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CLASSICAL AND WEAK SOLUTIONS FOR SEMILINEAR PARABOLIC EQUATIONS WITH PREISACH HYSTERESIS

Abstract. We consider the solvability of the semilinear parabolic differential equation

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + c(x,t)u(x,t) = \mathcal{P}(u) + \gamma(x,t)$$

in a cylinder $D = \Omega \times (0,T)$, where $\mathcal{P}$ is a hysteresis operator of Preisach type. We show that the corresponding initial boundary value problems have unique classical solutions. We further show that using this existence and uniqueness result, one can determine the properties of the Preisach operator $\mathcal{P}$ from overdetermined boundary data.

Keywords: hysteresis, parabolic, inverse problem, uniqueness.

Mathematics Subject Classification: 35K55, 35R30.

1. INTRODUCTION

The study of partial differential equations with hysteresis is mostly connected with the name of A. Visintin and his monograph [12], where an outline of the most common parabolic and hyperbolic partial differential equations with hysteresis is presented. There are also various results on the unique solvability and existence of weak solutions of these problems. Further results on weak solutions of hyperbolic problems have for example been investigated by Krejci [6] and Yamamoto and Longfeng [9]. In this work we show that in the semilinear, parabolic case, we get classical solutions for equations with general Preisach operators if these satisfy a growth estimate. In Section 2 we give a short definition of Preisach operators. In Section 3 we present an existence and uniqueness result for initial boundary value problems with Preisach hysteresis and in Section 4 we apply this result in the treatment of inverse problems with Preisach hysteresis. This is especially interesting for the application of hysteresis models. Often one can perceive some kind of hysteresis effect, but it is hard to analyse what kind of hysteresis is present.

2. PREISACH OPERATORS

This section provides a short introduction to hysteresis operators. We use the notation and the approach of Brokate and Sprekels ([1]) for this. The proofs for the results in this section can be found in [1].

Definition 2.1 (Hysteresis operator). *Let $C_{pm}[0,T]$ be the space of continous, piecewise monotone functions on $[0,T]$. Let $\mathcal{H} : C_{pm}[0,T] \mapsto \mathbb{R}$ be a rate-independent functional. We define the general hysteresis operator $\mathcal{W} : C_{pm}[0,T] \mapsto Map([0,T])$ as follows*

$$\mathcal{W}(v)(t) = \mathcal{H}(v_t), \quad t \in [0,T] \tag{2.1}$$

where v_t is the cut-off of v at t, i.e.,

$$v_t(\tau) = \begin{cases} v(\tau) & \text{for } 0 \le \tau \le t, \\ v(t) & \text{for } t \le \tau \le T. \end{cases}$$

Alternatively one can consider an operator defined on the set S of finite strings $(v_0, v_1, \ldots, v_N)$. $\mathcal{H}$ induces an operator $\tilde{\mathcal{W}} : S \mapsto S$ by

$$\tilde{\mathcal{W}}(s) = (\tilde{\mathcal{H}}(v_0), \tilde{\mathcal{H}}(v_0, v_1), \ldots, \tilde{\mathcal{H}}(s)) \quad \text{for } s = (v_0, \ldots, v_N) \in S.$$

We also write $\mathcal{W}_f$ for the functional $\mathcal{H}$.

The following operator plays an essential role in the definition of general Preisach operators.

Definition 2.2 (Play operator). *We define the scalar play operator $\mathcal{F}_r : C([0,T]) \mapsto C([0,T])$ as*

$$\mathcal{F}_r(v)(0) = f_r(v(0), 0), \quad \mathcal{F}_r(v)(t) = f_r(v(t), \mathcal{F}_r(v)(t_i)) \tag{2.2}$$
$$\text{for } t_i < t \le t_{i+1}, \quad 0 \le i \le N-1,$$

where $0 = t_0 < t_1 < \cdots < t_N = T$ is a monotonicity partition of v on $[0,T]$ and

$$f_r(v, s) = \max\{v - r, \min\{v + r, s\}\}.$$

The corresponding functional of $\mathcal{F}_r$ is thus defined as

$$\mathcal{F}_{r,f}(v_0) = f_r(v_0, 0), \tag{2.3}$$
$$\mathcal{F}_{r,f}(v_0, \ldots, v_n) = f_r(v_n, \mathcal{F}_{r,f}(v_0, \ldots, v_{n-1})), \tag{2.4}$$

where $v_i = v(t_i)$ and $0 = t_0 < t_1 < \cdots < t_N = T$ is a monotonicity partition of v on $[0,T]$.

We present some properties of the play operator here.

Lemma 2.3. *For all v, v_1, $v_2 \in C([0,T])$, all ω_{-1}, $\omega_{-1,1}$, $\omega_{-1,2} \in \mathbb{R}$, all $0 \le t' \le t \le T$ and $s \ge 0$*

$$\|\mathcal{F}_r(v_1;\omega_{-1,1}) - \mathcal{F}_r(v_2;\omega_{-1,2})\|_0^{[0,T]} \le \max\{\|v_1 - v_2\|_0^{[0,T]}, |\omega_{-1,1} - \omega_{-1,2}|\}, \quad (2.5)$$

$$|\mathcal{F}_r(v)(t) - \mathcal{F}_r(v)(t')| \le \sup_{t' \le \tau \le t} |v(\tau) - v(t')|, \quad (2.6)$$

$$\mathcal{F}_r(v;\omega_{-1}) = \mathcal{F}_r(v - \omega_{-1}) + \omega_{-1}, \quad (2.7)$$

$$\mathcal{F}_r(v_1;\omega_{-1,1}) \le \mathcal{F}_r(v_2;\omega_{-1,2}), \text{ if } v_1 \le v_2, \text{ and } \omega_{-1,1} \le \omega_{-1,2}. \quad (2.8)$$

To define general Preisach operators we will need a few additional auxiliary definitions.

Definition 2.4 (Preisach memory curves). *We define the set of admissable Preisach memory curves as*

$$\Psi_0 := \{\phi | \ \phi : \mathbb{R}_+ \mapsto \mathbb{R}, \ |\phi(r) - \phi(\bar{r})| \le |r - \bar{r}| \ \forall r, \bar{r} \ge 0, \ R_{supp}(\phi) < \infty\}$$

where

$$R_{supp}(\phi) := \sup\{r| \ r \ge 0, \ \phi(r) \ne 0\}.$$

Definition 2.5 (Evolution of Preisach memory curves). *Let $\psi_{-1} \in \Psi_0$ be an initial memory curve. Let $s = (v_0, \ldots, v_N) \in S$. The Preisach evolution is defined as*

$$\psi_i(r) := f_r(v_i, \psi_{i-1}(r)), \quad r \ge 0, \quad 0 \le i \le N$$

and it holds

$$\psi_N(r) = \mathcal{F}_{r,f}(s, \psi_i(r)) \quad \forall r \ge 0.$$

Definition 2.6 (Hysteresis operators of Preisach type). *Let $s = (v_0, \ldots, v_N) \in S$. The operator $\mathcal{F}_f : S \times \Psi_0 \mapsto \Psi_0$ with*

$$\mathcal{F}_f(s;\psi_{-1})(r) := \mathcal{F}_{r,f}(s;\psi_{-1}(r)), \quad \forall r > 0$$

is called Preisach memory operator. A hysteresis operator $\mathcal{W}$ with output functional $\mathcal{Q}$ defined by

$$\mathcal{W}_f(s;\psi_{-1}) := \mathcal{Q}(\mathcal{F}_f(s, \psi_{-1})), \quad s \in S$$

is called a hysteresis operator of Preisach type.

Definition 2.7 (Preisach operators). *A hysteresis operator $\mathcal{P}$ of Preisach type is called a Preisach operator if its output functional $\mathcal{Q} : \Psi_0 \mapsto \mathbb{R}$ is of the form*

$$\mathcal{Q}(\phi) = \int_0^\infty q(r, \phi(r)) d\nu(r) + \omega_{00},$$

where ν denotes a regular σ-finite Borel measure, $\omega_{00} \in \mathbb{R}$ and q is given by

$$q(r,s) = 2\int_0^s \omega(r,\sigma) d\sigma$$

for some given $\omega \in L_{1,loc}(\mathbb{R}_+ \times \mathbb{R}; \nu \otimes \lambda)$.

Preisach operators have some very important growth and regularity properties.

Proposition 2.8. *Let $\mathcal{W}$ be an operator of Preisach type with output functional $\mathcal{Q} : \Psi_0 \mapsto \mathbb{R}$. Then it holds*

- *Let $\eta(\delta; \mathcal{Q}) := \sup\{|Q(\phi) - Q(\psi)| : \phi, \psi \in \Psi_0, \ \|\phi - \psi\|_{L^\infty} < \delta\}$. Then for $\lim_{\delta \to 0} \eta(\delta; \mathcal{Q}) = 0$ the operator $\mathcal{W}$ is uniformly continuous on $C([0,T]) \times \Psi_0$ and maps bounded subsets of $C([0,T]) \times \Psi_0$ onto bounded subsets in $C([0,T])$.*
- *If there exist constants $C > 0$ and $\alpha \in (0,1]$ with*

$$\eta(\delta; \mathcal{Q}) \leq C\delta^\alpha, \tag{2.9}$$

  then $\mathcal{W}$ is Hölder continuous with exponent α on $C([0,T]) \times \Psi_0$. Further for every $\beta \in (0,1]$ $\mathcal{W}$ maps bounded subsets of $C^{0,\beta}([0,T]) \times \Psi_0$ onto bounded subsets in $C^{0,\alpha\beta}([0,T])$.
- *If (2.9) holds for $\alpha = 1$, then $\mathcal{W}$ is Lipschitz on $C([0,T]) \times \Psi_0$ and maps bounded subsets of $B \times \Psi_0$ onto bounded subsets of B, where $B = BV[0,T]$ or $B = W^1_p(0,T)$, $1 \leq p \leq \infty$. In particular we get*

$$|\mathcal{W}(v)'(t)| \leq C|v'(t)| \tag{2.10}$$

  for every $t \in [0,T]$ for which both derivatives exist.

Proposition 2.9 (Regularity properties of Preisach operators). *Let $\mathcal{P}$ be a Preisach operator with initial memory curve $\psi_{-1} \in \Psi_0$. Then*

- *If*

$$c_1 := \int_0^\infty \sup_{s \in \mathbb{R}} |\omega(r,s)| d|\nu|(r) < \infty$$

  holds it follows $\eta(\delta; \mathcal{Q}) \leq 2c_1\delta$ and the results of Proposition 2.8 hold. In particular for almost all $t \in [0,T]$

$$\mathcal{P}(v)'(t) = 2\int_0^\infty \omega(r, \mathcal{F}_r(v; \psi_{-1}(r))(t)) \cdot \mathcal{F}_r(v; \psi_{-1}(r))'(t) d\nu(r)$$

  and thus

$$|\mathcal{P}(v)'(t)| \leq 2c_1|v'(t)|$$

  for every $t \in [0,T]$ for which both exist.
- *If $\partial_s\omega(r,s)$ is measurable and in addition to the constant c_1 above, there exists a c_2 with*

$$c_2 := \int_0^\infty \sup_{s \in \mathbb{R}} |\partial_s\omega(r,s)| d|\nu|(r) < \infty,$$

  then it holds

$$\|\mathcal{P}(v_1)' - \mathcal{P}(v_2)'\|_{L_1} \leq 2(c_2\|v_1'\|_{L_1} + c_1)\|v_1 - v_2\|_{BV}.$$

In our analysis in this work we will restrict ourselves to Preisach operators with linear growth. This is justified by the following proposition.

Proposition 2.10 (Growth estimates for Preisach operators). *Let $\mathcal{P}$ be a Preisach operator with initial memory curve $\psi_{-1} \in \Psi_0$. Then for all $v \in C([0,T])$ it holds*

$$|\mathcal{P}(v)(t)| \leq \int_0^\infty 2 \left| \int_0^{\psi(t,r)} |\omega(r,s)| ds \right| d|\nu|(r) + |\omega_{00}|. \tag{2.11}$$

If v is large enough, i.e. $\|v\|_0 \geq \max\{\|\psi_{-1}\|_0, R_{supp}\}$, then it holds

$$|\mathcal{P}(v)(t)| \leq q_M \|v\|_0 + |\omega_{00}|$$

with

$$q_M(y) := 2 \int_0^y \int_{-y}^y |\omega(r,s)| ds d|\nu|(r).$$

In particular the Preisach operator has linear growth

$$\|\mathcal{P}(v)\|_0 \leq c_0 \|v\|_0 + c_1,$$

if

$$\int_0^y \int_{-y}^y |\omega(r,s)| ds d|\nu|(r) \leq c_2 |y| + c_3, \quad y \geq 0. \tag{2.12}$$

Since we are analyzing partial differential equations with Preisach operators, we need the notion of a space-dependent Preisach operator.

Definition 2.11 (Space dependent Preisach operators). *We define the space-dependent Preisach operators as*

$$\mathcal{P}(v; \psi_{-1})(x,t) := \mathcal{P}(v(x,\cdot), \psi_{-1})(t). \tag{2.13}$$

This definition is a special case of the so-called x-dependent Preisach operator

$$\mathcal{P}(v; \psi_{-1}; x)(x,t) := \mathcal{P}(v(x,\cdot); \psi_{-1}(x))(t), \tag{2.14}$$

where different values of x are associated with different initial memory curves. However, for our purposes, space-dependent Preisach operators will be sufficient.

3. THE DIRECT PROBLEM

Now we consider the solvability of direct semilinear problem with hysteresis. Let $\Omega \subset \mathbb{R}^n$ be open and simply connected, let $\partial\Omega$ be a C^3-boundary and let $T > 0$. We set $D := \Omega \times (0,T)$ and $S := \partial\Omega \times (0,T)$. In this section we consider the solvability of the initial-boundary value problems

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + cu(x,t) = \mathcal{P}(u)(x,t) + \gamma(x,t), \qquad (x,t) \in D, \tag{3.1}$$

$$\frac{\partial u}{\partial \nu}(x,t) + b(x,t)u(x,t) = g(x,t), \qquad (x,t) \in S, \tag{3.2}$$

$$u(x,0) = \psi(x), \qquad x \in \Omega, \tag{3.3}$$

and

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + c(x,t)u(x,t) = \mathcal{P}(u)(x,t) + \gamma(x,t), \qquad (x,t) \in D, \quad (3.4)$$

$$u(x,t) = h(x,t), \qquad (x,t) \in S, \quad (3.5)$$

$$u(x,0) = \psi(x), \qquad x \in \Omega, \quad (3.6)$$

where $\mathcal{P}(u)(x,t)$ is a Preisach operator. First we want to consider the existence and uniqueness of classical solutions for the initial-boundary value problems above and we make the following assumptions:

(a1) $\psi \in C^{2+\alpha}(\bar{\Omega})$;
(a2) $\mathcal{P}$ is a Preisach operator that satisfies $\|\mathcal{P}(v)\|_0 = C\|v\|_0 + C_0$ and $\|\mathcal{P}(v) - \mathcal{P}(u)\|_0 \le c_1\|u - v\|_0$;
(a3) $\gamma \in C^{\alpha,\alpha/2}(\bar{D})$;
(a4) $c(x,t) \ge c_0 > 0$, $c_0 > C$ and $c_0 > c_1$;
(a5) $b(x,t) > 0$;
(a6) $g \in C^{1+\alpha,1+\alpha/2}(S)$, $g(x,0) = \frac{\partial \psi}{\partial \nu}(x) + b(x,0)\psi(x)$, for $x \in \partial\Omega$;
(a7) $h \in C^{2+\alpha,1+\alpha/2}(S)$, $h(x,0) = \psi(x)$, for $x \in \partial\Omega$;

Before we present one of the main results, we show that the function $P(u)(x,t)$ is an element of $C^{\alpha,\alpha/2}(\bar{D})$ if u is an element of $C^{\alpha,\alpha/2}(\bar{D})$.

Lemma 3.1. *Let* $u \in C^{\alpha,\alpha/2}(\bar{D})$ *and* $\mathcal{P}$ *satisfy assumption* (a2). *Then* $\mathcal{P}(u)$ *is an element of* $C^{\alpha,\alpha/2}(\bar{D})$ *as well.*

Proof. $\|\mathcal{P}(v)(x,t)\|_0^{\bar{D}} \le C\|v(x,t)\|_0^{\bar{D}} + C_0$ is valid. We consider points $P = (x_1,t_1) \neq Q = (x_2,t_2)$, $t_1 \le t_2$, and $Q_1 = (x_2,t_1)$ as well as $Q' = (x_2,t')$ with $|v(x_2,t') - v(x_2,t_1)| = \|v(x_2,\cdot) - v(x_2,t_1)\|_0^{[t_1,t_2]}$. Then

$$\frac{|\mathcal{P}(v)(x_1,t_1) - \mathcal{P}(v)(x_2,t_2)|}{\rho(P,Q)^\alpha} = \frac{|\mathcal{P}(v(x_1,\cdot))(t_1) - \mathcal{P}(v(x_2,\cdot))(t_2)|}{\rho(P,Q)^\alpha} = \quad (3.7)$$

$$= \frac{|\mathcal{P}(v(x_1,\cdot))(t_1) - \mathcal{P}(v(x_2,\cdot))(t_1) + \mathcal{P}(v(x_2,\cdot))(t_1) - \mathcal{P}(v(x_2,\cdot))(t_2)|}{\rho(P,Q)^\alpha} \le \quad (3.8)$$

$$\le \frac{|\mathcal{P}(v(x_1,\cdot))(t_1) - \mathcal{P}(v(x_2,\cdot))(t_1)|}{\rho(P,Q)^\alpha} + \frac{|\mathcal{P}(v(x_2,\cdot))(t_1) - \mathcal{P}(v(x_2,\cdot))(t_2)|}{\rho(P,Q)^\alpha} \le \quad (3.9)$$

$$\le C\frac{\|v(x_1,\cdot) - v(x_2,\cdot)\|_0^{[0,T]}}{\|x_1 - x_2\|^\alpha} + C\frac{|v(x_2,t_1) - v(x_2,t')|}{\rho(Q_1,Q')^\alpha} \quad (3.10)$$

and since we have $\|\mathcal{P}(v)\|_0^{\bar{D}} \le C\|v\|_0^{\bar{D}} + C_0$ we can conclude that

$$\|\mathcal{P}(v)\|_{\alpha,\alpha/2}^{\bar{D}} \le 2C\|v\|_{\alpha,\alpha/2}^{\bar{D}} + C_0.$$

□

Now we present one of the main results.

Theorem 3.2. *Let conditions* $(a1)-(a6)$ *hold. Then there exists at least one solution* $u \in C^{2+\alpha,1+\alpha/2}(\bar{D})$ *of the initial-boundary value problem*

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + c(x,t)u(x,t) = \mathcal{P}(u)(x,t) + \gamma(x,t), \qquad (x,t) \in D, \tag{3.11}$$

$$\frac{\partial u}{\partial \nu}(x,t) + b(x,t)u(x,t) = g(x,t), \qquad (x,t) \in S, \tag{3.12}$$

$$u(x,0) = \psi(x), \qquad x \in \Omega. \tag{3.13}$$

Proof. We set $B := C^{\alpha,\alpha/2}(\bar{D})$. The initial-boundary value problem

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + cu(x,t) = \sigma\mathcal{P}(v)(x,t) + \sigma\gamma(x,t), \qquad (x,t) \in D, \tag{3.14}$$

$$\frac{\partial u}{\partial \nu}(x,t) + b(x,t)u(x,t) = \sigma g(x,t), \qquad (x,t) \in S \tag{3.15}$$

$$u(x,0) = \sigma\psi(x), \qquad x \in \Omega \tag{3.16}$$

therefore has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\bar{D})$ for every $v \in B$. With a parabolic standard estimate we see that a constant c_2 (see for example [8, Chapter IV]) exists, with

$$\|u\|^{\bar{D}}_{2+\alpha,1+\alpha/2} \leq c_2(\|\psi\|^{\bar{\Omega}}_{2+\alpha} + \|g\|^{S}_{1+\alpha,1+\alpha/2} + \|\gamma\|^{\bar{D}}_{\alpha,\alpha/2} + \|\mathcal{P}(v)\|^{\bar{D}}_{\alpha,\alpha/2}). \tag{3.17}$$

To prove the existence we define a map $\mathcal{T} : B \times [0,1] \mapsto C^{2+\alpha,1+\alpha/2}(\bar{D}) \subset\subset B$ by $\mathcal{T}(v,\sigma) = u$ and show that $\mathcal{T}$ satisfies all the conditions of the Leray-Schauder fixed-point theorem (see for example [2]).

— $\mathcal{T}(v,0) = 0$ for every $v \in B$.

— Let $\mathcal{T}(u,\sigma) = u$. It follows from equation (3.17) that

$$\|u\|^{\bar{D}}_{2+\alpha,1+\alpha/2} \leq c_2(\|\psi\|^{\bar{\Omega}}_{2+\alpha} + \|\gamma\|^{\bar{D}}_{\alpha,\alpha/2} + \|g\|^{S}_{1+\alpha,1+\alpha/2} + 2C\|u\|^{\bar{D}}_{\alpha,\alpha/2} + C_0). \tag{3.18}$$

Using a parabolic interpolation result (see [7, Chapter 8]), we get

$$2c_2C\|u\|^{\bar{D}}_{\alpha,\alpha/2} \leq \frac{1}{2}\|u\|^{\bar{D}}_{2+\alpha,1+\alpha/2} + L\|u\|^{\bar{D}}_{0} \tag{3.19}$$

for a $L \in \mathbb{R}_+$. From a parabolic standard estimate (see [8, Lemma 5.3]) it follows that a constant $c_3 > 0$ exists with

$$\|u\|^{\bar{D}}_{0} \leq \frac{1}{c_0}(\|\mathcal{P}(u)\|^{\bar{D}}_{0} + C_0 + \|\gamma\|^{\bar{D}}_{0}) + \|\psi\|^{\bar{\Omega}}_{0} + c_3\|g\|^{S}_{0} \tag{3.20}$$

and thus, because of $c > C$, for a constant $c_4 > 0$

$$\|u\|^{\bar{D}}_{0} \leq c_4(\|\gamma\|^{\bar{D}}_{0} + \|\psi\|^{\bar{\Omega}}_{0} + \|g\|^{S}_{0} + C_0). \tag{3.21}$$

Substituting (3.18) and (3.21) in (3.19) we see that

$$\|u\|_B \leq M \tag{3.22}$$

with a positive constant M.

— Now let $\sigma \in [0,1]$, $M_0 > 0$, $v_1, v_2 \in B$, and $\|v_1\|_B \le M_0$, $\|v_2\|_B \le M_0$ with $\mathcal{T}(v_1,\sigma) = u_1$, $\mathcal{T}(v_2,\sigma) = u_2$. Then for $u := u_1 - u_2$, we get

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + cu(x,t) &= \sigma(\mathcal{P}(v_1)(x,t) - \mathcal{P}(v_2)(x,t)), && (x,t) \in D,\\ \frac{\partial u}{\partial \nu}(x,t) + b(x,t)u(x,t) &= 0, && (x,t) \in S,\\ u(x,0) &= 0, && x \in \Omega. \end{aligned}$$

Thus we get

$$\|u_1 - u_2\|_0^{\bar{D}} \le \frac{1}{c_0}\|\mathcal{P}(v_1) - \mathcal{P}(v_2)\|_0^{\bar{D}} \le \frac{C}{c}\|v_1 - v_2\|_0^{\bar{D}}. \tag{3.23}$$

and the estimates

$$\|u_1 - u_2\|_{2+\alpha/2,1+\alpha/4}^{\bar{D}} \le K\|\mathcal{P}(v_1) - \mathcal{P}(v_2)\|_{\alpha/2,\alpha/4}^{\bar{D}} \tag{3.24}$$

with $K \in \mathbb{R}_+$,

$$\|u_1 - u_2\|_{\alpha,\alpha/2}^{\bar{D}} \le \|u_1 - u_2\|_{2+\alpha/2,1+\alpha/4}^{\bar{D}} + L_0\|u_1 - u_2\|_0^{\bar{D}} \tag{3.25}$$

and $L_0 \in \mathbb{R}_+$. Therefore

$$\|u_1 - u_2\|_{\alpha,\alpha/2}^{\bar{D}} \le K\|\mathcal{P}(v_1) - \mathcal{P}(v_2)\|_{\alpha/2,\alpha/4}^{\bar{D}} + \frac{C}{c_0}L_0\|v_1 - v_2\|_0^{\bar{D}}. \tag{3.26}$$

To show the continuity of $\mathcal{T}(v,\sigma)$ for a fixed σ, we estimate the term $\|\mathcal{P}(v_1) - \mathcal{P}(v_2)\|_{\alpha/2,\alpha/4}^{\bar{D}}$. Since $\|\mathcal{P}(v_1) - \mathcal{P}(v_2)\|_0^{\bar{D}} \le C\|v_1 - v_2\|_0^{\bar{D}}$, we only have to estimate the Hölder coefficient $H_{\alpha/2,\alpha/4}(\mathcal{P}(v_1) - \mathcal{P}(v_2))$.
Let $\varepsilon > 0$ and $\|v_1 - v_2\|_{\alpha,\alpha/2}^{\bar{D}} < \delta$ with $0 < \delta < \varepsilon^2/8C^2M_0$. We consider $P = (x_1,t_1)$, $Q = (x_2,t_2)$, $P \ne Q$. If $\rho(P,Q)^{\alpha/2} < \varepsilon/4CM_0$, it follows that

$$\begin{aligned} &\frac{|\mathcal{P}(v_1)(x_1,t_1) - \mathcal{P}(v_2)(x_1,t_1) - \mathcal{P}(v_1)(x_2,t_2) + \mathcal{P}(v_2)(x_2,t_2)|}{\rho(P,Q)^{\alpha/2}} \le\\ &\le \rho(P,Q)^{\alpha/2}(H_{\alpha,\alpha/2}^{\bar{D}}(\mathcal{P}(v_1)) + H_{\alpha,\alpha/2}^{\bar{D}}(\mathcal{P}(v_2))) \le\\ &\le d(P,Q)^{\alpha/2}(2CH_{\alpha,\alpha/2}^{\bar{D}}(v_1) + 2CH_{\alpha,\alpha/2}^{\bar{D}}(v_2)) \le\\ &\le \rho(P,Q)^{\alpha/2}4CM_0 < \varepsilon. \end{aligned}$$

If $\rho(P,Q)^{\alpha/2} > \varepsilon/4CM_0$, it follows that

$$\begin{aligned} &\frac{|\mathcal{P}(v_1)(x_1,t_1) - \mathcal{P}(v_2)(x_1,t_1) - \mathcal{P}(v_1)(x_2,t_2) + \mathcal{P}(v_2)(x_2,t_2)|}{\rho(P,Q)^{\alpha/2}} \le\\ &\le \frac{|\mathcal{P}(v_1(x_1,\cdot))(t_1) - \mathcal{P}(v_2(x_1,\cdot))(t_1)| + |\mathcal{P}(v_1(x_2,\cdot))(t_2) - \mathcal{P}(v_2(x_2,\cdot))(t_2)|}{\rho(P,Q)^{\alpha/2}} \le\\ &\le \frac{4C^2M_0}{\varepsilon}2\|v_1 - v_2\|_0^{\bar{D}} \le \varepsilon. \end{aligned}$$

Thus we have shown that $\mathcal{T}(v,\sigma)$ is continuous in v for a fixed σ. From the compactness of the embedding $C^{2+\alpha,1+\alpha/2}(\bar{D}) \subset\subset B$, we see that $\mathcal{T}(v,\sigma)$ is compact in v for fixed σ.

— Let us now consider $u_i = \mathcal{T}(v,\sigma_i)$, $i = 1,2$. Then the difference $u := u_1 - u_2$ satisfies

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + cu(x,t) = (\sigma_1 - \sigma_2)\mathcal{P}(v)(x,t), \qquad (x,t) \in D, \tag{3.27}$$

$$\frac{\partial u}{\partial \nu}(x,t) = (\sigma_1 - \sigma_2)g(x,t), \qquad (x,t) \in S, \tag{3.28}$$

$$u(x,0) = (\sigma_1 - \sigma_2)\psi(x), \qquad x \in \Omega. \tag{3.29}$$

It follows that

$$\|u\|^{\bar{D}}_{2+\alpha,1+\alpha/2} \leq c_2|\sigma_1 - \sigma_2|(2C\|v\|^{\bar{D}}_{\alpha,\alpha/2} + \|\psi\|^{\bar{\Omega}}_{2+\alpha} + \|g\|^{S}_{1+\alpha,1+\alpha/2}) \tag{3.30}$$

and we see that $\mathcal{T}$ is uniformly continuous with respect to σ.

Thus we have shown that all the conditions of the Leray-Schauder Theorem are satisfied and the proof is complete. □

The result above still holds under more general conditions if we replace some of the conditions (a1)–(a6) with alternative conditions. Apart from the available uniqueness results in [12] for weak solutions we can also easily prove a uniqueness result for classical solutions.

Theorem 3.3. *Let the conditions of Theorem 3.2 hold. Then there exists exactly one classical solution of the initial boundary value problem*

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + cu(x,t) = \mathcal{P}(u)(x,t) + \gamma(x,t), \qquad (x,t) \in D, \tag{3.31}$$

$$\frac{\partial u}{\partial \nu}(x,t) + b(x,t)u(x,t) = g(x,t), \qquad (x,t) \in S, \tag{3.32}$$

$$u(x,0) = \psi(x), \qquad x \in \Omega. \tag{3.33}$$

Proof. Let u_1 and u_2 be different solutions of (3.31). Then $u := u_1 - u_2$ satisfies

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + cu(x,t) = \mathcal{P}(u_1)(x,t) - \mathcal{P}(u_2)(x,t), \qquad (x,t) \in D,$$

$$\frac{\partial u}{\partial \nu}(x,t) = 0, \qquad (x,t) \in S,$$

$$u(x,0) = 0, \qquad x \in \Omega$$

and thus, due to assumption (a2),

$$\|u\|^{\bar{D}}_0 \leq \frac{1}{c_0}\|\mathcal{P}(u_1) - \mathcal{P}(u_2)\|^{\bar{D}}_0 \leq \frac{C}{c}\|u_1 - u_2\|^{\bar{D}}_0 = \frac{C}{c_0}\|u\|^{\bar{D}}_0.$$

Since $c_0 > C$, it follows that

$$\|u\|^{\bar{D}}_0 = 0.$$

□

The above results can also easily be adjusted to the case of Neumann or Dirichlet boundary conditions. In particular, the following result holds.

Corollary 3.4. *Assume the conditions of Theorem 3.2 hold, except for* (a6). *Instead we assume that* (a7) *holds. Then the initial boundary value problem*

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + c(x,t)u(x,t) = \mathcal{P}(u)(x,t) + \gamma(x,t), \qquad (x,t) \in D, \tag{3.34}$$

$$u(x,t) = h(x,t), \qquad (x,t) \in S, \tag{3.35}$$

$$u(x,0) = \psi(x), \qquad x \in \Omega \tag{3.36}$$

has at least one classical solution $u \in C^{2+\alpha,1+\alpha/2}(\bar{D})$. *The solution is unique if the conditions of Theorem 3.3 hold.*

In the next section, we show how these results can be used to hysteresis parameters from overdetermined boundary data in partial differential equations.

4. APPLICATIONS TO INVERSE PROBLEMS

Inverse problems for partial differential equations with classical solutions are often easier to solve than inverse problems without classical solutions (see for example [3, 4, 10, 11]). This is also the case for partial differential equations with hysteresis. Let conditions (a1)–(a6) hold and let the additional information

$$u(x_o,t) = \theta(t) \quad \text{for a } x_0 \in \partial\Omega, \quad 0 < t < T. \tag{4.1}$$

be given. To recover the properties of the Preisach operator $\mathcal{P}$ in (3.11), (3.12), (3.13), we use the following auxilliary result. Consider the initial boundary value problem

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + u(x,t) = p(u(x,t)) + \gamma(x,t), \qquad (x,t) \in D, \tag{4.2}$$

$$\frac{\partial u}{\partial \nu}(x,t) = g(x,t), \qquad (x,t) \in S, \tag{4.3}$$

$$u(x,0) = 0, \qquad x \in \Omega. \tag{4.4}$$

If the function p is Lipschitz and p, γ and g are given, the initial boundary value problem (4.2), (4.3), (4.4) has the unique classical solution u. The inverse problem now consists of recovering p on its maximal domain, where besides γ and g the additional information

$$u(x_o,t) = \theta(t) \quad \text{for a } x_0 \in \partial\Omega, \quad 0 < t < T, \tag{4.5}$$

is given. We further assume that p lies in the function space

$$LF_B := \{f : f \in Lip_1(\mathbb{R}),\ \ f(0) = 0,\ \ \|f\|_1 \le B\},$$

for some positive B and $Lip_1(\mathbb{R})$ is the space of Lipschitz functions on $\mathbb{R}$. To obtain sufficient conditions for the solvability of the inverse problem we consider another auxilliary problem

$$\psi_t(x,t) - \Delta\psi(x,t) = \gamma(x,t), \qquad (x,t) \in D, \tag{4.6}$$

$$\frac{\partial\psi}{\partial\nu}(x,t) = g(x,t), \qquad (x,t) \in S, \tag{4.7}$$

$$\psi(x,0) = 0, \qquad x \in \Omega. \tag{4.8}$$

Using the solution ψ of the above initial boundary value problem one can show the following.

Theorem 4.1. *Let g, γ and θ be as in* (4.2) *and* (4.5) *and let the following conditions hold:*

(c1) $\gamma \in Lip_1(D)$, $\gamma > 0$;
(c2) $g_t(x,t) \in Lip_1(D)$, $g(x,t) \le g(x_0,t)$, $g_t(x,t) > 0$, $g(x,0) = 0$, $g(x,t) \ge 0$;
(c3) $\theta'(t) \in Lip_1([0,T])$, $\theta(0) = 0$, $\theta'(t) > 0$, $\theta'(0) = \gamma(x,0)$;
(c4) $\|\theta'(t) - \psi_t(x_0,t)\|_1 < 1/2C_1$, *where ψ is the solution of* (4.6), (4.7), (4.8).

Then the inverse problem of finding $p \in LF_{C_1}$ such that

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + u(x,t) = p(u(x,t)) + \gamma(x,t), \qquad (x,t) \in D, \tag{4.9}$$

$$\frac{\partial u}{\partial\nu}(x,t) = g(x,t), \qquad (x,t) \in S, \tag{4.10}$$

$$u(x,0) = 0, \qquad x \in \Omega \tag{4.11}$$

has the unique solution p on the maximal domain $\mathcal{I}(u)$.

Proof. See [10]. □

Remark. *Conditions* (c1) *and* (c2) *secure that u obtains its maximum at $u(x_0,t)$ and thus the function p can be determined on its maximal domain. If $g(x,t) \le g(x_0,t)$ is not valid, then $p(u)$ can only be determined on $[0,\theta(T)]$. However this is sometimes sufficient to determine a hysteresis operator, as we will see later.*

Instead of LF_B we will work with a slightly weaker version to recover $\mathcal{P}$.

Definition 4.2. *For every $B \in \mathbb{R}_+$ and every interval I with $0 \in I$, we define the set $LF_{I,B}$ as*

$$LF_{I,B} := \{f : f \in Lip_1(I),\ \ f(0) = 0,\ \ \|f\|_1^I \le B\}.$$

The result in 4.1 obviously remains valid, if we substitute the condition $p \in LF_{\mathcal{I}(u),C_1}$ for

$$p \in LF_{C_1}.$$

Now we present two examples, where one can retrieve information about a hysteresis operator.

Example 4.3 (Play operator). *Here we consider a special inverse problem for the Play operator $\mathcal{P} = \mathcal{F}_r$. We want to recover the parameters β and r in the initial value problem*

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + cu(x,t) = \beta \mathcal{F}_r(u)(x,t) + \gamma(x,t), \quad (x,t) \in D, \tag{4.12}$$

$$\frac{\partial u}{\partial \nu}(x,t) = g(x,t), \quad (x,t) \in S, \tag{4.13}$$

$$u(x,0) = 0, \quad x \in \Omega. \tag{4.14}$$

The additional data for the inverse problem to g and γ is as in Theorem 4.1:

$$u(x_o,t) = \theta(t) \quad \text{for a } x_0 \in \partial\Omega, \quad 0 < t < T. \tag{4.15}$$

We assume that g, γ and θ satisfy the following conditions:

(b1) $g_t(x,t) \in Lip_1(\partial\Omega \times [0,T])$, $g(x,t) \geq 0$, $g(x,0) > 0$;
(b2) $\theta'(t) \in Lip_1([0,T])$, $\theta(0) = 0$, $\theta'(0) = \gamma(x,0)$;
(b3) $\|\theta'(t) - \psi_t(x_0,t)\|_1 < 1/2C_1$, *where ψ is the solution of* (4.6), (4.7), (4.8);
(b4) *There exists a $t_0 > 0$ with $\theta(t_0) > r$, $\theta'(t) > 0$ in $[0,t_0)$, where t_0 is a local maximum of θ and $g_t(x,t) \geq 0$ in $[0,t_0]$;*
(b5) $\gamma \in Lip_1(D)$, $\gamma > 0$, $\gamma_t(x,t) \geq 0$, *on* $[0,t_0]$;
(b6) $\beta \leq c$ *and* $p : \mathbb{R} \mapsto \mathbb{R}$ *with* $p(u) := cu - \beta h_r(u)$ *satisfies* $p \in LF_{\mathcal{I}(u),C_1}$, *where* $h_r(u) = \max\{u - r, 0\}$.

Theorem 4.4. *Under conditions* (b1)–(b6), *the additional data* (4.15) *is sufficient for the unique determination of the hysteresis-output $\beta\mathcal{F}_r(\theta)$ on $[0,t_0]$, which in turn can be used to uniquely determine the parameters β and r.*

Proof. We present a constructive proof here, which can be used as basis for a numerical algorithm. The main idea is not to calculate β and r directly, but to determine the hysteresis output $\beta\mathcal{F}_r(\theta)(t)$ first.
Using basic calculus one can show that the time-derivative u_t satisfies the positivity principle on $\Omega \times (0,t_0)$ (see [5, Theorem 5.1.7]), i.e.,

$$u_t(x,t) \geq 0 \quad \text{in} \quad \Omega \times (0,t_0).$$

This yields that on $\Omega \times (0,t_0)$

$$\mathcal{F}_r(u)(x,t) = h_r(u(x,t)), \quad h_r(u) = \max\{u - r, 0\}.$$

Now we set $p(u) := h_r(u) - cu$, which yields

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = p(u(x,t)) + \gamma(x,t), \quad (x,t) \in \Omega \times (0,t_0), \tag{4.16}$$

$$\frac{\partial u}{\partial \nu}(x,t) = g(x,t), \quad (x,t) \in \partial\Omega \times (0,t_0), \tag{4.17}$$

$$u(x,0) = 0, \quad x \in \Omega. \tag{4.18}$$

Since the functions g, γ, θ and p satisfy all conditions of Theorem 4.1 on $\Omega \times (0, t_0)$, we conclude that $p(u)$ is uniquely determined on $[0, \theta(t_0)]$. In particular $p(\theta(t)) = \beta\mathcal{F}_r(\theta)(t) - c\theta(t))$ is uniquely determined. To determine the unknown parameters we proceed as follows.

— Determine t_0.
— Set $p(u) := h_r(u) - cu$ and recover $p(u)$. This gives us knowledge of $\theta(t)$ and $w(t) := \mathcal{F}_r(\theta)(t)$ on $[0, t_0]$.
— To recover β and r, we determine the values $w(t)$ and θ at two points s_1 and s_2, with $w(s_1) \neq 0$, $w(s_2) \neq 0$ and $\theta(s_1) \neq \theta(s_2)$. This yields

$$w(s_1) = \beta\theta(s_1) - \beta r = \beta(\theta(s_1) - r),$$
$$w(s_2) = \beta\theta(s_2) - \beta r = \beta(\theta(s_2) - r).$$

Since $\beta \neq 0$, we get

$$r = \theta(s_1) - \frac{w(s_1)}{\beta}.$$

And since $\theta(s_1) \neq \theta(s_2)$, we get

$$\beta = \frac{w(s_2) - w(s_1)}{\theta(s_2) - \theta(s_1)}.$$

□

Remark. *The result above can easily be adjusted to recover the parameters α_i, r_i, $1 \leq i \leq m$ of the operator $\mathcal{P} = \sum_{i=1}^{m} \alpha_i \mathcal{F}_{r_i}$.*

Remark. *In the same way, one can also retrieve partial (or sometimes global) information about the shape function of a general Prandtl operator* [5]. In the following example, we consider the recovery of a special Preisach operator, the nonlinear play operator $q(\mathcal{F}_r)$. Here we want to determine the function q and the parameter r.

Example 4.5 (Nonlinear play operator). *The nonlinear play operator $\mathcal{P}$ is determined through a Lipschitz continuous function q, i.e.,*

$$\mathcal{P}(v) = q(\mathcal{F}_r(v)).$$

We consider the initial boundary value problem

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + cu(x,t) + q(\mathcal{F}_r(u)(x,t)) = \gamma(x,t), \qquad (x,t) \in D, \qquad (4.19)$$

$$\frac{\partial u}{\partial \nu}(x,t) = g(x,t), \qquad (x,t) \in S, \qquad (4.20)$$

$$u(x,0) = 0, \qquad x \in \Omega. \qquad (4.21)$$

Let conditions (b1)–(b5) *of Example 4.3 hold, as well as*

(b6') $q \circ h_r \in LF_{\mathcal{I}(u), C_1}$, *$q$ is strictly monotonically increasing.*

Analogously to the proof of Theorem 4.4, one can determine the parameter r and the function q as well on $[0, \mathcal{F}_r(\theta)(t_0)]$. *Here it is of course of interest to recover the function q on the largest interval possible. To show that this can be done, we prove the following theorem.*

Theorem 4.6 (A special positivity result). *Let u be a solution of* (4.19), (4.20), (4.21) *and let the above conditions hold, as well as, for some* $x_0 \in \partial\Omega$:

(e1) $\gamma(x_0,t) > \gamma(x,t)$ *for every* $x \in \Omega$, $t \in (0,T)$;
(e2) $g(x_0,t) > g(x,t)$ *for every* $x \in \partial\Omega$, $t \in (0,T)$, $g_t(x,t) > 0$ *on* S.

Then it also holds that $u(x,t) \leq u(x_0,t)$ *for every* $x \in \Omega$, $t \in (0,T)$.

Proof. The function $u(x_0,t)$ satisfies the equation

$$\frac{\partial u}{\partial t}(x_0,t) - \Delta u(x_0,t) + cu(x_0,t) + q(\mathcal{F}_r(u(x_0,\cdot))(t)) = \gamma(x_0,t), \quad t \in (0,T).$$

If we set $w(x,t) := u(x_0,t) - u(x,t)$, using the Lipschitz property of q and $q(0) = 0$, we conclude that

$$\frac{\partial w}{\partial t}(x,t) - \Delta w(x,t) + cw(x,t) + \alpha(x,t)(\mathcal{F}_r(u(x_0,\cdot))(t) - \mathcal{F}_r(u(x,\cdot))(t)) = \underbrace{\gamma(x_0,t) - \gamma(x,t)}_{>0}, \quad (x,t) \in D, \tag{4.22}$$

$$\frac{\partial w}{\partial \nu}(x,t) = g(x_0,t) - g(x,t) > 0, \quad (x,t) \in S, \tag{4.23}$$

$$w(x,0) = 0, \quad x \in \Omega \tag{4.24}$$

with some $\alpha(x,t) \in L_\infty$ and $\alpha(x,t) \geq 0$, $\forall (x,t) \in D$, since we assumed q to be strictly monotonically increasing. Now we assume that at some point in D there is $w < 0$ and set

$$t^* := \inf\{t : \ w(x,t) < 0 \ \text{for a} \ X = (x,t) \in \bar{D}\}.$$

Since $\frac{\partial w}{\partial \nu} > 0$ on S, there exists a $X^* = (x^*, t^*)$ with $w(X^*) = 0$ in $\bar{D}$. Due to $\mathcal{F}_r(u)(x_0,0) = \mathcal{F}_r(u)(x,0) = 0$ and $r > 0$, there exists a $t_0 > 0$ with

$$\frac{\partial w}{\partial t}(x,t) - \Delta w(x,t) + cw(x,t) = \gamma(x_0,t) - \gamma(x,t), \quad (x,t) \in \Omega \times (0,t_0),$$

$$\frac{\partial w}{\partial \nu}(x,t) = g(x_0,t) - g(x,t) > 0, \quad (x,t) \in \partial\Omega \times (0,t_0),$$

$$w(x,0) = 0, \quad x \in \Omega.$$

From Lemma 2.3[(2.8)], we can conclude that $w(x,t) \geq 0$ on $\Omega \times (0,t_0)$. Condition (e2) then gives that X^* is inside of D. Since $w(\cdot,t^*)$ has its minimum in X^* we get

$$w_t(X^*) \leq 0 \quad \Delta w(X^*) \geq 0.$$

Using (4.22) gives

$$w_t(X^*) - \Delta w(X^*) - \alpha(X^*)(\mathcal{F}_r(u(x_0,\cdot))(t^*) - \mathcal{F}_r(u(x^*,\cdot))(t^*)) > 0$$

and since $\alpha(X^*) \geq 0$ we conclude that

$$\mathcal{F}_r(u(x_0,\cdot))(t^*) - \mathcal{F}_r(u(x^*,\cdot))(t^*) < 0.$$

Since $\mathcal{F}_r(u)(x_0,0) = \mathcal{F}_r(u)(x,0) = 0$, using Lemma 2.3 there must be a t_1, $0 < t_1 < t^*$ with $u(x_0,t_1) < u(x,t_1)$. However, this is a contradiction to

$$t^* := \inf\{t : \ w(x,t) < 0 \ \text{ for a } X = (x,t) \in \bar{D}\}.$$

□

To ensure that the function q is recovered on a maximum interval, one can impose the conditions

$$q(\mathcal{F}_r(u)(x,t) \leq \gamma(x,t), \quad \theta(t_0) \geq \theta(t), \quad 0 < t < T,$$

The first condition ensures $u(x,t) \geq 0$ on D and the second one, together with Theorem 4.6, ensures that $u(x,t) \leq u(x_0,t)$ on D. All in all we get $0 \leq u(x,t) \leq \theta(t_0)$. Thus the maximum domain for u is given by the interval $[0, \mathcal{F}_r(\theta)(t_0)]$.

Remark. *We are aware that the conditions above seem quite severe. However, one has to bear in mind that we do not use global knowledge of the function u, but a simple overdetermination and we have shown that given classical solutions of partial differential equations with hysteresis, it is often possible to recover knowledge about the type of hysteresis. More examples for inverse problems with hysteresis using classical solutions can be found in* [5]. *There it is also shown how one can recover more general properties of Preisach operators.*

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