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Vladimir Samodivkin

ON EQUALITY IN AN UPPER BOUND FOR THE ACYCLIC DOMINATION NUMBER

Abstract. A subset A of vertices in a graph G is acyclic if the subgraph it induces contains no cycles. The acyclic domination number $\gamma_a(G)$ of a graph G is the minimum cardinality of an acyclic dominating set of G. For any graph G with n vertices and maximum degree $\Delta(G)$, $\gamma_a(G) \leq n - \Delta(G)$. In this paper we characterize the connected graphs and the connected triangle-free graphs which achieve this upper bound.

Keywords: dominating set, acyclic set, independent set, acyclic domination number.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected and without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes, et al. [6]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G \rangle$. For a vertex x of G, $N(x, G)$ denote the set of all neighbours of x in G, $N[x, G] = N(x, G) \cup \{x\}$ and the degree of x is $deg(x, G) = |N(x, G)|$. The maximum degree in the graph G is denoted by $\Delta(G)$. For a set of vertices $S \subseteq V(G)$, $N(S, G)$ is the union of $N(x, G)$, when $x \in S$.

A *dominating set* in a graph G is such a set of vertices D that every vertex of G is either in D or is adjacent to an element of D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of all dominating sets of G. A set $I \subseteq V(G)$ is said to be *independent* if every pair of vertices in I is nonadjacent. Let $i(G)$ denote the size of a smallest maximal independent set. The number $i(G)$ is called the *independent domination number*. Note that any maximal independent set is dominating (cf. [1]). A subset of vertices A in a graph G is said to be *acyclic* if $\langle A, G \rangle$ contains no cycles. The *acyclic domination number* $\gamma_a(G)$ of a graph G is the minimum cardinality of acyclic dominating set in G. The concept of acyclic domination in graphs was introduced by

Hedetniemi et al. [7]. Note that for any graph G, $\gamma(G) \le \gamma_a(G) \le i(G)$ (cf. [7]). Further results on acyclic domination in graphs may be found in [2, 3, 8, 9].

A classical result by Berge [1] states that for any graph G, $\gamma(G) \le |V(G)| - \Delta(G)$. Domke et al. [4] noted that $i(G) \le |V(G)| - \Delta(G)$ and they were first to consider the problem of characterization of the graphs with $\mu(G) = |V(G)| - \Delta(G)$, $\mu \in \{\gamma, i\}$. Favaron and Mynhardt [5] continued the study on the problem and gave necessary and sufficient conditions for $\mu(G) = |V(G)| - \Delta(G)$, $\mu \in \{\gamma, i\}$. Since $\gamma_a(G) \le i(G)$, it follows that $\gamma_a(G) \le |V(G)| - \Delta(G)$. In this paper, we deal with graphs G satisfying $\gamma_a(G) = |V(G)| - \Delta(G)$.

We need the following notation. Let G be a graph, $x \in V(G)$, $deg(x, G) = \Delta(G)$. Let $B = N(x, G), C = V(G) - N[x, G]$ and $R = B - N(C, G)$. For each $c \in C$, let $B_c = N(c, G) \cap B$. Now we define the following three properties, the first two of which were also given in [4] and [5]:

$A_1(x)$: C is independent.
$A_2(x)$: Every vertex of B has at most one neighbour in C.
$A_3(x)$: For every non-empty subset C' of C, the subset $B' = (\cup_{u \in C'} B_u) \cup R$ of B is not dominated by an acyclic set with exactly one vertex in each B_u, $u \in C'$.

We also need the following results:

Theorem 1 ([4]). *For any graph G, $i(G) + \Delta(G) \le |V(G)|$. If $i(G) + \Delta(G) = |V(G)|$ and x is a vertex of G of degree $\Delta(G)$, then $A_1(x)$ holds.*

Theorem 2 ([5]). *A connected triangle-free graph G satisfies $\gamma(G) = |V(G)| - \Delta(G)$ if and only if G is bipartite with bipartition $A \cup B$, where $|A| \le |B|$, $\Delta(G) = |B|$, $deg(u, G) \le 2$ for each $u \in B$, and if $deg(u, G) = 2$ for all $u \in B$, then $deg(v, G) \ge 2$ for each $v \in A$.*

If G is a disconnected graph with $k \ge 2$ components, $\Delta(G) \ge 1$ and $i(G) + \Delta(G) = |V(G)|$, then all but one component of G are K_1-components, because of Theorem 1. This shows that it is sufficient to consider the connected graphs G with $\gamma_a(G) + \Delta(G) = |V(G)|$.

2. GRAPHS WHICH SATISFY $\gamma_a(G) + \Delta(G) = |V(G)|$

Theorem 3. *Let G be a connected graph, $x \in V(G)$ and $deg(x, G) = \Delta(G)$. Let $A_1(x)$ and $A_2(x)$ hold. Then:*

(i) $|V(G)| - \Delta(G) - 1 \le \gamma_a(G)$;
(ii) *if $y \in B$ with $N[y, G] \subseteq R \cup \{x\}$ then $\gamma_a(G) = |V(G)| - \Delta(G)$;*
(iii) *if $R = \emptyset$ and C contains a vertex of degree 1, then $i(G) = \gamma_a(G) = |V(G)| - \Delta(G) - 1$.*

Proof. (i) By $A_1(x)$ and $A_2(x)$, every acyclic dominating set of G, if it is to dominate C, then it has to have at least one vertex in each of the $|C|$ disjoint sets $\{u\} \cup B_u$, $u \in C$. Hence $\gamma_a(G) \ge |C| = |V(G)| - \Delta(G) - 1$.

(ii) No vertex $y \in B$ with $N[y, G] \subseteq R \cup \{x\}$ is dominated by any subset of $B - R$ and hence, if such y exists, then $\gamma_a(G) \geq |C| + 1 = |V(G)| - \Delta(G)$.

(iii) Let $R = \emptyset$, $u \in C$ and $N(u, G) = \{v\}$. By $A_1(x)$, $C - \{u\}$ is independent and by $A_2(x)$, $M = \{v\} \cup (C - \{u\})$ is independent, too. Since $R = \emptyset$, M is a dominating set of G. Hence M is an independent dominating set of G, of cardinality $|M| = |C| = |V(G)| - \Delta(G) - 1$. Since $\gamma_a(G) \leq i(G) \leq |M|$, the result follows by (i). $\square$

We now characterize the graphs G for which $\gamma_a(G) = |V(G)| - \Delta(G)$.

Theorem 4. *Let G be a connected graph.*

(i) *If $\gamma_a(G) = |V(G)| - \Delta(G)$, then $A_1(x), A_2(x)$ and $A_3(x)$ hold for every vertex x of degree $\Delta(G)$;*

(ii) *If $A_1(x), A_2(x)$ and $A_3(x)$ hold for some vertex x of degree $\Delta(G)$, then $\gamma_a(G) = |V(G)| - \Delta(G)$.*

Proof. (i) Let $x \in V(G)$, $deg(x, G) = \Delta(G)$ and $\gamma_a(G) = |V(G)| - \Delta(G)$. By Theorem 1, $A_1(x)$ is satisfied. Let $y \in N(x, G)$ and suppose that y is adjacent to r vertices in C with $r > 1$. Then x and y, together with the $|V(G)| - \Delta(G) - 1 - r$ vertices of C that are not in $N(y, G)$, form an acyclic dominating set of G, say M. Then $\gamma_a(G) \leq |M| = (|V(G)| - \Delta(G) - 1 - r) + 2 < |V(G)| - \Delta(G)$, a contradiction. Hence $A_2(x)$ is satisfied, too.

Note that if $u \in C$, then B_u is non-empty, since G is connected. Moreover, the sets B_u form a partition of $N(C, G) = B - R$ by $A_2(x)$. Suppose $A_3(x)$ does not hold and $C^{'}$ is a non-empty subset of C such that there exists an acyclic dominating set D of $(\cup_{u \in C'} B_u) \cup R$, with exactly one vertex in each B_u, $u \in C^{'}$. Then $M = D \cup (C - C^{'})$ is a dominating set of G, of cardinality $|M| = |C| = |V(G)| - \Delta(G) - 1 = \gamma_a(G) - 1$. Since $C - C^{'}$ is independent, D is acyclic and $N(D, G) \cap (C - C^{'}) = \emptyset$, it follows that M is an acyclic dominating set of cardinality $\gamma_a(G) - 1$, a contradiction. Hence $A_3(x)$ holds.

(ii) Let $x \in V(G)$, $deg(x, G) = \Delta(G)$ and $A_1(x)$, $A_2(x)$ and $A_3(x)$ hold. Let D be any acyclic dominating set of G with $|D| = \gamma_a(G)$; let $C^{''} = D \cap C$ and $C^{'} = C - C^{''}$. In order to dominate C, D has to have at least one vertex in each of the $|C|$ disjoint sets $\{u\} \cup B_u$, $u \in C$ (by $A_1(x)$ and $A_2(x)$). Hence the set $D \cap B$ contains at least $|C^{'}|$ vertices, one vertex in each B_u with $u \in C^{'}$. If $|D \cap B| > |C^{'}|$, then $\gamma_a(G) = |D| > |C^{'}| + |C^{''}| = |C| = |V(G)| - \Delta(G) - 1$. So, let $|D \cap B| = |C^{'}|$. By $A_3(x)$, $D \cap B$ does not dominate $B^{'} = B - \cup_{u \in C''} B_u$ and thus, to dominate $B^{'}$, $x \in D$. It follows that $|C^{'}| + |C^{''}| < |D|$, i.e., $|V(G)| - \Delta(G) \leq |D| = \gamma_a(G) \leq |V(G)| - \Delta(G)$. $\square$

Theorem 5. *Let G be a connected triangle-free graph. Then $\gamma_a(G) = |V(G)| - \Delta(G)$ if and only if $\gamma(G) = |V(G)| - \Delta(G)$.*

Proof. Necessity: If $\gamma_a(G) = 1$, then the result is obvious. So, let $2 \leq \gamma_a(G) = |V(G)| - \Delta(G)$ and let $x \in V(G)$ be of maximum degree. By $A_1(x)$, C is independent and since G is triangle-free, $B = N(x, G)$ is also independent. Hence G is bipartite with bipartition $A \cup B$, where $A = \{x\} \cup C$ and $B = N(x, G)$. Since A is an independent dominating set of cardinality $|A| = |V(G)| - \Delta(G) = \gamma_a(G)$, then $|A| \leq |B|$. Moreover, $|B| = \Delta(G)$, by the choice of x. By $A_2(x)$, $deg(u, G) \leq 2$ for each $u \in B$.

Suppose $deg(u, G) = 2$ for each $u \in B$. Then $deg(x, G) \geq 2$, $R = \emptyset$ and if some vertex u of $A - \{x\} = C$ has degree 1, then $\gamma_a(G) < |V(G)| - \Delta(G)$ because of Theorem 3 (iii), a contradiction. Now, by Theorem 2, $\gamma(G) = |V(G)| - \Delta(G)$.
Sufficiency: Obvious. □

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Vladimir Samodivkin
vlsam_fte@uacg.bg

University of Architecture Civil Engineering and Geodesy
Department of Mathematics
Hristo Smirnenski 1 Blv., 1046 Sofia, Bulgaria

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