Opuscula Mathematica • Vol. 27 • No. 2 • 2007

Józef Kalinowski

ON THE EQUIVALENCE OF PRE-SCHRÖDER EQUATIONS

Abstract. In the paper the equivalence of the system of two pre-Schröder functional equations (equations (S_n), (S_m) for $m > n \geq 3$, $n, m \in \mathbb{N}$) and the whole system (S), is considered. The results solve the problem of Gy. Targonski [4] in a particular case.

Keywords: pre-Schröder equations, Targonski's problem, torsion free semigroups.

Mathematics Subject Classification: 39B62, 39B42.

1. INTRODUCTION

Let X be a set, and let $g : X \to X$ be a given function. Then the equation

$$f(g(x)) = s \cdot f(x), \quad x \in X,$$

for the eigenfunction $f : X \to Y$, of the operator of substitution $f \to f \circ g$, where $(Y, \cdot)$ is a commutative semigroup, corresponding to the eigenvalue $s \in Y$, is named the Schröder equation.

Iterating the Schröder equation n times, we obtain

$$f(g_n(x)) = s^n \cdot f(x) \quad x \in X,$$

where g_n denotes the n-th iterate of the function g for an integer $n \geq 0$, i.e.,

$$g_0(x) = x, \quad g_{n+1}(x) = g(g_n(x)), \quad x \in X.$$

Next, we raise both sides of the Schröder equation to the n-th power, getting

$$f^n(g(x)) = s^n \cdot f^n(x), \quad x \in X,$$

Eliminating the factor s^n from the above equations we arrive at the system

$$f^n(g(x)) = f(g_n(x)) \cdot f^{n-1}(x) \quad \text{for all integers } n \geq 2, \tag{S}$$

(for $n = 1$ system (S) is not interesting, since it becomes an identity).

This infinite system (S) of functional equations has been introduced by Gy. Targonski under the name of the pre-Schröder system. The n-th equation of system (S) will be denoted by (S_n).

If f fulfils the Schröder equation, then f also satisfies infinite system (S).

The problem of equivalence between the pre-Schröder equations was posed in 1970 by Gy. Targonski [4]. He also posed the question whether if a part of system (S) and whole system (S) are equivalent.

The positive solution of the problem was given in 1970 by Z. Moszner [3]. He proved that equation (S_2) and whole system (S) are equivalent under the assumption that Y is a countable set. In 1972 Gy. Targonski [5] proved the equivalence of (S_2) and (S) in the case where $(Y, \cdot)$ is a commutative group.

The equivalence of (S) and of particular equations (S_n), $n \geq 2$, has been investigated in 1975 by J. Drewniak, J. Kalinowski [1] (see also chapter 9.2 in the book by M. Kuczma, B. Choczewski, R. Ger [2]).

The paper is a continuation of that research. We will consider the question of when the system of two equations (S_m), (S_n), $m, n \in \mathbb{N}$, $m > n \geq 3$, and the whole system of the pre-Schröder equations (S) are equivalent.

2. PRELIMINARIES

Let $\mathbb{N}$ denote the set of positive integers. We put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $(Y, \cdot)$ be a commutative semigroup. We denote by 0 a zero element in Y, such that

$$\bigwedge_{y \in Y} 0 \cdot y = 0,$$

provided that it does exist. It is obvious, that if a zero exists, it is unique.

In [1] we have proved the following

Theorem 1. *Let $(Y, \cdot)$ be a commutative semigroup satisfying the following cancellation law*

$$\bigwedge_{x,y,z \in Y} (xy = xz \ \wedge \ x \neq 0) \Longrightarrow (y = z). \tag{1}$$

Then every function f verifying equation (S_2) satisfies all of the equations of system (S).

We also proved [1] that no equation (S_n) for $n \geq 3$ is equivalent to whole system (S). No two equations (S_m), (S_n), $m \neq n$ for $m, n \geq 3$, $m, n \in \mathbb{N}$, are equivalent, either.

3. MAIN RESULTS

In the paper, we suppose that for a fixed number $n > 1$, $n \in \mathbb{N}$, a semigroup $(Y, \cdot)$ is without the n-th degree torsion, i.e.,

$$\bigwedge_{x,y\in Y} (x^n = y^n) \Longrightarrow (x = y). \tag{2}$$

Theorem 2. *Let $(Y, \cdot)$ be a commutative semigroup satisfying cancellation law* (1) *and without the n-th degree torsion. If the function f satisfies system of two equations (S_n), (S_{n+1}) for $n \geq 3$, $n \in \mathbb{N}$, then f is a solution of whole system (S).*

Proof. Let $n \geq 3$. Multiplying, by $f^n(x)$, both sides of equation (S_n) with variable x replaced by $g(x)$ and using the commutativity of the multiplication, we obtain

$$f^n(g_2(x)) \cdot f^n(x) = f(g_{n+1}(x)) \cdot f^n(x) \cdot f^{n-1}(g(x)). \tag{3}$$

From (3), using the equation (S_{n+1}), we obtain

$$\begin{aligned}[f(g_2(x)) \cdot f(x)]^n &= [f(g_{n+1}(x)) \cdot f^n(x)] \cdot f^{n-1}(g(x)) = \\ &= f^{n+1}(g(x)) \cdot f^{n-1}(g(x)) = [f^2(g(x))]^n.\end{aligned}$$

Since $(Y, \cdot)$ is a semigroup without n-th degree torsion, the function f satisfies equation (S_2). By Theorem 1, we obtain the statement of the theorem. □

Theorem 3. *Let $(Y, \cdot)$ be a commutative semigroup satisfying cancellation law* (1) *and without the n-th degree torsion. If the function f satisfies the system of two equations (S_n), (S_{2n}) for $n \geq 3$, $n \in \mathbb{N}$, then f is a solution of whole system (S).*

Proof. Let $n \geq 3$. Multiplying both sides of equation (S_{2n}) by $f^{n-1}(g_n(x))$ and using equation (S_n) with variable x replaced by $g_n(x)$, we obtain

$$f^{2n}(g(x)) \cdot f^{n-1}(g_n(x)) = f^n(g_{n+1}(x)) \cdot f^{2n-1}(x). \tag{4}$$

Multiplying both sides of equation (4) by $f^{n(n-1)}(g(x))$ and using equation (S_n) with variable x replaced by $g(x)$, we obtain

$$\begin{aligned}f^{n(n+1)}(g(x)) \cdot f^{n-1}(g_n(x)) &= f^n(g_{n+1}(x)) \cdot f^{n(n-1)}(g(x)) \cdot f^{2n-1}(x) = \\ &= [f(g_{n+1}(x)) \cdot f^{n-1}(g(x))]^n \cdot f^{2n-1}(x) = \\ &= f^{n^2}(g_2(x)) \cdot f^{2n-1}(x).\end{aligned} \tag{5}$$

From (5) and the equation (S_n), there follows

$$\begin{aligned}[f(g_2(x)) \cdot f(x)]^{n^2} &= f^{n^2}(g_2(x)) \cdot f^{2n-1}(x) \cdot f^{(n-1)^2}(x) = \\ &= f^{n(n+1)}(g(x)) \cdot f^{n-1}(g_n((x)) \cdot f^{(n-1)^2}(x) = \\ &= f^{n(n+1)}(g(x)) \cdot [f(g_n(x)) \cdot f^{n-1}(x)]^{n-1} = \\ &= f^{n(n+1)}(g(x)) \cdot [f^n(g(x))]^{n-1} = [f^2(g(x))]^{n^2}.\end{aligned}$$

Because $(Y, \cdot)$ is a semigroup without the n-th degree torsion, the function f verifies equation (S_2). By Theorem 1, f is a solution of (S). This completes the proof. □

Remark 1. *Using the methods analogous to those in the proof of Theorem 3, we can prove that a solution of system (S_4), (S_6) is a solution of equation (S_2). By Theorem 1, we obtain that system (S_4), (S_6) is equivalent to whole system (S).*

Theorem 4. *Let* $(Y, \cdot)$ *be a commutative semigroup satisfying cancellation law* (1) *and without the n-th degree torsion. If the function f satisfies the system of two equations (S_{n+1}), (S_{2n+1}) for* $n \geq 2$, $n \in \mathbb{N}$, *then f is a solution of whole system (S).*

Proof. Let $n \geq 2$. Multiplying both sides of equation (S_{2n+1}) by $f^n(g_n(x))$ and using equation (S_{n+1}) with variable x replaced by $g_n(x)$, we obtain

$$f^{2n+1}(g(x)) \cdot f^n(g_n(x)) = [f(g_{2n+1}(x)) \cdot f^n(g_n(x))] \cdot f^{2n}(x) = f^{n+1}(g_{n+1}(x)) \cdot f^{2n}(x).$$

Multiplying both sides of the above equation by $f^{n(n-1)}(x)$ and using the commutativity of the multiplication and equation (S_{n+1}), we obtain

$$f^{2n+1}(g(x)) \cdot f^n(g_n(x)) \cdot f^{n(n-1)}(x) = f^{n+1}(g_{n+1}(x)) \cdot f^{n(n+1)}(x) =$$
$$= [f(g_{n+1}(x)) \cdot f^n(x)]^{n+1} = [f^{n+1}(g(x))]^{n+1} = f^{(n+1)^2}(g(x)).$$

Therefore

$$f^{2n+1}(g(x)) \cdot f^n(g_n(x)) \cdot f^{n(n-1)}(x) = f^{(n+1)^2}(g(x)). \tag{6}$$

Let us consider two possible cases:

(a) $f^{2n+1}(g(x)) \neq 0$.

Because $(Y, \cdot)$ satisfies cancellation law (1), from equation (6) we obtain

$$f^n(g_n(x)) \cdot f^{n(n-1)}(x) = f^{n^2}(g(x)). \tag{7}$$

(b) $f^{2n+1}(g(x)) = 0$.

Since $(Y, \cdot)$ is a semigroup satisfying cancellation law (1), then Y has no zero divisors and we obtain $f(g(x)) = 0$. Replacing the variable x by $g(x)$ in equation (S_{n+1}), we obtain

$$f^{n+1}(g_2(x)) = f(g_{n+2}(x)) \cdot f^n(g(x)),$$

whence $f(g_2(x)) = 0$. Replacing x by $g_2(x)$ in equation (S_{n+1}), we obtain $f(g_3(x)) = 0$. By induction, there is $f(g_n(x)) = 0$ for $n \in \mathbb{N}$. Then

$$f^n(g_n(x)) \cdot f^{n(n-1)}(x) = 0$$

and (b) yields $f^{n^2}(g(x)) = 0$. So in case (b), equation (7) is satisfied too.

The equation (7) can be written in the form

$$[f(g_n(x)) \cdot f^{n-1}(x)]^n = [f^n(g(x))]^n.$$

Owing to (2), the function f satisfies equation (S_n). Then the function f fulfils the system of equations (S_n), (S_{n+1}). By Theorem 2, the function f satisfies whole system (S). □

Remark 2. *Using the methods analogous to those in the proof of Theorem 4, we can prove that* (S_4), (S_7) *imply* (S_3), *as well as that* (S_4), (S_9) *imply* (S_3). *Then, by Theorem 2, we obtain that both systems* (S_4), (S_7) *and* (S_4), (S_9) *are equivalent to whole system (S).*

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Józef Kalinowski
kalinows@ux2.math.us.edu.pl

Silesian University
Department of Mathematics
ul. Bankowa 14, 40-007 Katowice, Poland

Received: December 6, 2004.