for \( i = 1, 2 \). In this case, \( D' = \{v\} \cup (D - \{x_1, y_1\}) \) is an \( \text{LDS} \) of \( T_v^- \) of size \( \gamma_L(T_v^-) - 1 \), a contradiction. Then \( v \in D \) and \( v \in A_L(T) \).

**Case 2.** \( k_3 = 1 \) and \( k_1 \geq 1 \).

Let \( x_3 \) and \( y_1 \) be two leaves at distances 3 and 1 from \( v \), respectively, and assume that \( v \notin D \). Then \( D \) must contain \( y_1 \) and two vertices from the \( x_1-x_3 \) path. Without loss of generality, we assume that \( x_i \in D \) for \( 1 \leq i \leq 2 \). Then \( D' = \{v\} \cup (D - \{y_1, x_1\}) \) is an \( \text{LDS} \) of \( T_v^- \) of size less than \( D \), a contradiction. Then \( v \in D \) and \( v \in A_L(T_v^-) \).

**Case 3.** \( k_1 + k_3 = 0 \) and \( k_2 + k_4 \geq 2 \).

**Subcase 3.1.** \( k_2 \geq 2 \) and \( k_1 + k_3 = 0 \).

Let \( x_2, y_2 \) be two leaves at distance 2 from \( v \) and suppose that \( v \notin D \). Then \( D \) must contain one vertex from each of \( \{x_1, x_2\} \) and \( \{y_1, y_2\} \). Without loss of generality, we assume that \( x_2, y_2 \in D \). In this case, \( D' = (D - \{v, x_2, y_2\}) \cup \{x_1, y_1\} \) is an \( \text{LDS} \) of \( T_v^- \) of size \( \gamma_L(T_v^-) - 1 \), because \( k_1 + k_3 = 0 \); a contradiction. Hence \( v \notin D \) and \( v \in N_L(T_v^-) \).

**Subcase 3.2.** \( k_4 \geq 2 \) and \( k_1 + k_3 = 0 \).

Let \( x_4, y_1 \) be two leaves at distance 4 from \( v \) and suppose that \( v \in D \). Then \( D \) must contain two vertices from each of \( \{x_1, \ldots, x_4\} \) and \( \{y_1, \ldots, y_4\} \). Without loss of generality, we assume that \( x_i, y_i \in D \) for \( i = 2, 3 \). In this case, \( D' = (D - \{v, x_2, x_3, y_2, y_3\}) \cup \{x_1, x_3, y_1, y_3\} \) is an \( \text{LDS} \) of \( T_v^- \) of size \( \gamma_L(T_v^-) - 1 \), because \( k_1 + k_3 = 0 \); a contradiction. Hence \( v \notin D \) and \( v \in N_L(T_v^-) \).

**Subcase 3.3.** \( k_1 = 1, k_2 = 1 \) and \( k_1 + k_3 = 0 \).

Let \( x_4, y_2 \) be two leaves at distances 4 and 2 from \( v \) and suppose that \( v \in D \). Then \( D \) must contain two vertices from \( \{x_1, \ldots, x_4\} \) and one vertex from \( \{y_1, y_2\} \). Without loss of generality, we assume that \( x_2, x_3, y_2 \in D \). In this case, \( D' = (D - \{v, x_2, y_2\}) \cup \{x_1, x_3, y_1, y_3\} \) is an \( \text{LDS} \) of \( T_v^- \) of size \( \gamma_L(T_v^-) - 1 \), because \( k_1 + k_3 = 0 \); a contradiction. Hence \( v \notin D \) and \( v \in N_L(T_v^-) \).
\( \{v, x_2, x_3, y_2\} \cup \{x_1, x_3, y_1\} \) is an LDS of \( T_v^+ \) of size \( \gamma_L(T_v^+) - 1 \), because \( k_1 + k_3 = 0 \); a contradiction. Hence \( v \not \in D \) and \( v \in N_L(T_v^+) \).

**Case 4.** \( k_1 + k_2 + k_3 = 0 \) and \( k_4 = 1 \).

Let \( x_4 \) be a leaf at distance 4 from \( v \) and suppose that \( v \in D \). Then \( D \) must contain two vertices from \( \{x_1, \ldots, x_4\} \). Without loss of generality, we assume that \( x_2, x_3 \in D \). In this case, \( D' = (D - \{x_2, x_3\}) \cup \{x_1, x_3\} \) is an LDS of \( T_v^+ \) of size \( \gamma_L(T_v^+) - 1 \), because \( k_1 + k_2 + k_3 = 0 \); a contradiction. Hence \( v \not \in D \) and \( v \in N_L(T_v^+) \).

Conversely, according to cases 1, 2, 3 above and the fact that \( A_L(T_v^+) \cap N_L(T_v^+) = \emptyset \), it remains to examine the following cases to complete the proof:

**Case 5.** \( k_3 = 3 \) and \( k_1 \geq 1 \).

All the leaves are at distances 1, 2 or 4 from \( v \). Since \( k_1 \geq 1 \), let \( x_1 \) be a leaf adjacent to \( v \); by Observation 1, there exists \( D \) such that \( v \in D \). Clearly we may deduce a \( \gamma_L(T_v^+) \)-set \( D' \) which contains \( x_1 \) and not \( v \), implying that \( v \not \in A_L(T_v^+) \cap N_L(T_v^+) \).

**Case 6.** \( k_3 = 1 \) and \( k_1 = 0 \).

Let \( x_3 \) be a leaf at distance 3 from \( v \). If \( v \in D \), then \( D \) must contain \( x_2 \) or \( x_3 \). In this case, we can deduce a \( \gamma_L(T_v^+) \)-set \( D' \) which contains \( x_1 \) and not \( v \). If \( v \not \in D \), then \( D \) must contain two vertices from \( \{x_1, x_2, x_3\} \). Without loss of generality, we assume that \( x_2, x_3 \in D \). In this case, \( D' = (D - \{x_3\}) \cup \{v\} \) is a \( \gamma_L(T_v^+) \)-set which contains \( v \), implying in all cases that \( v \not \in A_L(T_v^+) \cap N_L(T_v^+) \).

**Case 7.** \( k_3 + k_1 = 0 \) and \( k_2 + k_4 = 1 \).

**Subcase 7.1.** \( k_4 + k_3 + k_1 = 0 \) and \( k_2 = 1 \). Thus \( T_v^+ \) is \( P_3 \). Therefore, this case has been considered at the beginning, implying that \( v \not \in A_L(T_v^+) \cap N_L(T_v^+) \).

**Subcase 7.2.** \( k_3 + k_2 + k_1 = 0 \) and \( k_4 = 1 \). Thus \( T_v^+ \) is \( P_4 \). This case too has already been considered at the beginning, implying that \( v \in N_L(T_v^+) \).

**Case 8.** \( k_3 + k_4 + k_1 = 1 \).

Finally, this case has also been considered at the beginning, implying that \( v \not \in A_L(T_v^+) \cap N_L(T_v^+) \).

From Lemmas 2 and 3, our main result follows:

**Theorem 3.** Let \( v \) be a vertex of the tree \( T \), then:

- \( v \in A_L(T) \) if and only if \( v \in A_L(T_v^+) \).
- \( v \in N_L(T) \) if and only if \( v \in N_L(T_v^+) \).

Therefore, the following corollary holds true.

**Corollary 1.** \( T \) is a \( \gamma_L \)-excellent tree if and only if \( N_L(T_v^+) = \emptyset \) for every vertex \( v \) of \( T \). (That is in a pruning tree \( T_v^+ \) there is:

\[
|L^1(v) \cup L^1(v)| \neq 0 \quad \text{or} \quad |L^2(v) \cup L^1(v)| \leq 1 \quad \text{and} \quad |L^3(v) \cup L^2(v) \cup L^1(v)| \neq 0 \quad \text{or} \quad |L^4(v)| \neq 1).
\]

It is easy to verify that a pruning tree can be found in a polynomial time with the process defined above. So, if \( N_L(T_v^+) = \emptyset \) for every vertex \( v \) of \( T \), then the \( \gamma_L \)-excellence property of a tree can be verified in a polynomial time.

**Corollary 2.** \( \gamma_L \)-excellent trees can be recognized in a polynomial time.
Acknowledgements
We thank the referee for his/her careful reading of this paper and a number of valuable comments.

REFERENCES


Mostafa Blidia
m_blidia@yahoo.fr

University of Blida
LAMDA-RO, Department of Mathematics
B.P. 270, Blida, Algeria

Rahma Lounes
rlounes@yahoo.fr

University of Blida
LAMDA-RO, Department of Mathematics
B.P. 270, Blida, Algeria

Received: October 2, 2007.
Revised: May 2, 2008.
Accepted: May 11, 2008.