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SIGN-CHANGING LYAPUNOV FUNCTIONS IN REGULARITY OF LINEAR EXTENSIONS OF DYNAMICAL SYSTEMS ON A TORUS

Abstract. In this paper we consider some sign-changing Lyapunov function in research on regularity of sets of linear extensions of dynamical systems on a torus.

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1. INTRODUCTION

Let us consider a system of differential equations:

$$\frac{dy}{dt} = f(x), \qquad \frac{dy}{dt} = A(x)y, \tag{1}$$

where: $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$, $A(x)$ is a square, n-dimensional matrix whose elements are real and periodic with period 2π, continuous with respect to each variable x_j, $j = \overline{1, m}$, that is defined on an m – dimensional torus T_m. We denote the set of all such functions which are real, continuous and periodic with period 2π with respect to each variable x_j, $j = \overline{1, m}$ with $C^0(T_m)$. We assume that the function $f(x)$ satisfies the Lipschitz inequality. We use $C_{Lip}(T_m)$ to stand for the space of functions $f(x) \in C^0(T_m)$ which satisfy the Lipschitz inequality. It follows that $f(x) \in C_{Lip}(T_m)$ and $A(x) \in C^0(T_m)$.

We also use the following notations: $\|y\|^2 = \langle y, y\rangle$ is the Euclidean norm in the space R^n; $\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j$ is an inner product in R^n; $C'(T_m; f)$ is the subspace of $C^0(T_m)$ functions $F(x)$ such that the superposition $F(x(t; x))$ as a function in the variable t is continuously differentiable with respect to $t \in R$, when $x(t; x_0)$ is a solution to the Cauchy problem:

$$\frac{dx}{dt} = f(x), \qquad x\big|_{t=x_0}, \qquad \forall x_0 \in T_m,$$

and we define:

$$\dot{F}(x) =^{df} \frac{dF(x(t;x_0))}{dt}\Big|_{t=0}, \qquad \dot{F}(x) \in C^0(T_m);$$

$C^1(T_m)$ is the subspace of $C^0(T_m)$ functions $F(x)$ with a continuous first derivative.

Definition 1. *We say that the differential equation system:*

$$\frac{dx}{dt} = f(x), \qquad \frac{dy}{dt} = A(x)y + h(x), \qquad h(x) \in C^0(T_m), \tag{2}$$

possesses an invariant torus determined by equatity:

$$y = u(x),$$

when $u(x) \in C'(T_m; f)$ and the identity:

$$\dot{u}(x) \equiv A(x)u(x) + h(x), \qquad \forall x \in T_m$$

holds.

Therefore, when $u(x) \in C^1(T_m)$:

$$\dot{u}(x) = \frac{du}{dx} \cdot f(x) = \sum_{j=1}^{m} \frac{\partial u(x)}{\partial x_j} \cdot f_j(x).$$

In such a case, an invariant torus for system (2) is periodic with period 2π solution to a partial differential equation:

$$\sum_{j=1}^{m} \frac{\partial u(x)}{\partial x_j} \cdot f_j(x) = A(x)u(x) + h(x).$$

Example 1. *Let us consider the differential equation:*

$$\begin{cases} \dfrac{dx_1}{dt} = 1, \qquad \dfrac{dx_2}{dt} = \sqrt{2}, \\ \dfrac{dy}{dt} = (\lambda + \cos(x_1 + x_2))y + h(x_1, x_2), \end{cases}$$

where $\lambda = const \in R$, $h(x_1, x_2) \in C^1(T_2)$. An invariant torus for the system has the following form:

I. *When $\lambda > 0$:*

$$y = u(x_1, x_2) =$$
$$= -\int_0^{\infty} e^{-\lambda\tau - \frac{1}{1+\sqrt{2}}\left[\sin\left((1+\sqrt{2})\tau + x_1 + x_2\right) - \sin(x_1 + x_2)\right]} h(\tau + x_1, \sqrt{2}\tau + x_2) d\tau.$$

II. *When $\lambda < 0$:*

$$y = u(x_1, x_2) =$$
$$= \int_{-\infty}^{0} e^{\lambda\tau + \frac{1}{1+\sqrt{2}}\left[\sin\left((1+\sqrt{2})\tau + x_1 + x_2\right) - \sin(x_1+x_2)\right]} h(\tau + x_1, \sqrt{2}\tau + x_2) d\tau.$$

III. *When $\lambda = 0$, the invariant torus for the system fails to exist for every $h(x) \in C^1(T_2)$. For example, when $h \equiv 1$, a torus does not exist.*

Definition 2. *Let $C(x)$ be an $(n \times n)$-dimensional continuous matrix, $C(x) \in C^0(T_m)$. Then the function $G_0(\tau; x)$:*

$$G_0(\tau; x) = \begin{cases} \Omega_\tau^0(x) C(x(\tau; x)), & \tau \leq 0, \\ \Omega_\tau^0(x) \left[C(x(\tau; x)) - I_n\right], & \tau > 0, \end{cases} \tag{3}$$

which fulfils the estimate:

$$\|G_0(\tau; x)\| \leq K e^{-\gamma|x|}, \tag{4}$$

where K and γ are positive constants, is called a Green function of an invariant torus for system (1).

Where $\Omega_\tau^t(x)$ is the fundamental matrix of the solutions of system $\frac{dy}{dt} = A(x(t; x))y$, which takes the value of n-dimensional identity matrix $\Omega_\tau^t(x)\big|_{t=\tau} = I_n$. We use the norm $\|A\| = \max\limits_{\|y\|=1} \|Ay\|$. The existence of Green function is connected with exponential dichotomy of the linear system $\frac{dy}{dt} = A(x(t; x))y$. Interesting investigations of the systems can be found in [1, 2].

When Green function (3) with estimate (4) exists, then for every vector function $h(x) \in C^0(T_m)$ an invariant torus for system (2) can be represented in integral form:

$$y = u(x) = \int_{-\infty}^{\infty} G_0(\tau; x) h(x(\tau; x)) d\tau.$$

When Green function for system (1) is unique, the system is called regular [3]. When Green function is not unique, then the system (1) is called sharply-weak regular.

From the other hand, the properties of regularity of systems can be researched by means of Lyapunov functions in the form of quadratic forms which can change sign. It is obvious [3] that system (1) is regular when the square form:

$$V = \langle S_0(x) y, y \rangle, \tag{5}$$

with the symmetric, non-degenerate ($\det S_0(x) \neq 0$) matrix $S_0(x) \in C^1(T_m)$, exists and its derivative along the solutions of system (1) is positively defined:

$$\dot{V} = \left\langle \left[\frac{\partial S_0(x)}{\partial x} f(x) + S_0(x) A(x) + A^T(x) S_0(x) \right] y, y \right\rangle \geq \epsilon \|y\|^2, \tag{6}$$

$\epsilon = const > 0$.

If the matrix $S_0(x)$ vanishes at some points, then even if (6) holds, system (1) does not possess any Green function and the conjoint system:

$$\frac{dx}{dt} = f(x), \qquad \frac{dy}{dt} = -A^T(x)y,$$

is sharply-weak regular. It means that many different Green functions for the conjoint system exists.

2. MAIN RESULTS

Dealing with problems of regularity of systems, we encounter issues which have not been considered in literature. We research some Lyapunov functions in the form of a quadratic form which can change sign and whose derivative along solutions of some sets of systems (1) are positively defined. It means we are searching for sets of regular systems.

It can be checked that a set of systems such that the derivative of quadratic form (5) along the solutions of the systems is positively defined exists for every non-degenerated matrix $S_0(x)$.

Let matrices $B(x), M(x) \in C^0(T_m)$ be arbitrary matrices which satisfy the following conditions:

$$B^T(x) \equiv B(x), \quad \langle B(x)x, x\rangle \geq \lambda \|x\|^2, \quad \lambda = const > 0, \tag{7}$$

$$M^T(x) \equiv -M(x). \tag{8}$$

Then the following theorem holds true.

Theorem 1. *To any non-degenerated matrix $S_0(x) \in C^1(T_m)$, there corresponds the set of regular systems:*

$$\frac{dx}{dt} = b(x), \qquad \frac{dy}{dt} = S_0^{-1}(x)\left[B(x) + M(x) - 0.5\frac{\partial S_0(x)}{\partial x}b(x)\right]y, \tag{9}$$

where $b(x)$ is an arbitrary vector function, $b(x) \in C_{Lip}(T_m)$, $B(x), M(x)$ are continuous matrices which satisfy (7), (8).

Remark 1. *The derivative of square form* (5)*, with the symmetric, non-degenerate matrix $S_0(x) \in C^1(T_m)$, along the solutions of system* (9) *is of the following form: $\dot{V} = 2\langle B(x)y, y\rangle$.*

Let matrices $B(x,\xi), M(x,\xi) \in C^0(T_m \times T_k)$ fulfil the following identities:

$$B(x,\xi)^T \equiv B(x,\xi), M(x,\xi)^T \equiv -M(x,\xi).$$

Then the following remark is true.

Remark 2. *In system* (9)*, a variable* $\xi \in T_k$ *can be added to the variable* x*. Then we consider the following system:*

$$\begin{aligned}
&\frac{dx}{dt} = b(x,\xi), \qquad \frac{d\xi}{dt} = \bar{b}(x,\xi),\\
&\frac{dy}{dt} = S_0^{-1}(x)\left[B(x,\xi) + M(x,\xi) - 0.5\frac{\partial S_0(x)}{\partial x}b(x,\xi)\right]y,
\end{aligned} \tag{10}$$

where $b(x,\xi), \bar{b}(x,\xi) \in C_{Lip}(T_{m+k})$ *are any vector functions and for matrices* $B(x,\xi), M(x,\xi) \in C^0(T_m \times T_k)$*, identities* $B^T \equiv B, M^T \equiv -M$ *hold. The derivative of square form* (5) *along the solutions of* (10) *is positively defined:* $\dot{V} = 2\langle B(x,\xi)y, y\rangle \geq \lambda\|y\|^2$*,* $\lambda = const > 0$*.*

Remark 3. *If for non-degenerate square form* (5)*, we consider the set of systems* (1)*, that the derivative to the form along the solutions of these systems is positively defined, we are able only to increase the number of variables* x*. Decreasing of the number of variables* x*, even to one variable, is not always possible.*

Two kinds of $2n$-dimensional square form, with matrices $S_0(x)$ below, are examined:

$$S_0(x) = \begin{pmatrix} 0 & I_n \\ I_n & S(x) \end{pmatrix}, \tag{11}$$

$$S_0(x) = \begin{pmatrix} I_n \cos x & I_n \sin x \\ I_n \sin x & -I_n \cos x \end{pmatrix}. \tag{12}$$

Research on such systems for which the derivative of quadratic form (5) with matrix (12) along the solutions of the system is positively defined leads to the following problem. Let us notice that when a matrix $A(x)$ in system (1) is constant $A(x) \equiv A$ then unique Green function for the system exists if and only if when all eigenvalues λ_i of the matrix A fulfil the condition $Re\lambda_i \neq 0$. Therefore, when $\det A = 0$ system (1) is not regular. In the case of non-constant $A(x)$ $\det A(x)$ may vanish at some points x. Moreover, examples exist of regular system (1) with matrix $A(x)$ such that $\det A(x) \equiv 0$. Thus we obtain the following proposition.

Proposition 1. *Systems in the form* (1) *exist for which* $\det A(x) \equiv 0$, $\forall x \in T_m$.

Examples of systems satisfying the proposition include:

$$\begin{cases}
\dfrac{dx}{dt} = \sqrt{2},\\
\dfrac{dy_1}{dt} = (\cos 2x + \sin 2x)y_1 + (-\cos 2x + \sin 2x - \sqrt{2})y_2,\\
\dfrac{dy_2}{dt} = (-\cos 2x + \sin 2x + \sqrt{2})y_1 + (-\cos 2x - \sin 2x)y_2.
\end{cases}$$

Let $(2n \times 2n)$-dimensional $B(x), M(x) \in C^0(T_m)$ of the form

$$B(x) = \begin{bmatrix} B_1(x) & B_{12}(x) \\ B_{12}^T(x) & B_2(x) \end{bmatrix}, \qquad M(x) = \begin{bmatrix} 0 & M_1(x) \\ -M_1^T(x) & 0 \end{bmatrix}, \tag{13}$$

satisfy conditions (7), (8).

The search for such systems that the derivative along the solutions of the systems of quadratic form (5) with matrix (11) is positively defined lead to the following proposition.

Proposition 2. *Let* $B(x) \in C^0(T_m)$ *be given in form* (13) *and fulfil conditions* (7). *Then the following system:*

$$\begin{aligned}
\frac{dx}{dt} &= f(x),\\
\frac{dy_1}{dt} &= \left[-SB_1 \sin\psi + p^{-1}\left[B_{12}^T(x) - M_1^T(x)\right] y_1\right] +\\
&\quad + \Bigg[- S\left[B_{12}(x) + M_1(x)\right] \sin\psi +\\
&\qquad + \left(B_2(x) - 0.5S\left(\sum_{j=1}^{m} f_j(x)\right)\cos\psi\right)\Bigg] y_2,\\
\frac{dy_2}{dt} &= B_1(x)y_1 + \left[B_{12}(x) + M_1(x)\right] y_2,\\
\psi &= \sum_{j=1}^{m} x_j,
\end{aligned} \tag{14}$$

is regular for any given vector functions $f(x) \in C_{Lip}(T_m)$, *matrices* $M(x) \in C^0(T_m)$ *and constant matrix* $S, S = S^T$.

Now let us come return to quadratic form (5) and assume that for some $x = x_0 \in T_m$:

$$\det S_0(x_0) = 0 \tag{15}$$

and (6) is satisfied. The following question arises. Under what changes does system (1) remain regular? That is the derivative of quadratic form (5) along the solutions of the system remains positively defined. As it turned out, in the case of system (9), we cannot choose any vector function $b(x) \in C_{Lip}(T_m)$.

Let us examine the one dimensional case $n = 1$ with $A(x)$ a scalar function. In this case, condition (6) takes form:

$$2S_0(x)A(x) + \sum_{j=1}^{m} \frac{\partial S_0(x)}{\partial x_j} f_j(x) = \bar{w}(x) > 0. \tag{16}$$

Let us explain how we may change functions $A(x)$, $f_1(x), f_2(x), \ldots, f_m(x)$ so that inequality (16) still holds for a given $S_0(x)$. Let us put $r_1 = 2A(x)$, $r_j = f_j(x)$, $j = \overline{2, m+1}$; this way we obtain an algebraic equation:

$$S_0(x)r_1 + \frac{\partial S_0(x)}{\partial x_1} r_2 + \ldots + \frac{\partial S_0(x)}{\partial x_m} r_{m+1} = w(x). \tag{17}$$

Our task is to write down every solution of (17) for any function $w(x) > 0$. If one of the coefficients of equation (17) does not vanish for each x, we can directly write the solution of the equation. More frequently, the coefficients vanish for some $x \in R$ (not simultaneously). We know one solution to equation (17), so for some $\bar{w}(x) > 0$ the following inequality:

$$S_0^2(x) + \left(\frac{\partial S_0(x)}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial S_0(x)}{\partial x_m}\right)^2 > 0 \tag{18}$$

is satisfied. To write down all solutions to equation (17) for any function $w(x) > 0$, we suggest the method below. Equation (17) is complemented to a system of equations by adding joining one more unknown r_{m+2}:

$$\begin{cases} S_0(x)r_1 + \frac{\partial S_0(x)}{\partial x_1} r_2 + \ldots + \frac{\partial S_0(x)}{\partial x_m} r_{m+1} = w(x), \\ r_1 - S_0(x) r_{m+2} = q_1(x), \\ r_2 - \frac{\partial S_0(x)}{\partial x_1} r_{m+2} = q_2(x), \\ \vdots \\ r_{m+1} - \frac{\partial S_0(x)}{\partial x_m} r_{m+2} = q_{m+1}(x). \end{cases} \tag{19}$$

The system has a unique solution $(r_1, \ldots, r_{m+1})$ for any fixed function $w(x)$, $q_j(x) \in C_{Lip}(T_m)$, $j = \overline{1, m+1}$. The determinant of the matrix of coefficients of the algebraic system equals to the left-hand side of inequality (18) so it does not vanish and the system is determinate. When we solve system (19) we obtain all solutions to (17) which depend on x for any $q_1, \ldots, q_m + 1$:

$$\begin{aligned} r_1 &= q_0(x) + \frac{w(x)}{\sigma(x)} S_0(x) - \frac{S_0^2(x)}{\sigma(x)} q_0(x) - S_0(x) \sum_{i=1}^{m} \frac{\partial S_0(x)}{\partial x_i} \frac{q_i(x)}{\sigma(x)}, \\ r_j &= q_j(x) + \frac{\partial S_0(x)}{\partial x_j} \left[\frac{w(x)}{\sigma(x)} - \frac{S_0(x)}{\sigma(x)} q_0(x) - \sum_{i=1}^{m} \frac{\partial S_0(x)}{\partial x_i} \frac{q_i(x)}{\sigma(x)} \right], \\ j &= \overline{2, m+1}, \end{aligned} \tag{20}$$

where $\sigma(x)$ is the left-hand side of the inequality (18). If we have any solution to equation (17), let it be $(\bar{r}_1, \ldots, \bar{r}_{m+1})$, we can choose such right-hand sides $q_1, \ldots, q_{m+1}$ that the solution to system (19) is a vector $(\bar{r}_1, \ldots, \bar{r}_{m+1}, 0)$. Namely, we can replace $q_j = \bar{r}_j$, $j = \overline{1, m+1}$. Therefore (20) shows all solutions to equation (17).

Functions $q_j(x)$, $j = \overline{1, m}$ may be chosen randomly so we can use the substitution $\frac{q_j}{\sigma} \to q_j$ to obtain the following theorem.

Theorem 2. *For any scalar function $S_0(x) \in C^1(T_m)$ which satisfies* (15) *and for any functions $q_j \in C_{Lip}(T_m)$, $j = \overline{1, m+1}$ and $w(x) \in C_{Lip}(T_m)$, $w(x) > 0$, the system:*

$$\frac{dx_j}{dt} = w(x)\frac{\partial S_0(x)}{\partial x_j} - S_0(x)\frac{\partial S_0(x)}{\partial x_j}q_0(x) + \left(q_j(x)\sigma(x) - \frac{\partial S_0(x)}{\partial x_j}\sum_{i=1}^{m}\frac{\partial S_0(x)}{\partial x_i}q_i(x)\right),$$
$$\frac{dy}{dt} = -0.5\left[w(x)S_0(x) + \left(\sigma(x) - S_0^2(x)\right)q_0(x) - S_0(x)\sum_{i=1}^{m}\frac{\partial S_0(x)}{\partial x_i}q_i(x)\right]y,$$
$$j = \overline{1, m+1} \tag{21}$$

is sharply-weak regular.

In order to understand the theory better, let us consider the following example.

Example 2. *Let us consider the system:*

$$\frac{dx_1}{dt} = \sin x_1, \qquad \frac{dx_2}{dt} = -2\sin x_2, \qquad \frac{dy}{dt} = -\left(3\cos x_1 - 4\cos x_2\right)y, \tag{22}$$

where $x_1 \in R, x_2 \in R, y \in R$. The derivative of the square form:

$$V = (4\cos x_2 - 3\cos x_1)y^2 \tag{23}$$

along the solutions to system (22) *is positively defined. Now let us take square form* (23)*. We will answer the following question. How can we change system* (22)*, so that the derivative of the square form along the solutions of the obtained system remains positively defined? To this end, let us write down system* (22) *in more general form:*

$$\frac{dx_1}{dt} = b_1(x_1, x_2), \qquad \frac{dx_2}{dt} = b_2(x_1, x_2), \qquad \frac{dy}{dt} = a(x_1, x_2)y, \tag{24}$$

where $(x_1, x_2) \in T_2, y \in R$, $b_1(x_1, x_2), b_2(x_1, x_2) \in C_{Lip}(T_2)$, $a(x_1, x_2) \in C^0(T_2)$. The derivative of square form (23) *along the solutions of system* (24) *has the form:*

$$\begin{aligned} 3\sin x_1 \cdot b_1(x_1, x_2) - 4\sin x_2 \cdot b_2(x_1, x_2) + \\ + 2\left(4\cos x_2 - 3\cos x_1\right)\cdot a(x_1, x_2) = \\ = w(x_1, x_2), \qquad w(x_1, x_2) > 0. \end{aligned} \tag{25}$$

Let us put:

$$2a(x_1, x_2) = r_1, \qquad b_1(x_1, x_2) = r_2, \qquad b_2(x_1, x_2) = r_3. \tag{26}$$

We obtain an algebraic equation in three unknowns:

$$3\sin x_1 \cdot r_1 - 4\sin x_2 \cdot r_2 + (4\cos x_2 - 3\cos x_1)\cdot r_3 = w(x_1, x_2).$$

Making use of (20)*, we obtain a solution:*

$$\begin{aligned} r_1 &= q_1 + 3\sin x_1 \cdot \left[w - 3\sin x_1 \cdot q_1 + 4\sin x_2 \cdot q_2 - (4\cos x_2 - 3\cos x_1)\cdot q_3\right], \\ r_2 &= q_2 - 4\sin x_2 \cdot \left[w - 3\sin x_1 \cdot q_1 + 4\sin x_2 \cdot q_2 - (4\cos x_2 - \ \cos x_1)\cdot q_3\right], \\ r_3 &= q_3 + (4\cos x_2 - 3\cos x_1)\cdot \\ &\qquad \cdot \left[w - 3\sin x_1 \cdot q_1 + 4\sin x_2 \cdot q_2 - (4\cos x_2 - 3\cos x_1)\cdot q_3\right], \end{aligned} \tag{27}$$

where $q_1, q_2, q_3 \in C_{Lip}(T_m)$ *are arbitrary functions. For such functions* (27) *numerous different Green function for the system conjoint to system* (24) *with respect to* y *exist. Functions* q_j *can be chosen randomly, so we can substitute:*

$$q_1(x_1, x_2) = \sin x_1, \qquad q_2(x_1, x_2) = -2\sin x_2, \qquad q_3(x_1, x_2) = (4\cos x_2 - 3\cos x_1)$$

to obtain system (22).

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