To the memory of Professor Andrzej Lasota

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REMARKS ON THE STABILITY
OF SOME QUADRATIC FUNCTIONAL EQUATIONS

Abstract. Stability problems concerning the functional equations of the form
\[ f(2x + y) = 4f(x) + f(y) + f(x + y) - f(x - y), \]
and
\[ f(2x + y) + f(2x - y) = 8f(x) + 2f(y) \]
are investigated. We prove that if the norm of the difference between the LHS and the RHS of one of equations (1) or (2), calculated for a function \( g \) is say, dominated by a function \( \varphi \) in two variables having some standard properties then there exists a unique solution \( f \) of this equation and the norm of the difference between \( g \) and \( f \) is controlled by a function depending on \( \varphi \).

Keywords: quadratic functional equations, stability.

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1. INTRODUCTION

In paper [8], C. Park and J. Su An considered the following functional equations
\[ f(2x + y) = 4f(x) + f(y) + f(x + y) - f(x - y), \] (1)
and
\[ f(2x + y) + f(2x - y) = 8f(x) + 2f(y) \] (2)
in the class of functions transforming a real linear space \( X \) into another real linear space \( Y \). They have proved that any of equations (1) and (2) and the quadratic functional equation
\[ Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y), \quad x, y \in X \] (3)
are equivalent. Stability of equation (3) was widely considered (cf., e.g., [2, 3, 5, 6]). In the case of \( Y \) a Banach space, the authors of [8] also considered the problem of
Hyers-Ulam-Rassias stability (see [4]) of equations (1) and (2). In particular, they have proved that if \( \alpha \in (0, 2) \) is a constant and
\[
\|g(2x + y) - 4g(x) - g(y) - g(x + y) + g(x - y)\| \leq \theta(\|x\|^\alpha + \|y\|^\alpha),
\]
for all \( x, y \in X \), then there exists a unique quadratic function \( Q : X \to Y \) such that
\[
\|g(x) - Q(x)\| \leq \frac{\theta}{2^\alpha - 4}\|x\|^\alpha, \quad x \in X.
\]
In this note we will show that equations (1), (2) and (3) are equivalent in a more general case. We will also show that in the stability results one may replace the right-hand-side of (4) by a function \( \varphi \) in two variables having some natural properties (cf. [1,6]). In particular, we cover the case of inequality (4) with \( \alpha \neq 2 \). However, we obtain somewhat larger estimation constant.

2. EQUIVALENCE OF EQUATIONS (1), (2) AND (3)

**Theorem 1.** Let \( X \) and \( Y \) be commutative groups, the latter without elements of order two. Then in the class of functions transforming \( X \) into \( Y \), equations (1), (2) and (3) are equivalent.

**Proof.** Assume that \( f : X \to Y \) is a solution of equation (1). Putting \( x = y = 0 \) in (1) we obtain \( 4f(0) = 0 \), whence \( f(0) = 0 \). Setting \( y = 0 \) in (1), we get \( f(2x) = 4f(x) \) and for \( x = 0 \) it follows from (1) that \( f(y) = f(-y) \). For arbitrary \( x, y \in X \), there is
\[
4f(x + y) + 4f(x - y) = f(2x + 2y) + f(2x - 2y) =
\]
\[
= 4f(x) + f(2y) + f(x + 2y) - f(x - 2y) +
\]
\[
+ 4f(x) + f(2y) + f(x - 2y) - f(x + 2y) =
\]
\[
= 8f(x) + 8f(y),
\]
which means that \( f \) satisfies equation (3).

Assume that \( Q \) is a solution of equation (3). Then \( Q \) is even and
\[
Q(2x + y) + Q(-y) = Q(x + (x + y)) + Q(x - (x + y)) =
\]
\[
= 2Q(x) + 2Q(x + y) =
\]
\[
= 2Q(x) + Q(x + y) + [2Q(x) + 2Q(y) - Q(x - y)] =
\]
\[
= 4Q(x) + Q(x + y) + 2Q(y) - Q(x - y).
\]
Therefore, \( Q \) fulfils equation (1).

Assume that \( f \) satisfies (2). Setting \( x = y = 0 \), we get \( 8f(0) = 0 \) and hence \( f(0) = 0 \). If \( y = 0 \), then \( f(2x) = 4f(x) \) and if \( x = 0 \), then \( f(-y) = f(y) \). Therefore,
\[
4f(x + y) + 4f(x - y) = f(2x + 2y) + f(2x - 2y) = 8f(x) + 2f(2y) =
\]
\[
= 8f(x) + 8f(y),
\]
which means that \( f \) satisfies (3).
Now assume that $Q$ satisfies (3). Then $Q(0) = 0$, $Q(y) = Q(-y)$ and $Q(2x) = 4Q(x)$. Thus

$$Q(2x + y) + Q(2x - y) = 2Q(2x) + 2Q(y) = 8Q(x) + 2Q(y),$$

which ends the proof.

**Remark 1.** The assumption that $Y$ has no elements of order two is essential. To see this, consider the group \{0, 1, 2, 3, 4, 5, 6, 7\} with the usual addition mod 8. Then $f \equiv 6$ is a solution of (1) but it does not satisfy equation (3). Moreover, $f \equiv 1$ is a solution of (2) but it satisfies neither equation (1) nor (3).

3. GENERAL LEMMA ON STABILITY

In the rest of the paper we assume that:

— $X$ is a commutative group,
— $Y$ is a real Banach space.

Moreover, we use the convention:

— $X^* = X \setminus \{0\}$;
— If not stated otherwise, any formula containing variables $x$ and/or $y$ is valid for all $x, y \in X^*$.

We start with some lemmas.

**Lemma 1.** Let $g : X \to Y$ be a function satisfying the inequality

$$\left\| \sum_{i=1}^{r} \alpha_i g(\gamma_i x + \delta_i y) \right\| \leq \varphi(x, y),$$

where we are given: a positive integer $r$, real constants $\alpha_i$, integer constants $\gamma_i, \delta_i, i \in \{1, \ldots, r\}$ such that $\gamma_i \delta_i \neq 0$ for some $i \in \{1, \ldots, r\}$ and $\delta_i \gamma_j \neq \delta_j \gamma_i$ for every $j \neq i, j \in \{1, \ldots, r\}$, real constants $\lambda_n, n \in \mathbb{N}, \lambda_0 = 1$, integer constants $\beta_n, n \in \mathbb{N}, \beta_0 = 1$, while $\varphi : X^* \times X^* \to [0, \infty)$ is a function satisfying the conditions

$$\begin{cases} 
\lim_{n \to \infty} \lambda_n \varphi(\beta_n x, \beta_n y) = 0; \\
\sum_{n=0}^{\infty} \lambda_n \varphi(\beta_n x, \beta_n x) < \infty.
\end{cases}$$

If there exists a constant $K > 0$ such that

$$\|\lambda_{n+1} g(\beta_{n+1} x) - \lambda_n g(\beta_n x)\| \leq K \lambda_n \varphi(\beta_n x, \beta_n x), n \in \mathbb{N} \cup \{0\},$$

then for every $x \in X$ the sequence $(\lambda_n g(\beta_n x))_{n \in \mathbb{N}}$ converges to a function $f : X \to Y$ fulfilling the equation

$$\sum_{i=1}^{r} \alpha_i f(\gamma_i x + \delta_i y) = 0$$

in $X^* \times X^*$. 

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and the estimate
\[ \|g(x) - f(x)\| \leq K \sum_{n=0}^{\infty} \lambda_n \varphi(\beta_n x, \beta_n x). \] \hspace{1cm} (9)

**Proof.** It follows from (7) that for given positive integers \( n, k \) there is
\[ \|\lambda_{n+k} g(\beta_{n+k} x) - \lambda_n g(\beta_n x)\| \leq \sum_{j=0}^{k-1} \|\lambda_{n+j+1} g(\beta_{n+j+1} x) - \lambda_{n+j} g(\beta_{n+j} x)\| \leq K \sum_{j=n}^{n+k-1} \lambda_j \varphi(\beta_j x, \beta_j x). \]

Therefore \((\lambda_n g(\beta_n x))_{n \in \mathbb{N}}\) is a Cauchy sequence, whence it is convergent. Define a function \( f^* : X^* \to Y \) by the equality
\[ f^*(x) = \lim_{n \to \infty} \lambda_n g(\beta_n x). \]

In (5) let us put \( \beta_n x \) instead of \( x \), \( \beta_n y \) instead of \( y \) and multiply both sides of (5) by \( \lambda_n \). On account of (6), taking the limit as \( n \) tends to infinity, we obtain
\[ \sum_{i=1}^{r} \alpha_i f(\gamma_i x + \delta_i y) = 0, \]
where
\[ f(x) = \begin{cases} f^*(x), & x \in X^*, \\ \lim_{n \to \infty} \lambda_n g(0), & x = 0. \end{cases} \]

Moreover,
\[ \|g(x) - \lambda_n g(\beta_n x)\| \leq \sum_{k=0}^{n-1} \|\lambda_{k+1} g(\beta_{k+1} x) - \lambda_k g(\beta_k x)\| \leq K \sum_{k=0}^{\infty} \lambda_k \varphi(\beta_k x, \beta_k x). \]

As \( n \) tends to infinity, we get
\[ \|g(x) - f(x)\| \leq K \sum_{n=0}^{\infty} \lambda_n \varphi(\beta_n x, \beta_n x). \]

\[ \square \]

4. LEMMAS ON EQUATIONS (1) AND (2)

**Lemma 2.** If \( f : X \to Y \) satisfies (1) for all \( x, y \in X^* \), then it satisfies (1) for all \( x, y \in X \).
Proof. Setting, successively, \( y = x, y = 2x, y = -x, x \neq 0 \) in (1), we get
\[
\begin{align*}
  f(3x) &= 5f(x) + f(2x) - f(0), \\
  f(4x) &= 4f(x) + f(2x) + f(3x) - f(-x), \\
  f(2x) &= 3f(x) + f(-x) + f(0).
\end{align*}
\]
Adding (10) and (11), we obtain
\[
f(4x) = 9f(x) + 2f(2x) - f(-x) - f(0),
\]
and, thanks to (12),
\[
3f(2x) + f(-2x) + f(0) = 9f(x) + 2f(2x) - f(-x) - f(0).
\]
Applying (12) once more, we observe that
\[
3f(x) + f(-x) + f(0) + 3f(-x) + f(x) + f(0) + f(0) = 9f(x) - f(-x) - f(0),
\]
which implies that
\[
f(0) = 5[f(x) - f(-x)].
\]
Consequently, \( f(0) = 0, f(-x) = f(x) \) and \( f(2x) = 2f(x) \). Now it is easy to verify that (1) is fulfilled for all \( x, y \in X \).

Lemma 3. If \( f : X \to Y \) satisfies (2) for all \( x, y \in X^* \), then it satisfies (2) for all \( x, y \in X \).

Proof. Setting \( y = x, x \neq 0 \) in (2), we get
\[
f(3x) = 9f(x),
\]
and the substitution \( y = -x, x \neq 0 \) in (2) yields \( f(3x) = 7f(x) + 2f(-x) \), whence
\[
f(x) = f(-x).
\]
If we put, successively, \( y = 2x, y = 4x, x \neq 0 \) in (2), then
\[
\begin{align*}
  f(4x) + f(0) &= 8f(x) + 2f(2x), \\
  f(6x) + f(2x) &= 8f(x) + 2f(4x).
\end{align*}
\]
On account of (14) and (13) we obtain
\[
9f(2x) + f(2x) = 8f(x) + 16f(x) + 4f(2x) - 2f(0)
\]
whence
\[
f(2x) = 4f(x) - \frac{1}{3}f(0).
\]
According to (15) and (13),
\[
f(6x) = 4f(3x) - \frac{1}{3}f(0) = 36f(x) - \frac{1}{3}f(0).
\]
On the other hand, by virtue of (14) and (15), we get
\[36f(x) - \frac{1}{3}f(0) + f(2x) = 8f(x) + 2[8f(x) + 2f(2x) - f(0)]\]
whence
\[f(0) = 0 \quad \text{and} \quad f(2x) = 4f(x),\]
by (15). Using these equalities together with \(f(x) = f(-x)\), one can easily verify that (2) is fulfilled for all \(x, y \in X\).

5. STABILITY OF EQUATION (1)

We use these Lemmas in the proofs of Theorems 2 and 3, in which we put
\[D := \{(x, x), (-x, x), (x, -x), (-x, -x); x \in X^*\}.\]

**Theorem 2.** Let \(g : X \to Y\) be a function satisfying the inequality
\[\|g(2x + y) - 4g(x) - g(y) - g(x + y) + g(x - y)\| \leq \omega(x, y),\]
where \(\omega : X^* \times X^* \to [0, \infty)\) is a function fulfilling the following conditions:
\[
\begin{align*}
\lim_{n \to \infty} \frac{1}{9^n} \omega(3^n x, 3^n y) &= 0; \\
\sum_{n=0}^{\infty} \frac{1}{9^n} \omega(3^n u, 3^n v) &< \infty \quad \text{for all} \quad (u, v) \in D.
\end{align*}
\]
Then there exists a unique quadratic function \(Q : X \to Y\) satisfying the estimate
\[\|Q(x) - g(x)\| \leq \sum_{n=0}^{\infty} \frac{1}{9^{n+1}} \varphi(3^n x, 3^n x) + \psi(x),\]
where
\[\varphi(x, y) = \frac{1}{2} [\omega(x, y) + \omega(-x, y) + \omega(x, -y) + \omega(-x, -y)] \quad \text{(18)}\]
and
\[\psi(x) = \frac{1}{6} \left[ \frac{1}{2} [\omega(x, x) + \omega(-x, -x)] + \omega(x, -x) + \omega(-x, x) + \frac{1}{2} [\omega(x, -2x) + \omega(-x, 2x)] \right]. \quad \text{(19)}\]

**Proof.** First observe that, because of (18) and the limit properties of the function \(\omega\) stated in (16), the function \(\varphi\) satisfies (6). Let \(p\) and \(h\) be the even and the odd part, respectively, of the function \(g\), i.e.,
\[p(x) = \frac{g(x) + g(-x)}{2}, \quad h(x) = \frac{g(x) - g(-x)}{2}, \quad x \in X.\]

It is not hard to check that
\[\|p(2x + y) - 4p(x) - p(x + y) - p(y) + p(x - y)\| \leq \frac{1}{2} [\omega(x, y) + \omega(-x, -y)] \quad \text{(20)}\]
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and

$$
\|h(2x + y) - 4h(x) - h(x + y) - h(y) + h(x - y)\| \leq \frac{1}{2} \|\omega(x, y) + \omega(-x, -y)\|. \quad (21)
$$

Setting $y = x$ and then $y = -x$ in inequality (20), we get

$$
\|p(3x) - 5p(x) - p(2x) + p(0)\| \leq \frac{1}{2} \|\omega(x, x) + \omega(-x, -x)\|
$$

and

$$
\| - 4p(x) - p(0) + p(2x)\| \leq \frac{1}{2} \|\omega(x, -x) + \omega(-x, x)\|.
$$

Consequently,

$$
\|p(3x) - 9p(x)\| \leq \frac{1}{2} \|\omega(x, x) + \omega(-x, -x) + \omega(x, -x) + \omega(-x, x)\|.
$$

Thus, because of (18),

$$
\| \frac{1}{9} p(3x) - p(x) \| \leq \frac{1}{9} \varphi(x, x). \quad (22)
$$

It follows from inequality (22) that

$$
\| \frac{1}{9^n+1} p(3^{n+1}x) - \frac{1}{9^n} p(3^n x) \| \leq \frac{1}{9} \|\omega(3^n x, 3^n x)\|.
$$

Taking $\lambda_n = 9^{-n}, \beta_n = 3^n, n \in \mathbb{N}$, from Lemmas 1 and 2 and Theorem 1, we infer that there exists a quadratic function $Q : X \to Y$ fulfilling the estimate

$$
\|Q(x) - p(x)\| \leq \sum_{n=0}^{\infty} \frac{1}{9^n+1} \|\omega(3^n x, 3^n x)\|. \quad (23)
$$

(Note that (20) is of form (5) with $r = 5$.)

Now we are going to check the inequality

$$
\|h(x)\| \leq \psi(x). \quad (24)
$$

Since $h$ is odd, then $h(0) = 0$. Setting $y = x, y = -x$ and finally $y = -2x$ in (21), we obtain

$$
\|h(3x) - 5h(x) - h(2x)\| \leq \frac{1}{2} \|\omega(x, x) + \omega(-x, -x)\|,
$$

$$
\|2h(2x) - 4h(x)\| \leq \|\omega(x, -x) + \omega(-x, x)\|,
$$

$$
\|h(3x) - 3h(x) + h(2x)\| \leq \frac{1}{2} \|\omega(x, -2x) + \omega(-x, 2x)\|.
$$

Consequently, by the triangle inequality

$$
\|h(x)\| \leq \frac{1}{6} \|\omega(x, x) + \omega(-x, -x)\| + \|\omega(x, -x) + \omega(-x, x) + \omega(x, -2x) + \omega(-x, 2x)\|. \quad (25)
$$
By virtue of (19) and (25), we obtain estimate (24). Because of (23), this is (17).

To prove the uniqueness of $Q$ assume that $Q_1 : X \to Y$ is a quadratic function satisfying estimate (17). On account of a theorem proved in [7], $Q(3x) = 9Q(x)$ as well as $Q_1(3x) = 9Q_1(x)$, $x \in X$.

Thus
\[
\|Q(x) - Q_1(x)\| = \frac{1}{9^k}\|Q(3^kx) - Q_1(3^kx)\| \leq \\
\leq \frac{1}{9^k}\{\|Q(3^kx) - f(3^kx)\| + \|Q_1(3^kx) - f(3^kx)\|\} \leq \\
\leq \frac{1}{9^k}\{2\sum_{n=0}^{\infty} \frac{1}{9^{n+1}}\varphi(3^{n+k},3^{n+k}) + \psi(3^kx)\} = \\
= 2\sum_{j=k}^{\infty} \frac{1}{9^{j+1}}\varphi(3^j x, 3^j x) + \frac{1}{9^k}\psi(3^kx).$

By our assumption, the last expression tends to zero, as $k \to \infty$. This completes the proof of Theorem 2. \hfill \Box

**Theorem 3.** Assume that $X$ is a commutative group uniquely divisible by 3. Let $g : X \to Y$ be a function satisfying the inequality
\[
\|g(2x + y) - 4g(x) - g(y) - g(x + y) + g(x - y)\| \leq \omega(x,y),
\]
where $\omega : X^* \times X^* \to [0, \infty)$ is a function fulfilling the following conditions
\[
\begin{align*}
\lim_{n \to \infty} 9^n \omega(3^{-n}x, 3^{-n}y) &= 0; \\
\sum_{n=0}^{\infty} 9^n \omega(3^{-n}u, 3^{-n}v) &< \infty \quad \text{for all } (u, v) \in D.
\end{align*}
\]
Then there exists a unique quadratic function $Q : X \to Y$ satisfying the estimate
\[
\|Q(x) - g(x)\| \leq \sum_{n=0}^{\infty} 9^n \varphi(3^{-n-1} x, 3^{-n-1} x) + \psi(x),
\]
where $\varphi$ and $\psi$ are defined as in Theorem 2.

**Proof.** The proof runs similarly to the proof of Theorem 2. Let $\varphi$ and $\psi$ be defined as in Theorem 2. We consider inequalities (20) and (21). In the proof of Theorem 2, we obtained the following inequalities
\[
\|h(x)\| \leq \psi(x)
\]
and
\[
\|p(3x) - 9p(x)\| \leq \varphi(x, x).
\]
Replacing $x$ by $x^3$ in (26), we get
\[ \|9p(x^3) - p(x)\| \leq \varphi(x, x) \]
whence
\[ \|9^{n+1}p\left(\frac{x}{3^{n+1}}\right) - 9^n p\left(\frac{x}{3^n}\right)\| \leq 9^n \varphi(\frac{x}{3^{n+1}}, \frac{x}{3^n}). \]

It follows from the assumptions of Theorem 3 that
\[
\begin{cases}
\lim_{n \to \infty} 9^n \varphi(3^{-n-1}x, 3^{-n-1}y) = 0, \\
\sum_{n=0}^{\infty} 9^n \varphi(3^{-n-1}x, 3^{-n-1}x) < \infty.
\end{cases}
\]

On account of Lemmas 1 and 3, as well as Theorem 1, there exists a quadratic function $Q : X \to Y$ satisfying the following estimate
\[ \|Q(x) - p(x)\| \leq \sum_{n=0}^{\infty} 9^n \varphi\left(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}\right). \]

Therefore,
\[ \|Q(x) - g(x)\| \leq \sum_{n=0}^{\infty} 9^n \varphi\left(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}\right) + \psi(x). \]

The proof of the uniqueness of $Q$ is quite similar to that of Theorem 2.

6. STABILITY OF EQUATION (2)

**Theorem 4.** Let $g : X \to Y$ be a function satisfying the inequality
\[ \|g(2x + y) - g(2x - y) - 8g(x) - 2g(y)\| \leq \varphi(x, y), \tag{27} \]
where $\varphi : X^* \times X^* \to [0, \infty)$ is a function fulfilling the conditions:
\[
\begin{cases}
\lim_{n \to \infty} \frac{1}{9^n} \varphi(3^n x, 3^n y) = 0, \\
\sum_{n=0}^{\infty} \frac{1}{9^n} \varphi(3^n x, 3^n x) \text{ is convergent.}
\end{cases}
\]

Then there exists a unique quadratic function $Q : X \to Y$ satisfying the following estimate
\[ \|Q(x) - g(x)\| \leq \sum_{n=0}^{\infty} \frac{1}{9^{n+1}} \varphi(3^n x, 3^n x)). \tag{28} \]

**Proof.** Putting $y = x$ in (27), we get
\[ \|g(3x) - 9g(x)\| \leq \varphi(x, x). \tag{29} \]

Now we argue quite similarly as in the proof of Theorem 2 (the even case), obtaining the existence of a unique quadratic function $Q : X \to Y$ fulfilling estimate (28).
**Theorem 5.** Assume that $X$ is a commutative group uniquely divisible by 3. Let $g : X \to Y$ be a function satisfying the inequality

$$
\|g(2x + y) + g(2x - y) - 8g(x) - 2g(y)\| \leq \varphi(x, y),
$$

where $\varphi : X^* \times X^* \to [0, \infty)$ is a function fulfilling the following conditions:

- $\lim_{n \to \infty} 9^n \varphi(3^{-n}x, 3^{-n}y) = 0$;
- $\sum_{n=0}^{\infty} 9^n \varphi(3^{-n}x, 3^{-n}x) < \infty$.

Then there exists a unique quadratic function $Q : X \to Y$ satisfying the estimate

$$
\|Q(x) - g(x)\| \leq \sum_{n=0}^{\infty} 9^n \varphi(3^{-n-1}x, 3^{-n-1}x).
$$

**Proof.** Setting $\frac{x}{3}$ instead of $x$ in (29), we obtain

$$
\|g(x) - 9g(\frac{x}{3})\| \leq \varphi(\frac{x}{3}, \frac{x}{3}).
$$

Now we argue as in the proof of Theorem 3, obtaining a unique quadratic function $Q$ fulfilling the following estimate

$$
\|Q(x) - g(x)\| \leq \sum_{n=0}^{\infty} 9^n \varphi(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}).
$$

**Concluding remark.** Let $\theta \geq 0$ and $\omega(x, y) = \theta(\|x\|^\alpha + \|y\|^\alpha)$, or $\omega(x, y) = \theta\|x\|^\beta \|y\|^\beta$. Theorems 2 and 4 can be applied to these functions $\omega$ with $\alpha < 2$ and $\beta < 1$, whereas Theorems 3 and 5 – with $\alpha > 2$ and $\beta > 1$. Thus our theorems cover the cases considered by several other authors, and in particular, by C. Park and J. Su An.

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