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SINGULAR INTEGRAL EQUATION
WITH A MULTIPLICATIVE CAUCHY KERNEL
IN THE HALF-PLANE

Abstract. In this paper the explicit solutions of singular integral equation with a multiplicative Cauchy kernel in the half-plane are presented.

Keywords: singular integral equation, exact solution, Cauchy kernel, multiplicative kernel.

Mathematics Subject Classification: 45E05.

1. INTRODUCTION

In the literature \[2,\ 5–7\] formulae describing a solution of the following equation

\[
\frac{1}{\pi} \int_{-1}^{1} \frac{\varphi(\tau)}{\tau - t} dt = f(t), \quad -1 < x < 1,
\]

are very well known. Some problems of aeroelasticity \[1\] can be reduced to the equation of the form

\[
\frac{1}{\pi^2} \int_{-1}^{1} \int_{-1}^{1} \frac{\varphi(\sigma_1, \sigma_2)}{(\sigma_1 - x) (\sigma_2 - y)} d\sigma_1 d\sigma_2 = f(x, y), \quad -1 < x, y < 1.
\]

Theory of this equation is well developed in \[3,\ 4,\ 9\]. In paper \[8\], a theory of following equations

\[
\frac{1}{\pi i} \int_{D_1} \int \frac{\varphi(\sigma_1, \sigma_2)}{(\sigma_1 - x) (\sigma_2 - y)} d\sigma_1 d\sigma_2 = f(x, y), \quad (x, y) \in D_1,
\]

are well developed.
\[ \frac{1}{(\pi i)^2} \int_{D_2} \frac{\varphi(\sigma_1, \sigma_2)}{(\sigma_1-x)(\sigma_2-y)} d\sigma_1 d\sigma_2 = f(x, y), \quad (x, y) \in D_2, \]

where \( D_1, D_2 \) are the quarter-plane and the whole complex plane, respectively, is presented. We have not found in the literature any study of the equation in which the surface of integration is the half-plane. In this paper, we consider the equation

\[ \frac{1}{(\pi i)^2} \int_{D} \frac{\varphi(\sigma_1, \sigma_2)}{(\sigma_1-x)(\sigma_2-y)} d\sigma_1 d\sigma_2 = f(x, y), \quad (x, y) \in D_2, \quad (1) \]

where \((x, y) \in D = \{(x, y) : 0 < \text{Re} \, z < \infty, -\infty < \text{Im} \, z < \infty, z = x + iy\}, f(x, y)\) is a given function and \( \varphi(x, y) \) is an unknown function.

2. FUNCTION CLASSES

Let us introduce function classes that will be used in this paper.

We write \( \varphi(x) \in h(\infty), x > 0, \) if the function

\[ \varphi^*(t) = \varphi\left(\frac{1+t}{1-t}\right), \quad t \in (-1, 1), \]

satisfies the inequality

\[ |\varphi^*(t') - \varphi^*(t'')| \leq K |t' - t''|^\mu, \quad (2) \]

where \( K > 0, 0 < \mu \leq 1 \) are constants independent of the arrangement of the points \( t', t'' \) in each closed interval contained in \((-1, 1)\), and in a neighbourhood of the point \( t = -1 \) the following condition is satisfied:

\[ \varphi^*(t) = \varphi^{**}(t) |t+1|^{-\alpha}, \quad 0 \leq \text{Re} \, \alpha < 1. \]

Here \( \varphi^{**}(t) \) is a Hölder continuous function on the interval \([-1, 1)\), and

\[ \lim_{t \to -1-0} \varphi^*(t) = \lim_{x \to -\infty} \varphi(x) = 0. \quad (3) \]

We write \( \varphi(x) \in h(\infty), -\infty < x < \infty, \) if the function

\[ \varphi^*(t) = \varphi\left(\frac{i+1}{1-t}\right), \quad |t| = 1, \]

satisfies inequality \((2)\).

We write \( \varphi(x, y) \in h(\infty) \times h(\infty), x > 0, -\infty < y < \infty, \) if the function

\[ \varphi^*(t_1, t_2) = \varphi\left(\frac{1+t_1}{1-t_1}, \frac{1+t_2}{1-t_2}\right), \quad (t_1, t_2) \in (-1, 1) \times L, \quad L = \{t_2 : |t_2| = 1\}, \]

satisfies the inequality
Singular integral equation with a multiplicative Cauchy kernel in the half-plane

\[ |\varphi^*(t_1', t_2') - \varphi^*(t_1''', t_2''')| \leq K_1 |t_1' - t_1'''|^{\mu_1} + K_2 |t_2' - t_2'''|^{\mu_2}, \quad (4) \]

\( K_1, K_2 > 0, \, 0 < \mu_1, \mu_2 \leq 1, \) in each closed domain contained in \((-1, 1) \times L,\) and

\[ \varphi^*(t_1, t_2) = \varphi^{**}(t_1, t_2)|t_1 + 1|^{-\alpha}, \quad 0 \leq \Re \alpha < 1, \]

where \( \varphi^{**}(t_1, t_2) \) satisfies the Hölder condition with respect to both variables on \([-1, 1] \times L,\) and moreover,

\[ \lim_{t_1 \to 1 - 0} \varphi^*(t_1, t_2) = \lim_{x \to \infty} \varphi(x, y) = 0 \quad \text{for} \quad t_2 \in L \quad \text{(for} \quad y \in (-\infty, \infty)). \quad (5) \]

We write \( \varphi(x) \in h(0, \infty), \, x \geq 0, \) if the function

\[ \varphi^*(t) = \varphi \left( \frac{1 + t}{1 - t} \right), \quad t \in [-1, 1), \]

satisfies (2) and the condition of the form (3).

We write \( \varphi(x, y) \in h(0, \infty) \times h(\infty), \) \( 0 \leq x \leq \infty, \, -\infty \leq y \leq \infty, \) if the function

\[ \varphi^*(t_1, t_2) = \varphi \left( \frac{1 + t_1}{1 - t_1}, \frac{1 + t_2}{1 - t_2} \right), \quad (t_1, t_2) \in [-1, 1) \times L, \quad L = \{t_2 : |t_2| = 1\}, \]

satisfies conditions (4) and (5).

3. SOLUTION IN THE CLASS \( h(\infty) \times h(\infty) \)

**Theorem 3.1.** Let \( f(x, y) \in h(0, \infty) \times h(\infty) \) and let

\[ \lim_{|y| \to \infty} f(x, y) = 0, \quad x \in [0, \infty). \quad (6) \]

Then each solution \( \varphi(x, y) \) of (1) in the function class \( h(\infty) \times h(\infty) \) is given by the formula

\[ \varphi(x, y) = R(f; x, y) + C_1(x) + \frac{C_2(y)i}{\sqrt{x}}, \quad (7) \]

where

\[ R(f; x, y) = \frac{(x + 1)(y + i)}{\sqrt{x}(\pi^2)} \int_{0}^{\infty} \int_{-\infty}^{\infty} \frac{\sqrt{\sigma_1 f(\sigma_1, \sigma_2)}}{(\sigma_1 + 1)(\sigma_2 + i)(\sigma_1 - x)(\sigma_2 - y)} d\sigma_1 d\sigma_2, \]

\( C_1(x) > 0, \, C_2(y), \, -\infty < y < \infty, \) are arbitrary functions of class \( h(\infty). \)

If we seek for a solution \( \varphi(x, y) \) in the class of functions satisfying the following conditions:

\[ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi(x, \sigma_2)}{\sigma_2 + i} d\sigma_2 = p(x), \quad (8) \]

\[ \frac{1}{\pi i} \int_{0}^{\infty} \frac{\varphi(\sigma_1, y)}{\sigma_1 + 1} d\sigma_1 = q(y), \quad (9) \]

where \( p(x) \in h(\infty), \, x > 0, \, q(y) \in h(\infty), \, -\infty < y < \infty \) are the functions fulfilling the relation
\[
\frac{1}{\pi i} \int_0^\infty \frac{p(\sigma_1)}{\sigma_1 + 1} d\sigma_1 = \frac{1}{\pi i} \int_{-\infty}^\infty \frac{q(\sigma_2)}{\sigma_2 + i} d\sigma_2 = \omega, \tag{10}
\]
then the solution is given by the formula
\[
\varphi(x, y) = R(f; x, y) - p(x) + \frac{q(y) i}{\sqrt{x}} + \frac{\omega i}{\sqrt{x}}. \tag{11}
\]

**Proof.** We can rewrite (1) in the form
\[
\frac{1}{\pi i} \int_0^\infty \frac{d\sigma_1}{\sigma_1 - x} \frac{1}{\pi i} \int_{-\infty}^\infty \frac{\varphi(\sigma_1, \sigma_2)}{\sigma_2 - y} d\sigma_2 = f(x, y), \tag{12}
\]
or
\[
\frac{1}{\pi i} \int_{-\infty}^\infty \frac{d\sigma_2}{\sigma_2 - y} \frac{1}{\pi i} \int_{0}^\infty \frac{\varphi(\sigma_1, \sigma_2)}{\sigma_1 - x} d\sigma_1 = f(x, y). \tag{13}
\]

Let us consider equation (12). It can be represented in the following form
\[
\frac{1}{\pi i} \int_0^\infty \frac{\psi_1(\sigma_1, y)}{\sigma_1 - x} d\sigma_1 = f(x, y), \tag{14}
\]
where
\[
\psi_1(\sigma_1, y) = \frac{1}{\pi i} \int_{-\infty}^\infty \frac{\varphi(\sigma_1, \sigma_2)}{\sigma_2 - y} d\sigma_2. \tag{15}
\]

Let us find the function \(\psi_1(x, y)\) appearing in (14). We solve (14) in the function class \(h(\infty)\), \(0 < x < \infty\). By [8], we obtain
\[
\psi_1(x, y) = \frac{x + 1}{\sqrt{x}} \frac{1}{\pi i} \int_{0}^\infty \frac{\sqrt{\sigma_1}}{\sigma_1 + 1} \frac{f(\sigma_1, y)}{\sigma_1 - x} d\sigma_1 + \frac{C_3(y) i}{\sqrt{x}},
\]
where \(C_3(y), y \in (-\infty, \infty)\), is an arbitrary function of class \(h(\infty)\).

Next, solving equation (15), we obtain the solution \(\varphi(x, y)\):
\[
\varphi(x, y) = \frac{1}{\pi i} \int_{-\infty}^\infty \frac{y + i}{\sigma_2 + i} \frac{\psi_1(x, \sigma_2)}{\sigma_2 - y} d\sigma_2 - C_4(x) = \\
= \frac{1}{(\pi i)^2} \frac{(x + 1)(y + i)}{\sqrt{x}} \int_{0}^\infty \int_{-\infty}^\infty \frac{\sqrt{\sigma_1 f(\sigma_1, \sigma_2)}}{(\sigma_1 + 1)(\sigma_2 + i)(\sigma_1 - x)(\sigma_2 - y)} d\sigma_1 d\sigma_2 + \\
+ \frac{(y + i)}{\pi i \sqrt{x}} \int_{-\infty}^\infty \frac{C_3(\sigma_2)}{(\sigma_2 + i)(\sigma_2 - y)} d\sigma_2 - C_4(x), \tag{16}
\]
where \(C_4(x), 0 < x < \infty\), is an arbitrary function of class \(h(\infty)\).
Now we consider equation (13). We rewrite it in the form

\[ \frac{1}{\pi i} \int_{-\infty}^{\infty} \psi_2(x, \sigma_2) \frac{d\sigma_2}{\sigma_2 - y} = f(x, y), \quad (17) \]

where

\[ \psi_2(x, \sigma_2) = \frac{1}{\pi i} \int_{0}^{\infty} \frac{\varphi(\sigma_1, \sigma_2)}{\sigma_1 - x} d\sigma_1. \quad (18) \]

Solving (17) in the function class \( h(\infty) \) with respect to the second variable, we obtain

\[ \psi_2(x, y) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{y + i f(x, \sigma_2)}{\sigma_2 + i} d\sigma_2 - C_5(x), \quad (19) \]

where \( C_5(x), \ x > 0, \) is an arbitrary function of class \( h(\infty) \).

Using (19), we solve equation (18). We derive

\[ \varphi(x, y) = \frac{x + 1}{\sqrt{x}} \frac{1}{\pi i} \int_{0}^{\infty} \frac{\sqrt{\sigma_1} \psi_2(\sigma_1, \sigma_2) d\sigma_1 + C_6(\sigma)}{\sigma_1 + 1} \frac{\sqrt{\sigma_1}}{\sigma_1 - x} = \]

\[ = \frac{(x + 1)(y + i)}{\sqrt{x}} \frac{1}{(\pi i)^2} \int_{0}^{\infty} \int_{-\infty}^{\infty} \frac{\sqrt{\sigma_1} f(\sigma_1, \sigma_2)}{(\sigma_1 + 1)(\sigma_2 + i)(\sigma_2 - y)} d\sigma_1 d\sigma_2 - (20) \]

\[ - \frac{x + 1}{\sqrt{x}} \frac{1}{\pi i} \int_{0}^{\infty} C_5(\sigma_1) \frac{1}{(\sigma_1 + 1)(\sigma_1 - x)} d\sigma_1 + \frac{C_6(y + i)}{\sqrt{x}}, \]

where \( C_6(y), \ -\infty < y < \infty, \) is an arbitrary function of class \( h(\infty) \).

Owing to (16) and (20), the general solution \( \varphi(x, y) \) of (1) in the considered class of functions is given by (7). Let us check it by substituting (7) into (1). Substituting \( R(f; x, y) \) into equation (1) and using the Poincaré-Bertrand formula for an infinite surface of integration:

\[ \frac{1}{(\pi i)^2} \int_{0}^{\infty} \int_{0}^{\infty} \frac{\varphi(\sigma_1, \sigma'_1)}{(\sigma_1 - x)(\sigma'_1 - \sigma_1)} d\sigma'_1 d\sigma_1 = \]

\[ = \varphi(x, x) + \frac{1}{(\pi i)^2} \int_{0}^{\infty} \int_{0}^{\infty} \frac{\varphi(\sigma_1, \sigma'_1)}{(\sigma_1 - x)(\sigma'_1 - \sigma_1)} d\sigma_1 d\sigma'_1 - \varphi(\infty, \infty), \]

and its similar version for the whole plane as the surface of integration, and the following formulæ

\[ \frac{1}{\pi i} \int_{0}^{\infty} \frac{\sigma_1 + 1}{\sqrt{\sigma_1}(\sigma_1 - x)} d\sigma_1 = 0, \quad \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\sigma_2 + i}{(\sigma_2 - y)(\sigma_2 - \sigma_1)} d\sigma_2 = 0, \]
we get
\[
\frac{1}{(\pi i)^2} \int_{0}^{\infty} \int_{0}^{\infty} R(f, \sigma_{1}, \sigma_{2}) \, d\sigma_{1} d\sigma_{2} = \frac{1}{(\pi i)^2} \int_{0}^{\infty} \int_{0}^{\infty} \frac{(\sigma_{1}+1)(\sigma_{2}+i)}{\sqrt{\pi i \left(\sigma_{1}-\sigma_{2}\right)}} \times \\
\times \left( \int_{0}^{\infty} \frac{x}{(\sigma_{1}+1)(\sigma_{2}+i)(\sigma_{1}-\sigma_{2})} \, d\sigma_{1} \right) \times \\
\times \left( \int_{0}^{\infty} \frac{x}{(\sigma_{2}+i)(\sigma_{1}-\sigma_{2})} \, d\sigma_{2} \right) \\
= \frac{1}{(\pi i)^2} \int_{0}^{\infty} \int_{0}^{\infty} \frac{\sigma_{2}+i}{(\sigma_{1}+1)(\sigma_{2}-y)(\sigma_{1}-x)} \times \\
\times \left( \int_{0}^{\infty} \frac{x}{(\sigma_{1}+1)(\sigma_{2}-y)(\sigma_{2}-x)} \, d\sigma_{1} \right) \times \\
\times \left( \int_{0}^{\infty} \frac{y}{(\sigma_{2}-y)(\sigma_{1}-x)} \, d\sigma_{2} \right) \\
= \frac{1}{(\pi i)^2} \int_{0}^{\infty} \int_{0}^{\infty} \frac{(\sigma_{2}+i)(\sigma_{1}-\sigma_{2})}{(\sigma_{2}-y)(\sigma_{1}-x)} \times \\
\times \left( \int_{0}^{\infty} \frac{x}{(\sigma_{1}+1)(\sigma_{2}-y)(\sigma_{1}-x)} \, d\sigma_{1} \right) \times \\
\times \left( \int_{0}^{\infty} \frac{y}{(\sigma_{2}-y)(\sigma_{1}-x)} \, d\sigma_{2} \right) \\
= f(x, y) + \frac{1}{(\pi i)^2} \int_{0}^{\infty} \left( \int_{0}^{\infty} \frac{\sigma_{2}+i}{(\sigma_{1}+1)(\sigma_{2}-y)(\sigma_{1}-x)} \, d\sigma_{1} \right) \frac{f(x, \sigma_{1})}{\sigma_{2}+i} \, d\sigma_{2} = f(x, y).
\]

Now we substitute the function $C_{1}(x), x > 0$ appearing in (7) into (1). We obtain
\[
\frac{1}{(\pi i)^2} \int_{0}^{\infty} \int_{0}^{\infty} C_{1}(\sigma_{1}) \, d\sigma_{1} d\sigma_{2} = \frac{1}{(\pi i)^2} \int_{0}^{\infty} C_{1}(\sigma_{1}) \, d\sigma_{1} \frac{1}{(\pi i)^2} \int_{0}^{\infty} d\sigma_{2} = 0.
\]

Finally, substituting $C_{2}(y)i$, $-\infty < y < \infty$ into (1) we get
\[
\frac{1}{(\pi i)^2} \int_{0}^{\infty} \int_{0}^{\infty} \sqrt{\pi i \left(\sigma_{1}-\sigma_{2}\right)} \, d\sigma_{1} d\sigma_{2} = \frac{1}{(\pi i)^2} \int_{0}^{\infty} C_{2}(\sigma_{2}) \, d\sigma_{2} \frac{1}{(\pi i)^2} \int_{0}^{\infty} \sqrt{\pi i \left(\sigma_{1}-\sigma_{2}\right)} \, d\sigma_{2} = 0.
\]

The above calculations justify formula (7). Now we prove formula (11). To this end, we substitute (7) into conditions (8), (9). We derive
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} R(f, \sigma_{2}+i) \, d\sigma_{2} = \frac{1}{\pi} \int_{-\infty}^{\infty} R(f, \sigma_{1}+1) \, d\sigma_{1} = p(x), \quad (21)
\]
\[
\frac{1}{\pi} \int_{0}^{\infty} R(f, \sigma_{1}+1) \, d\sigma_{1} = \frac{1}{\pi} \int_{0}^{\infty} \frac{C_{1}(\sigma_{1})}{\sigma_{1}+1} \, d\sigma_{1} = q(y), \quad (22)
\]

Since
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} R(f, \sigma_{2}+i) \, d\sigma_{2} = 0, \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{C_{1}(\sigma_{1})}{\sigma_{2}+i} \, d\sigma_{1} = -C_{1}(x),
\]
\[
\frac{1}{\pi} \int_{0}^{\infty} R(f, \sigma_{1}+1) \, d\sigma_{1} = 0, \quad C_{2}(y)i \int_{0}^{\infty} \frac{d\sigma_{1}}{\sqrt{\pi i \left(\sigma_{1}+1\right)}} = C_{2}(y),
\]
it follows that (21) and (22) take the forms

\[ \begin{align*}
C_1(x) &= \frac{i}{\sqrt{x}} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{C_2(\sigma_2)}{\sigma_2 + i} d\sigma_2 - p(x), \\
C_2(y) &= q(y) - \frac{1}{\pi i} \int_{0}^{\infty} C_1(\sigma_1) \frac{1}{\sigma_1 + 1} d\sigma_1.
\end{align*} \]

Hence

\[ \varphi(x, y) = R(f; x, y) - p(x) + \frac{q(y) i}{\sqrt{x}} + \frac{i}{\sqrt{x}} \left( \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{C_2(\sigma_2)}{\sigma_2 + i} d\sigma_2 - \frac{1}{\pi i} \int_{0}^{\infty} \frac{C_1(\sigma_1)}{\sigma_1 + 1} d\sigma_1 \right). \]

Let us denote

\[ \gamma = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{C_2(\sigma_2)}{\sigma_2 + i} d\sigma_2 - \frac{1}{\pi i} \int_{0}^{\infty} \frac{C_1(\sigma_1)}{\sigma_1 + 1} d\sigma_1. \]

Taking into account (10), we obtain

\[ \frac{1}{\pi i} \int_{0}^{\infty} \frac{p(\sigma_1)}{\sigma_1 + 1} d\sigma_1 = \frac{1}{\pi i} \int_{0}^{\infty} \left( \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi(\sigma_1, \sigma_2)}{\sigma_2 + i} d\sigma_2 \right) \frac{d\sigma_1}{\sigma_1 + 1} = \]

\[ = \frac{1}{\pi i} \int_{0}^{\infty} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{R(f; \sigma_1, \sigma_2)}{\sigma_2 + i} d\sigma_2 \frac{d\sigma_1}{\sigma_1 + 1} + \]

\[ - \frac{1}{\pi i} \int_{0}^{\infty} \frac{p(\sigma_1)}{\sigma_1 + 1} \int_{-\infty}^{\infty} \frac{d\sigma_2}{\sigma_2 + i} + \]

\[ + \frac{i}{\sqrt{\pi i}} \int_{0}^{\infty} \frac{d\sigma_1}{\sqrt{\sigma_1 (\sigma_1 + 1)}} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{q(\sigma_2)}{\sigma_2 + i} d\sigma_2 + \]

\[ + \frac{\gamma i}{\pi i} \int_{0}^{\infty} \frac{d\sigma_1}{\sqrt{\sigma_1 (\sigma_1 + 1)}} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{d\sigma_2}{\sigma_2 + i} = \]

\[ = \omega + \omega - \gamma. \]

Therefore \( \gamma = \omega \), and formula (11) is proved. \( \square \)

**Example 3.1.** Let the functions \( f(x, y) \), \( p(x) \), \( q(y) \) be given by the following formulae

\[ f(x, y) = \frac{1}{x + 2 y + 1 + i}, \quad p(x) = 0, \quad q(y) = \frac{1}{y + 1 + i}. \]
Then the solution of \((1)\) in the function class \(h(\infty) \times h(\infty)\) has the form

\[
\varphi(x, y) = \frac{i\sqrt{2} (x + 1)}{\sqrt{x} (x + 2)} (y + 1 + i).
\]

4. SOLUTION IN THE CLASS \(h(0, \infty) \times h(\infty)\)

**Theorem 4.1.** Let \(f(x, y) \in h(0, \infty) \times h(\infty)\) satisfy condition \((6)\). Then a solution \(\varphi(x, y)\) of \((1)\) in the function class \(h(0, \infty) \times h(\infty)\), satisfying the relations

\[
\varphi(x, \infty) = 0, \quad x \in [0, \infty),
\]

is given by the following formula:

\[
\varphi(x, y) = \frac{\sqrt{x}}{\pi i} \int_0^\infty \frac{\varphi(\sigma_1, y)}{\sigma_1 + 1} \, d\sigma_1 = \frac{i(y + i)}{(\pi i)^2} \int_0^\infty \int_{-\infty}^\infty \frac{f(\sigma_1, \sigma_2)}{\sqrt{\sigma_1 (\sigma_1 + 1)(\sigma_2 + i) (\sigma_2 - y)}} \, d\sigma_1 \, d\sigma_2,
\]

\[
\varphi(x, y) = \int_0^\infty \varphi(\sigma_1, y) \frac{d\sigma_1}{\sqrt{\sigma_1 (\sigma_1 + 1)(\sigma_2 + i) (\sigma_2 - y)}}.
\]

**Proof.** As in the proof of Theorem 3.1, equation \((1)\) can be rewritten in form \((14)\). Solving \((14)\) in the function class \(h(0, \infty)\), we obtain (cf. \([8]\))

\[
\psi_1(x, y) = \frac{\sqrt{x}}{\pi i} \int_0^\infty \frac{f(\sigma_1, y)}{\sqrt{\sigma_1 (\sigma_1 + 1)(\sigma_2 + i) (\sigma_2 - y)}} \, d\sigma_1,
\]

with the condition

\[
\frac{1}{\pi i} \int_0^\infty \frac{\psi_1(\sigma_1, y)}{\sigma_1 + 1} \, d\sigma_1 = \frac{1}{\pi} \int_0^\infty \frac{f(\sigma_1, y)}{\sqrt{\sigma_1 (\sigma_1 + 1)}} \, d\sigma_1.
\]

Substituting \((15)\) into \((27)\), we derive

\[
\frac{1}{(\pi i)^2} \int_0^\infty \int_{-\infty}^\infty \frac{\varphi(\sigma_1, \sigma_2)}{\sigma_1 + 1} \, d\sigma_1 \, d\sigma_2 = \frac{1}{\pi} \int_0^\infty \frac{f(\sigma_1, y)}{\sqrt{\sigma_1 (\sigma_1 + 1)}} \, d\sigma_1.
\]

Multiplying each side of \((28)\) by \(\frac{1}{y+i}\), on account of \((23)\), using Hilbert transform \([8]\), and finally multiplying both sides of the equation by \(y + i\), we obtain condition \((24)\).

Now we solve \((15)\). By \([8]\), there is

\[
\varphi(x, y) = \frac{y + i}{\pi i} \int_{-\infty}^\infty \frac{\psi_1(x, \sigma_2)}{(\sigma_2 + i)(\sigma_2 - y)} \, d\sigma_2 = \int_0^\infty \int_{-\infty}^\infty \frac{f(\sigma_1, \sigma_2)}{\sqrt{\sigma_1 (\sigma_1 + 1)(\sigma_2 + i) (\sigma_2 - y)}} \, d\sigma_1 \, d\sigma_2.
\]
We can rewrite (1) in form (17), (18). We solve (17) in the function class $h(\infty)$ vanishing at infinity:

$$
\psi_2(x, y) = \frac{y + i}{\pi i} \int_{-\infty}^{\infty} \frac{f(x, \sigma_2)}{(\sigma_2 + i)(\sigma_2 - y)} \, d\sigma_2.
$$

Now we find the solution of (18) in the class of bounded functions:

$$
\varphi(x, y) = \frac{\sqrt{x}}{\pi i} \int_{0}^{\infty} \frac{\psi_2(\sigma_1, y)}{\sqrt{\sigma_1}(\sigma_1 - x)} \, d\sigma_1 =
$$

$$
= \frac{\sqrt{x}(y + i)}{(\pi i)^3} \int_{0}^{\infty} \int_{-\infty}^{\infty} \frac{f(\sigma_1, \sigma_2) \, d\sigma_1 \, d\sigma_2}{\sqrt{\sigma_1}(\sigma_2 + i)(\sigma_1 - x)(\sigma_2 - y)},
$$

with the condition

$$
\frac{1}{\pi i} \int_{0}^{\infty} \frac{\varphi(\sigma_1, y)}{\sigma_1 + 1} \, d\sigma_1 = \frac{1}{\pi} \int_{0}^{\infty} \frac{\psi_2(\sigma_1, y)}{\sqrt{\sigma_1}(\sigma_1 + 1)} \, d\sigma_1 =
$$

$$
= \frac{i(y + i)}{(\pi i)^6} \int_{0}^{\infty} \int_{-\infty}^{\infty} \frac{f(\sigma_1, \sigma_2) \, d\sigma_1 \, d\sigma_2}{\sqrt{\sigma_1}(\sigma_1 + 1)(\sigma_2 + i)(\sigma_2 - y)}.
$$

Formulae (29) and (30) coincide with (25) and (24), respectively.

As in the proof of Theorem 3.1, one can substitute (25) into (1) and check that function (25) is a solution of equation (1). \hfill \Box

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