Vladimir Samodivkin

ON EQUALITY IN AN UPPER BOUND
FOR THE ACYCLIC DOMINATION NUMBER

Abstract. A subset $A$ of vertices in a graph $G$ is acyclic if the subgraph it induces contains no cycles. The acyclic domination number $\gamma_a(G)$ of a graph $G$ is the minimum cardinality of an acyclic dominating set of $G$. For any graph $G$ with $n$ vertices and maximum degree $\Delta(G)$, $\gamma_a(G) \leq n - \Delta(G)$. In this paper we characterize the connected graphs and the connected triangle-free graphs which achieve this upper bound.

Keywords: dominating set, acyclic set, independent set, acyclic domination number.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected and without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes, et al. [6]. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G \rangle$. For a vertex $x$ of $G$, $N(x, G)$ denote the set of all neighbours of $x$ in $G$, $N[x, G] = N(x, G) \cup \{x\}$ and the degree of $x$ is $\deg(x, G) = |N(x, G)|$. The maximum degree in the graph $G$ is denoted by $\Delta(G)$. For a set of vertices $S \subseteq V(G)$, $N(S, G)$ is the union of $N(x, G)$, when $x \in S$.

A dominating set in a graph $G$ is such a set of vertices $D$ that every vertex of $G$ is either in $D$ or is adjacent to an element of $D$. The domination number $\gamma(G)$ of a graph $G$ is the minimum cardinality of all dominating sets of $G$. A set $I \subseteq V(G)$ is said to be independent if every pair of vertices in $I$ is nonadjacent. Let $i(G)$ denote the size of a smallest maximal independent set. The number $i(G)$ is called the independent domination number. Note that any maximal independent set is dominating (cf. [1]). A subset of vertices $A$ in a graph $G$ is said to be acyclic if $\langle A, G \rangle$ contains no cycles. The acyclic domination number $\gamma_a(G)$ of a graph $G$ is the minimum cardinality of acyclic dominating set in $G$. The concept of acyclic domination in graphs was introduced by
Further results on acyclic domination in graphs may be found in [2, 3, 8, 9]. Domke et al. [4] noted that the problem of characterization of the graphs with $\mu(G) = |V(G)| - \Delta(G)$ and they were first to consider the problem of characterization of the graphs with $\mu(G) = |V(G)| - \Delta(G)$, $\mu \in \{\gamma, i\}$. Favaron and Mynhardt [5] continued the study on the problem and gave necessary and sufficient conditions for $\mu(G) = |V(G)| - \Delta(G)$, $\mu \in \{\gamma, i\}$. Since $\gamma(G) \leq i(G)$, it follows that $\gamma_a(G) \leq |V(G)| - \Delta(G)$. In this paper, we deal with graphs $G$ satisfying $\gamma_a(G) = |V(G)| - \Delta(G)$. We need the following notation. Let $G$ be a graph, $x \in V(G)$, $\deg(x, G) = \Delta(G)$. Let $B = N(x, G), C = V(G) - N[x, G]$ and $R = B - N(C, G)$.

A classical result by Berge [1] states that for any graph $G$, $\gamma(G) \leq |V(G)| - \Delta(G)$.

Let $G$ be a connected triangle-free graph $\gamma_a(G) = \Delta(G)$. If $\deg(u, G) = 2$ for all $u \in B$, then $\deg(v, G) \geq 2$ for each $v \in A$.

If $G$ is a disconnected graph with $k \geq 2$ components, $\Delta(G) \geq 1$ and $i(G) + \Delta(G) = |V(G)|$, then all but one component of $G$ are $K_1$-components, because of Theorem 1. This shows that it is sufficient to consider the connected graphs $G$ with $\gamma_a(G) + \Delta(G) = |V(G)|$.

2. GRAPHS WHICH SATISFY $\gamma_a(G) + \Delta(G) = |V(G)|$

**Theorem 3.** Let $G$ be a connected graph, $x \in V(G)$ and $\deg(x, G) = \Delta(G)$. Let $A_1(x)$ and $A_2(x)$ hold. Then:

(i) $|V(G)| - \Delta(G) - 1 \leq \gamma_a(G)$;
(ii) if $y \in B$ with $N[y, G] \subseteq R \cup \{x\}$ then $\gamma_a(G) = |V(G)| - \Delta(G)$;
(iii) if $R = \emptyset$ and $C$ contains a vertex of degree 1, then $i(G) = \gamma_a(G) = |V(G)| - \Delta(G) - 1$.

**Proof.** (i) By $A_1(x)$ and $A_2(x)$, every acyclic dominating set of $G$, if it is to dominate $C$, then it has to have at least one vertex in each of the $|C|$ disjoint sets $\{u\} \cup B_u, u \in C$. Hence $\gamma_a(G) \geq |C| = |V(G)| - \Delta(G) - 1$. 
Theorem 5. Let $y$ be a connected triangle-free graph. Then $\gamma_a(G) = |V(G)| - \Delta(G)$ if and only if

(i) $\gamma_a(G) = |V(G)| - \Delta(G)$;

(ii) $\gamma_a(G) = |V(G)| - \Delta(G)$ and $\gamma_a(G) = |V(G)| - \Delta(G)$. By Theorem 1, $A_1(x)$ is satisfied. Let $y \in N(x, G)$ and suppose that $y$ is adjacent to $r$ vertices in $C$ with $r > 1$. Then $x$ and $y$, together with the $|V(G)| - \Delta(G) - 1 - r$ vertices of $C$ that are not in $N(y, G)$, form an acyclic dominating set of $G$, say $M$. Then $\gamma_a(G) \leq |M| = (|V(G)| - \Delta(G) - 1 - r) + 2 < |V(G)| - \Delta(G)$, a contradiction. Hence $A_2(x)$ is satisfied, too.

Note that if $u \in C$, then $B_u$ is non-empty, since $G$ is connected. Moreover, the sets $B_u$ form a partition of $N(C, G) = B - R$ by $A_2(x)$. Suppose $A_3(x)$ does not hold and $C'$ is a non-empty subset of $C$ such that there exists an acyclic dominating set $D$ of $\bigcup_{u \in C'} B_u \cup R$, with exactly one vertex in each $B_u$, $u \in C'$. Then $M = D \cap (C - C')$ is a dominating set of $G$, of cardinality $|M| = |C| - |V(G)| - \Delta(G) - 1 = \gamma_a(G) - 1$. Since $C - C'$ is independent, $D$ is acyclic and $N(D, G) \cap (C - C') = \emptyset$, it follows that $M$ is an acyclic dominating set of cardinality $\gamma_a(G) - 1$, a contradiction. Hence $A_3(x)$ holds.

(ii) Let $x \in V(G)$, $deg(x, G) = \Delta(G)$ and $A_1(x)$, $A_2(x)$ and $A_3(x)$ hold. Let $D$ be any acyclic dominating set of $G$ with $|D| = \gamma_a(G)$; let $C' = D \cap C$ and $C'' = C - C'$. In order to dominate $C$, $D$ has to have at least one vertex in each of the $|C|$ disjoint sets $\{u\} \cup B_u$, $u \in C$ (by $A_1(x)$ and $A_2(x)$). Hence the set $D \cap B$ contains at least $|C'|$ vertices, one vertex in each $B_u$ with $u \in C'$. If $|D \cap B| > |C'|$, then $\gamma_a(G) = |D| > |C'| + |C''| = |C| = |V(G)| - \Delta(G) - 1$. So, let $|D \cap B| = |C'|$. By $A_3(x)$, $D \cap B$ does not dominate $B' = B - \bigcup_{u \in C''} B_u$ and thus, to dominate $B', x \in D$. It follows that $|C'| + |C''| < |D|$, i.e., $|V(G)| - \Delta(G) \leq |D| = \gamma_a(G) \leq |V(G)| - \Delta(G)$. 

Theorem 5. Let $G$ be a connected triangle-free graph. Then $\gamma_a(G) = |V(G)| - \Delta(G)$ if and only if $\gamma(G) = |V(G)| - \Delta(G)$.

Proof. Necessity: If $\gamma_a(G) = 1$, then the result is obvious. So, let $2 \leq \gamma_a(G) = |V(G)| - \Delta(G)$ and let $x \in V(G)$ be of maximum degree. By $A_1(x)$, $C$ is independent and since $G$ is triangle-free, $B = N(x, G)$ is also independent. Hence $G$ is bipartite with bipartition $A \cup B$, where $A = \{x\} \cup C$ and $B = N(x, G)$. Since $A$ is an independent dominating set of cardinality $|A| = |V(G)| - \Delta(G) = \gamma_a(G)$, then $|A| \leq |B|$. Moreover, $|B| = \Delta(G)$, by the choice of $x$. By $A_2(x)$, $deg(u, G) \leq 2$ for each $u \in B$. 

We now characterize the graphs $G$ for which

$$\gamma_a(G) = |V(G)| - \Delta(G).$$

Suppose $\deg(u, G) = 2$ for each $u \in B$. Then $\deg(x, G) \geq 2$, $R = \emptyset$ and if some vertex $u$ of $A - \{x\} = C$ has degree 1, then $\gamma_a(G) < |V(G)| - \Delta(G)$ because of Theorem 3 (iii), a contradiction. Now, by Theorem 2, $\gamma(G) = |V(G)| - \Delta(G)$.

Sufficiency: Obvious.

REFERENCES