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INTEGRABLE THREE-DIMENSIONAL COUPLED NONLINEAR DYNAMICAL SYSTEMS RELATED WITH CENTRALLY EXTENDED OPERATOR LIE ALGEBRAS

Abstract. A hierarchy of Lax-type flows on a dual space to the centrally extended Lie algebra of integral-differential operators with matrix-valued coefficients is considered. By means of a specially constructed Backlund transformation the Hamiltonian representations for these flows coupled with suitable eigenfunctions and adjoint eigenfunctions evolutions of associated spectral problems are obtained. The Hamiltonian description of the corresponding set of additional symmetry hierarchies is represented. The relation of these hierarchies with Lax integrable (3+1)-dimensional nonlinear dynamical systems and their triple Lax-type linearizations is analysed.

Keywords: centrally extended operator Lie algebra, Lax-type flows, Backlund transformation, “ghost” symmetries.

Mathematics Subject Classification: Primary 37K05, 37K10, 37K30, 37K35; Secondary 37K15.

1. INTRODUCTION

For the first time Lax representations [8] for integrable (1+1)-dimensional nonlinear dynamical system hierarchies [4, 13, 18] on functional manifolds were interpreted as Hamiltonian flows on a dual space to the Lie algebra of integral-differential operators in [1]. An algebraic method for constructing Lax integrable (2+1)-dimensional nonlinear dynamical systems by means of two commuting flows from the hierarchy on the suitable coadjoint action orbit of an integral-differential operator with an infinite integral part was proposed in [5, 27]. The relation of some Lax integrable (1+1)- and (2+1)-dimensional systems with corresponding hierarchies of Hamiltonian flows on dual spaces to centrally extended by means of the standard Maurer-Cartan two-cocycle Lie algebras was intensively investigated in [6, 19, 22, 23].

Every Hamiltonian flow of such a type on a dual space either to the operator Lie algebra or to its central extension can be written as a compatibility condition of the spectral relationship for the corresponding integral-differential operator and the suitable eigenfunction evolution. If the above spectral relationship admits a finite set of eigenvalues, an important problem of finding the Hamiltonian representation for the Lax-type hierarchy coupled with the evolutions of eigenfunctions and appropriate adjoint eigenfunctions naturally arises. It was partly solved in the papers [7, 16, 17, 20, 25] for the Lie algebra of integral-differential operators and its supergeneralization by means of the variational Casimir functionals property under some Lie-Backlund transformation.

Section 2 deals with a general Lie-algebraic scheme for constructing a hierarchy of Lax-type integrable flows as Hamiltonian ones on a dual space to the centrally extended Lie algebra of integral-differential operators with matrix-valued coefficients.

In section 3 the Hamiltonian structure for the related coupled Lax-type hierarchy is obtained by means of the Backlund transfrmation technique developed in [7,17,20,25].

In section 4 the corresponding hierarchies of additional or so called "ghost" symmetries [2, 7, 12] for the coupled Lax-type flows are stated to be Hamiltonian too. It is established that the additional hierarchy of Hamiltonian flows is generated by the Poisson structure being equal to the tensor product of the $\mathcal{R}$-deformed canonical Lie-Poisson bracket [4,14,17,20,25,28] and the standard Poisson bracket on the related eigenfuction and adjoint eigenfunction space [3,17,20,25], and the corresponding natural powers of a suitable eigenvalue are their Hamiltonian functions. The method for introducing another independent variable into (2+1)-dimensional nonlinear dynamical systems by use of the additional symmetries, which preserves their Lax integrability, is proposed and an integrable $(3 + 1)$-dimensional analog of the Davey-Stewartson system [26, 29] is constructed.

2. THE LIE-ALGEBRAIC STRUCTURE OF LAX-TYPE INTEGRABLE (2+1)-DIMENSIONAL DYNAMICAL SYSTEMS

Let $\tilde{\mathcal{G}} := C^{\infty}(\mathbb{S} \times \mathbb{S}; \mathcal{G})$ be a Lie algebra of smooth mappings taking values in a semi-simple matrix Lie algebra $\mathcal{G}$. By means of $\tilde{\mathcal{G}}$ one constructs a Lie algebra $\hat{\mathcal{G}}$ of matrix integral-differential operators:

$$a := \xi^m + \sum_{j<m} a_j \xi^j,$$

where $a_j \in \tilde{\mathcal{G}}$, $j < m$, $j \in \mathbb{Z}$, $m \in \mathbb{N}$, the symbol $\xi := \partial/\partial x$ denotes the differentiation with respect to the independent variable $x \in \mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}$. The Lie structure in $\hat{\mathcal{G}}$ is defined as

$$[a, b] := a \circ b - b \circ a$$

for all $a,\ b \in \hat{\mathcal{G}}$, where "$\circ$" is the composition of integral-differential operators taking the form:

$$a \circ b := \sum_{\alpha \in \mathbb{Z}_+} \frac{1}{\alpha!} \frac{\partial^{\alpha} a}{\partial \xi^{\alpha}} \frac{\partial^{\alpha} b}{\partial x^{\alpha}} \ .$$

On the Lie algebra $\hat{\mathcal{G}}$ there exists the *ad*-invariant nondegerated symmetric bilinear form:

$$(a,b) := \int_0^{2\pi}\int_0^{2\pi} Tr\,(a\circ b)\;dxdy, \tag{1}$$

where Tr-operation for all $a\in\hat{\mathcal{G}}$ is given by the expression:

$$Tr\,a := res_\xi\, tr\,a = tr\,a_{-1},$$

and tr is the matrix trace. With use of scalar product (1) the Lie algebra $\hat{\mathcal{G}}$ is transformed into a metrizable one. As a consequence, its dual linear space of matrix integral-differential operators $\hat{\mathcal{G}}^*$ is identified with the Lie algebra, that is, $\hat{\mathcal{G}}^*\simeq\hat{\mathcal{G}}$.

The linear subspaces $\hat{\mathcal{G}}_+\subset\hat{\mathcal{G}}$ and $\hat{\mathcal{G}}_-\subset\hat{\mathcal{G}}$ such that

$$\begin{aligned}
\hat{\mathcal{G}}_+ &:= \left\{ a := \xi^{n(\hat{a})} + \sum_{j=0}^{n(\hat{a})-1} a_j\xi^j :\ a_j\in\tilde{\mathcal{G}}, j=\overline{0,n(\hat{a})} \right\},\\
\hat{\mathcal{G}}_- &:= \left\{ b := \sum_{j=0}^{\infty}\xi^{-(j+1)}b_j :\ b_j\in\tilde{\mathcal{G}}, j\in\mathbb{Z}_+ \right\},
\end{aligned} \tag{2}$$

are Lie subalgebras in $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}}=\hat{\mathcal{G}}_+\oplus\hat{\mathcal{G}}_-$. Owing to splitting $\hat{\mathcal{G}}$ into the direct sum of its Lie subalgebras (2), one can construct a Lie-Poisson structure on $\hat{\mathcal{G}}^*$ by use of the special linear endomorphism $\mathcal{R}$ of $\hat{\mathcal{G}}$ [4,14,28]:

$$\mathcal{R} := (P_+ - P_-)/2,\quad P_\pm\hat{\mathcal{G}} := \hat{\mathcal{G}}_\pm,\quad P_\pm\hat{\mathcal{G}}_\mp = 0.$$

The central extended Lie commutator on $\hat{\mathcal{G}}_c := \hat{\mathcal{G}}\oplus\mathbb{C}$ is given as [6,19,23]:

$$[(a,\alpha),(b,\beta)] := ([a,b],\omega(\hat{a},\hat{b})), \tag{3}$$

where $\alpha,\beta\in\mathbb{C}$, being generated by means of the standard Maurer-Cartan two-cocycle on $\hat{\mathcal{G}}$:

$$\omega(a,b) := (a,[\partial/\partial y,b]),$$

where $\partial/\partial y$ is the differentiation with respect to the independent variable $y\in\mathbb{S}$ and $[\partial/\partial y,b] := \partial b/\partial y$. Commutator (3) can be deformed by means of the endomorphism $\mathcal{R}$ of $\hat{\mathcal{G}}$ defined above:

$$[(a,\alpha),(b,\beta)]_{\mathcal{R}} := ([a,b]_{\mathcal{R}},\omega_{\mathcal{R}}(a,b)), \tag{4}$$

where the $\mathcal{R}$-commutator takes the form:

$$[a,b]_{\mathcal{R}} := [\mathcal{R}a,b]+[a,\mathcal{R}b],$$

and the $\mathcal{R}$-deformed two-cocycle is determined in the following way:

$$\omega(a,b)_{\mathcal{R}} := \omega(\mathcal{R}a,b)+\omega(a,\mathcal{R}b)\ .$$

For any Frechet smooth functionals $\gamma, \mu \in \mathcal{D}(\hat{\mathcal{G}}_c^*)$ the Lie-Poisson bracket on $\hat{\mathcal{G}}_c^*$ related with commutator (4) and the extended scalar product:

$$((a, \alpha), (b, \beta)) := (a, b) + \alpha\beta,$$

where $a, b \in \hat{\mathcal{G}}$ and $\alpha, \beta \in \mathbb{C}$, is given as

$$\{\gamma, \mu\}_{\mathcal{R}}(l) = (l, [\nabla\gamma(l), \nabla\mu(l)]_{\mathcal{R}}) + c\omega_{\mathcal{R}}(\nabla\gamma(l), \nabla\mu(l)), \tag{5}$$

where $l \in \hat{\mathcal{G}}^*$ and $c \in \mathbb{C}$. Based on scalar product (1) the gradient $\nabla\gamma(l) \in \hat{\mathcal{G}}$ of some functional $\gamma \in \mathcal{D}(\hat{\mathcal{G}}_c^*)$ at the point $l \in \hat{\mathcal{G}}^*$ is naturally defined as

$$\delta\gamma(l) := (\nabla\gamma(l), \delta l) \ .$$

Construct now the Casimir functionals $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, as

$$\gamma_n(l) := \int_0^{2\pi}\int_0^{2\pi} Tr\,(\xi^n \hat{l}_0)dxdy, \tag{6}$$

being invariant with respect to Ad^*-action of the corresponding to $\hat{\mathcal{G}}_c^*$ abstract Lie group $\hat{G}_c$ and satisfying the following condition [22]

$$(l - c\partial/\partial y) \circ \Phi = \Phi \circ (l_0 - c\partial/\partial y) \tag{7}$$

at a point $l \in \hat{\mathcal{G}}^*$. In (7)

$$\hat{l}_0 := \xi^m + \sum_{j<m} c_j \xi^j \in \hat{\mathcal{G}}^*,$$

where $c_j \in \tilde{\mathcal{G}}$, $[\xi, c_j] = 0$, $j < m$, $j \in \mathbb{Z}$ and $m \in \mathbb{N}$,

$$\Phi = 1 + \sum_{r>0} \Phi_r \xi^{-r} \in G_-,$$

where $\Phi_r \in \tilde{\mathcal{G}}$, $r \in \mathbb{N}$, and G_- means the suitable abstract Lie group [5, 6, 22], generated by the Lie subalgebra $\hat{\mathcal{G}}_-$. As in [22] one can show that condition (7) is equivalent to the following relationship

$$[l - c\partial/\partial y, \nabla\gamma_n(l)] = 0, \tag{8}$$

for all $n \in \mathbb{N}$. In the case of $c = 0$ the Casimir functionals take Adler's form [1, 17].

Lie-Poisson bracket (5) generates the hierarchy of Hamiltonian dynamical systems on $\hat{\mathcal{G}}_c^*$ with Casimir functionals $\gamma_n \in I(\mathcal{G}_c^*)$, $n \in \mathbb{N}$, as Hamiltonian functions taking the form:

$$d\hat{l}/dt_n := [\mathcal{R}\nabla\gamma_n(l),\ l - c\partial/\partial y] = [(\nabla\gamma_n(l))_+,\ l - c\partial/\partial y] \ . \tag{9}$$

where the subscript "+" denotes a differential part of the corresponding integral-differential operator. The latter equation is equivalent to the usual commutator Lax-type representation. It is easy to verify that for every $n \in \mathbb{N}$ the above relationship is the compatibility condition of the following system of linear integral-differential equations:

$$(l - c\partial/\partial y)f = \lambda f, \tag{10}$$

and

$$df/dt_n = (\nabla\gamma_n(l))_+ f, \tag{11}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $f \in W := W(\mathbb{S}\times\mathbb{S};\mathbf{H})$ and $\mathbf{H}$ is a matrix representation space of the Lie algebra $\mathcal{G}$. The dynamical system related to (11) on the adjoint function space $W^* := W^*(\mathbb{S}\times\mathbb{S};\mathbf{H})$ takes the form:

$$df^*/dt_n = -(\nabla\gamma_n(l))_+^* f^*, \tag{12}$$

where $f^* \in W^*$ is a solution of the adjoint spectral relationship

$$(l^* + c\partial/\partial y)f^* = \nu f^*, \tag{13}$$

with a spectral parameter $\nu \in \mathbb{C}$.

Further we shall assume that spectral relationship (10) admits $N \in \mathbb{N}$ different eigenvalues $\lambda_i \in \mathbb{C}$, $i = \overline{1,N}$, and study algebraic properties of equation (9) combined with $N \in \mathbb{N}$ copies of (11):

$$df_i/dt_n = (\nabla\gamma_n(\hat{l}))_+ f_i, \tag{14}$$

for the corresponding eigenfunctions $f_i \in W(\mathbb{S}\times\mathbb{S};\mathbf{H})$, $i = \overline{1,N}$, and the same number of copies of (12):

$$df_i^*/dt_n = -(\nabla\gamma_n(\hat{l}))_+^* f_i^*, \tag{15}$$

for the suitable adjoint eigenfunctions $f_i^* \in W^*(\mathbb{S}\times\mathbb{S};\mathbf{H})$ related with N different eigenvalues $\nu_i \in \mathbb{C}$, $i = \overline{1,N}$ of (13), being considered as a coupled evolution system on the space $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$. The same problem has been studied for $c = 0$ and $N = 1$ before in the papers [16,17].

3. THE POISSON BRACKET ON THE EXTENDED PHASE SPACE

To compactify the description below we shall use the following notation of the gradient vector:

$$\nabla\gamma(\tilde{l},\tilde{\mathrm{f}},\tilde{\mathrm{f}}^*) := (\delta\gamma/\delta\tilde{l},\ \delta\gamma/\delta\tilde{\mathrm{f}}, \delta\gamma/\delta\tilde{\mathrm{f}}^*)^\top,$$

where $\tilde{\mathrm{f}} := (\tilde{f}_1,\ldots,\tilde{f}_N)$, $\tilde{\mathrm{f}}^* := (\tilde{f}_1^*,\ldots,\tilde{f}_N^*)$ and $\delta\gamma/\delta\tilde{\mathrm{f}} := (\delta\gamma/\delta\tilde{f}_1,\ldots,\ \delta\gamma/\delta\tilde{f}_N)$, $\delta\gamma/\delta\tilde{\mathrm{f}}^* := (\delta\gamma/\delta f_1^*,\ldots,\delta\gamma/\delta\tilde{f}_N^*)$, at a point $(\tilde{l},\tilde{\mathrm{f}},\mathrm{f}^*)^\top \in \hat{\mathcal{G}}^* \oplus W^N \oplus W^{*N}$ for any smooth functional $\gamma \in \mathcal{D}(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$.

On the spaces $\hat{\mathcal{G}}_c^*$ and $W^N \oplus W^{*N}$ there exist canonical Poisson structures in the forms

$$\delta\gamma/\delta\tilde{l} :\xrightarrow{\tilde{\theta}} [\tilde{l} - c\partial/\partial y, (\delta\gamma/\delta\tilde{l})_+] - [\tilde{l} - c\partial/\partial y, \delta\gamma/\delta\tilde{l}]_+, \tag{16}$$

where $\tilde{\theta}: T^*(\hat{\mathcal{G}}_c^*) \to T(\hat{\mathcal{G}}_c^*)$ is an implectic operator corresponding to (5) at a point $\tilde{l} \in \hat{\mathcal{G}}^*$ and

$$(\delta\gamma/\delta\tilde{\mathrm{f}},\, \delta\gamma/\delta\tilde{\mathrm{f}}^*)^\top :\xrightarrow{\tilde{J}} (-\delta\gamma/\delta\tilde{\mathrm{f}}^*,\, \delta\gamma/\delta\tilde{\mathrm{f}})^\top, \tag{17}$$

where $\tilde{J}: T^*(W^N \oplus W^{*N}) \to T(W^N \oplus W^{*N})$ is an implectic operator corresponding to the symplectic form $\omega^{(2)} = \sum_{i=1}^N d\tilde{f}_i^* \wedge d\tilde{f}_i$ at a point $(\tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*) \in W^N \oplus W^{*N}$. It should be noted here that Poisson structure (16) generates equation (9) for any Casimir functional $\gamma \in I(\hat{\mathcal{G}}_c^*)$.

Thus, on the extended phase space $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ one can obtain a Poisson structure as the tensor product $\tilde{\Theta} := \tilde{\theta} \otimes \tilde{J}$ of (16) and (17).

Consider the following Backlund transformation:

$$(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*)^\top :\xrightarrow{B} (l(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*),\, \mathrm{f} = \tilde{\mathrm{f}},\, \mathrm{f}^* = \tilde{\mathrm{f}}^*)^\top, \tag{18}$$

generating some Poisson structure $\Theta: T^*(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}) \to T(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$ on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$. The main condition imposed on mapping (18) is the coincidence of the resulting dynamical system

$$(dl/dt_n,\, d\mathrm{f}/dt_n,\, d\mathrm{f}^*/dt_n)^\top := -\Theta\, \nabla\overline{\gamma}_n(l, \mathrm{f}, \mathrm{f}^*) \tag{19}$$

with equations (9), (14) and (15) in the case of $\overline{\gamma}_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, independent of variables $(\mathrm{f}, \mathrm{f}^*) \in W^N \oplus W^{*N}$.

To satisfy that condition we shall find a variation of a Casimir functional $\overline{\gamma}_n := \gamma_n|_{l=l(\tilde{l},\mathrm{f},\mathrm{f}^*)} \in \mathcal{D}(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$, $n \in \mathbb{N}$, under the constraint $\delta\tilde{l} = 0$, taking into account evolutions (14), (15) and Backlund transformation definition (18). There follows

$$\begin{aligned}
\delta\overline{\gamma}_n(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*)\Big|_{\delta\tilde{l}=0} &= \sum_{i=1}^N \Big(\langle \delta\overline{\gamma}_n/\delta\tilde{f}_i, \delta\tilde{f}_i\rangle + \langle \delta\overline{\gamma}_n/\delta\tilde{f}_i^*, \delta\tilde{f}_i^*\rangle\Big) = \\
&= \sum_{i=1}^N \Big(\langle -d\tilde{f}_i^*/dt_n, \delta\tilde{f}_i\rangle + \langle d\tilde{f}_i/dt_n, \delta\tilde{f}_i^*\rangle\Big)\Big|_{\tilde{\mathrm{f}}=\mathrm{f},\, \tilde{\mathrm{f}}^*=\mathrm{f}^*} = \\
&= \sum_{i=1}^N \big(\langle (\delta\gamma_n/\delta l)_+^* f_i^*, \delta f_i\rangle + \langle (\delta\gamma_n/\delta l)_+ f_i, \delta f_i^*\rangle\big) = \\
&= \sum_{i=1}^N \big(\langle f_i^*, (\delta\gamma_n/\delta l)_+ \delta f_i\rangle + \langle (\delta\gamma_n/\delta l)_+ f_i, \delta f_i^*\rangle\big) = \\
&= \sum_{i=1}^N \big((\delta\gamma_n/\delta l, (\delta f_i)\xi^{-1} \otimes f_i^*) + (\delta\gamma_n/\delta l, f_i\xi^{-1} \otimes \delta f_i^*)\big) = \\
&= \left(\delta\gamma_n/\delta l, \delta\sum_{i=1}^N f_i\xi^{-1} \otimes f_i^*\right) := (\delta\gamma_n/\delta l, \delta l),
\end{aligned} \tag{20}$$

where $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$ and the brackets $\langle .,. \rangle$ denote the paring of the spaces W^* and W.

As a result of expression (20) one obtains the relationship:

$$\delta l|_{\delta\tilde{l}=0} = \sum_{i=1}^{N} \delta(f_i \xi^{-1} \otimes f_i^*). \tag{21}$$

If the linear dependence of l on $\tilde{l} \in \hat{\mathcal{G}}^*$ are chosen, there directly follows from (21) that

$$l = \tilde{l} + \sum_{i=1}^{N} f_i \xi^{-1} \otimes f_i^* \ . \tag{22}$$

Thus, Backlund transformation (18) can be written as

$$(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*)^\top :\xrightarrow{B} (l = \tilde{l} + \sum_{i=1}^{N} f_i \xi^{-1} \otimes f_i^*, \mathrm{f}, \mathrm{f}^*)^\top \ . \tag{23}$$

Expression (23) generalizes results obtained both for the scalar Lie algebra of integral-differential operators in [20] and for the matrix one in [17]. The existence of Backlund transformation (23) enables the following theorem to be proved.

Theorem 1. *Dynamical system* (19) *on* $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ *is equivalent to the following system of evolution equations:*

$$\begin{aligned} d\tilde{l}/dt_n &= [(\nabla\overline{\gamma}_n(\tilde{l}))_+, \tilde{l}] - [\nabla\overline{\gamma}_n(\tilde{l}), \tilde{l}]_+, \\ d\tilde{\mathrm{f}}/dt_n &= \delta\overline{\gamma}_n/\delta\tilde{\mathrm{f}}^*, \quad d\tilde{\mathrm{f}}^*/dt_n = -\delta\overline{\gamma}_n/\delta\tilde{\mathrm{f}}, \end{aligned}$$

where $\overline{\gamma}_n := \gamma_n|_{l=l(\tilde{l},\mathrm{f},\mathrm{f}^*)} \in \mathcal{D}(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$ *and* $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$ *is a Casimir functional at a point* $l \in \mathcal{G}^*$ *for every* $n \in \mathbb{N}$*, under Backlund transformation* (23).

Now by means of simple calculations via the formula:

$$\Theta = B^{'} \tilde{\Theta} B^{'*},$$

where $B^{'} : T(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}) \to T(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$ is a Frechet derivative of (23), one easily finds the following form of the Backlund transformed Poisson structure Θ on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$:

$$\nabla\gamma(l, \mathrm{f}, \mathrm{f}^*) :\xrightarrow{\Theta} \left(\begin{array}{c} [l - c\partial/\partial y, (\delta\gamma/\delta l)_+] - [l - c\partial/\partial y, \delta\gamma/\delta l]_+ + \\ \sum_{i=1}^{N} (f_i \xi^{-1} \otimes (\delta\gamma/\delta f_i) - (\delta\gamma/\delta f_i^*) \xi^{-1} \otimes f_i^*) \\ -\delta\gamma/\delta\mathrm{f}^* - (\delta\gamma/\delta l)_+ \mathrm{f} \\ \delta\gamma/\delta\mathrm{f} + (\delta\gamma/\delta l)_+^* \mathrm{f}^* \end{array} \right), \tag{24}$$

where $\gamma \in \mathcal{D}(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$ is an arbitrary smooth functional. Thereby, one can formulate the following theorem.

Theorem 2. *The hierarchy of dynamical systems* (9), (14) *and* (15) *is Hamiltonian with respect to the Poisson structure* Θ *as in* (24) *and the functionals* $\overline{\gamma}_n := \gamma_n \in I(\hat{\mathcal{G}}_c^*)$*,* $n \in \mathbb{N}$ *being Casimir invariants on* $\hat{\mathcal{G}}_c^*$*.*

Based on expression (19) one can construct a new hierarchy of Hamiltonian evolution equations describing commutative flows generated by Casimir invariants $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, involutive with respect to Poisson bracket (5) on the extended phase space $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$.

4. THE HIERARCHIES OF ADDITIONAL SYMMETRIES

Hierarchy (9), (14) and (15) of evolution equations possesses another natural set of invariants including all higher powers of the eigenvalues λ_k, $k = \overline{1,N}$. They can be considered as Frechet smooth functionals on the extended phase space $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$, owing to the evident representation:

$$\lambda_k^s = \langle f_k^*, (l - c\partial/\partial y)^s f_k \rangle, \tag{25}$$

where $s \in \mathbb{N}$, holding under the normalizing constraints

$$\langle f_k^*, f_k \rangle = 1 \ .$$

In the case of Backlund transformation (22), where

$$l := l_+ + \sum_{i=1}^{N} f_i \xi^{-1} \otimes f_i^* \tag{26}$$

formula (25) gives rise to the following variation of the functionals $\lambda_k^s \in \mathcal{D}(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$, $k = \overline{1,N}$:

$$\begin{aligned}
\delta\lambda_k^s = {} & \langle \delta f_k^*, (l - c\partial/\partial y)^s f_k \rangle + \\
& + \langle f_k^*, (\delta (l - c\partial/\partial y)^s) f_k \rangle + \langle f_k^*, (l - c\partial/\partial y)^s (\delta f_k) \rangle = \\
= {} & (M_k^s, \delta l_+) + \sum_{i=1}^{N} \langle (-M_k^s + \delta_k^i (l - c\partial/\partial y)^s)^* f_i^*, \delta f_i \rangle + \\
& + \sum_{i=1}^{N} \langle (-M_k^s + \delta_k^i (l - c\partial/\partial y)^s) f_i, \delta f_i^* \rangle,
\end{aligned}$$

where δ_k^i is the Kronecker symbol and the operators M_k^s, $s \in \mathbb{N}$, are defined as follows:

$$M_k^s := \sum_{p=0}^{s-1} ((l - c\partial/\partial y)^p f_k) \xi^{-1} \otimes ((l^* + c\partial/\partial y)^{s-1-p} f_k^*) \ .$$

Thus, one obtains the exact forms of gradients for the functionals $\lambda_k^s \in \mathcal{D}(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$, $k = \overline{1,N}$:

$$\begin{aligned}
\nabla \lambda_k^s (l_+, \mathrm{f}, \mathrm{f}^*) = {} & (M_k^s, (-M_k^s + \delta_k^i (l - c\partial/\partial y)^s)^* f_i^*, \\
& (-M_k^s + \delta_k^i (l - c\partial/\partial y)^s) f_i : \ i = \overline{1,N})^\top .
\end{aligned} \tag{27}$$

By means of expressions (27), (16) and (17) one finds a new hierarchy of coupled evolution equations on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$:

$$dl_+/d\tau_{s,k} = -[M_k^s, l_+ - c\partial/\partial y]_+ \ , \tag{28}$$

$$df_i/d\tau_{s,k} = (-M_k^s + \delta_k^i (l - c\partial/\partial y)^s) f_i, \tag{29}$$

$$df_i^*/d\tau_{s,k} = (M_k^s - \delta_k^i (l - c\partial/\partial y)^s)^* f_i^* \ , \tag{30}$$

where $i = \overline{1,N}$ and $\tau_{s,k} \in \mathbb{R}$, $s \in \mathbb{N}$, $k = \overline{1,N}$, are evolution parameters. Owing to Backlund transformation (26), equation (28) can be rewritten in the following equivalent commutator form:

$$\begin{aligned} dl/d\tau_{s,k} &= -[M_k^s, l - c\partial/\partial y] = \\ &= -\lambda_k^p \nu_k^{s-1-p} [M_k^1, l - c\partial/\partial y] = \lambda_k^p \nu_k^{s-1-p} dl/d\tau_{1,k}, \end{aligned} \tag{31}$$

where $p = \overline{0, s-1}$. Thereby, one can formulate the following theorem.

Theorem 3. *For $k = \overline{1,N}$ and $s \in \mathbb{N}$ dynamical systems* (31), (29) *and* (30) *are Hamiltonian with respect to the Poisson structure Θ as in* (24) *and the invariant functionals $\overline{\gamma}_s := \lambda_k^s \in \mathcal{D}(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$.*

Theorem 4. *Dynamical systems* (31), (29) *and* (30) *describe flows on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ commuting both with each other and with the hierarchy of Lax-type dynamical systems* (9), (14) *and* (15).

Proof. To prove the latter theorem it is sufficient to show that

$$[d/dt_n, d/d\tau_{1,k}] = 0, \quad [d/d\tau_{1,k}, d/d\tau_{1,q}] = 0, \tag{32}$$

where $k, q = \overline{1,N}$ and $n \in \mathbb{N}$. The first equality in formula (32) follows from the identities:

$$d(\nabla\gamma_n(l))_+/d\tau_{1,k} = [(\nabla\gamma_n(l))_+, M_1^1]_+, \quad dM_1^1/dt_n = [(\nabla\gamma_n(l))_+, M_1^1]_-,$$

and the second one is a consequence of the relationship:

$$dM_k^1/d\tau_{1,q} - dM_q^1/d\tau_{1,k} = [M_k^1, M_q^1] \ .$$

Thus, for every $k = \overline{1,N}$ and all $s \in \mathbb{N}$ dynamical systems (31), (29) and (30) on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ form a hierarchy of additional homogeneous or so called "ghost" symmetries for Lax-type flows (9), (14) and (15) on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$. For the first time the additional symmetry hierarchies for integrable (1|1+1)-dimensional nonlinear dynamical systems associated with the Lie algebra of super-integral-differential operators were described as commutator-type flows in [2]. They were used to construct Lax-type integrable $(2|1+1)$-dimensional dynamical systems in [12].

If $N \geq 2$, one can obtain a new class of nontrivial Hamiltonian flows $d/dT_n := d/dt_n \pm \sum_{k=1}^{N-1} d/d\tau_{n,k}$, $n \in \mathbb{N}$, on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ in the Lax-type form by use of the invariants considered above for the centrally extended Lie algebra $\hat{\mathcal{G}}_c$ of integral-differential operators. Acting on the eigenfunctions $(f_i, f_i^*) \in W \oplus W^*$, $i = \overline{1,N}$, these flows generate some integrable $(N+1)$-dimensional nonlinear dynamical systems.

For example, in the case of the element $l := \partial/\partial x + f_1\xi^{-1} \otimes f_1^* + f_2\xi^{-1} \otimes f_2^* \in \hat{\mathcal{G}}^*$ with $(f_1, f_2, f_1^*, f_2^*) \in W^2 \times W^{*2}$ the flows $d/d\tau := d/d\tau_{1,1}$ and $d/dT := d/dT_2 = d/dt_2 + d/d\tau_{2,1}$ on $\hat{\mathcal{G}}_c^* \oplus W^2 \oplus W^{*2}$ acting on the functions f_i, f_i^*, $i = \overline{1,2}$, give rise to such dynamical systems as

$$\begin{aligned} f_{1,\tau} &= f_{1,x} - cf_{1,y} + f_2 u, & f_{1,\tau}^* &= f_{1,x}^* - cf_{1,y}^* + f_2^* \bar{u}, \\ f_{2,\tau} &= -f_1 \bar{u}, & f_{2,\tau}^* &= -f_1^* u, \end{aligned} \tag{33}$$

and

$$\begin{aligned} f_{1,T} &= \quad f_{1,xx} + f_{1,\tau\tau} + wf_1 + 2f_1 v_\tau, \\ f_{1,T}^* &= -f_{1,xx}^* - f_{1,\tau\tau}^* - wf_1^* - 2f_1^* v_\tau, \\ f_{2,T} &= \quad f_{2,xx} + wf_2 - f_{1,\tau}\bar{u} + f_1 \bar{u}_\tau, \\ f_{2,T}^* &= -f_{2,xx}^* - wf_2^* + f_{1,\tau}^* u - f_1^* u_\tau, \\ cw_y &= w_x - 2(f_1 \otimes f_1^* + f_2 \otimes f_2^*)_x, \\ u_x &= f_1^T f_2^*, \quad \bar{u}_x = f_1^{*T} f_2, \quad v_x = f_1^T f_1^*, \end{aligned} \tag{34}$$

where one puts $(\nabla\gamma_2(l))_+ := \partial^2/\partial x^2 + w$ for some function $w \in \tilde{\mathcal{G}}$ depending parametrically on variables $\tau, T \in \mathbb{R}$. Systems (33) and (34) represent a Lax-type integrable (3+1)-dimensional generalization of the (2+1)-dimensional system being equivalent to the Davey-Stewartson one [26, 29] with an infinite sequence of conservation laws which can be derived from formula (6) in the form

$$\gamma_n(l) := tr \int_0^{2\pi} \int_0^{2\pi} (f_1 \partial^{n-1} f_1^*/\partial x^{n-1} + f_2 \partial^{n-1} f_2^*/\partial x^{n-1}) dx dy,$$

where $n \in \mathbb{N}$. Its Lax-type linearization is given by spectral problem (10) and the following evolution equations:

$$f_\tau = -M_1^1 f, \tag{35}$$

$$f_T = ((\nabla\gamma_2(l))_+ - M_1^2) f, \tag{36}$$

for an arbitrary eigenfunction $f \in W(\mathbb{S} \times \mathbb{S}; \mathbf{H})$. Relationships (35) and (36) give rise to the additional nonlinear constraint:

$$w_\tau = 2(f_1 \otimes f_1^*)_x. \tag{37}$$

In the case of $\dim \mathbf{H} = 1$ Lax-type representation (10), (35) and (36) for above mentioned (3+1)-dimensional generalization (33), (34) and (37) of the Davey-Stewartson system [26, 29] has an equivalent matrix form:

$$\frac{dF}{dx} = \begin{pmatrix} 0 & 0 & f_1^* \\ 0 & 0 & f_2^* \\ -f_1 & -f_2 & \lambda + c\partial/\partial y \end{pmatrix} F,$$

$$\frac{dF}{d\tau} = \begin{pmatrix} -(\lambda + c\partial/\partial y) & \bar{u} & f_1^* \\ -u & 0 & 0 \\ -f_1 & 0 & 0 \end{pmatrix} F,$$

$$\frac{dF}{dT} = CF,$$

where $F = (F^1, F^2, F^3 = f)^\top \in W(\mathbb{S} \times \mathbb{S}; \mathbb{C}^3)$, $C := \{C_{mn} \in \mathrm{gl}(3;\mathbb{C}) : \ m, n = \overline{1,3}\}$, and

$$\begin{aligned}
C_{11} &= -(\lambda + c\partial/\partial y)^2 - u\bar{u} - 2f_1 f_1^*, \\
C_{12} &= -f_1 f_2^* - (\lambda + c\partial/\partial y)\bar{u} - \bar{u}_\tau, \\
C_{13} &= 2((\lambda + c\partial/\partial y) f_1^* - f_{1,x}^*) - \bar{u} f_2^*, \\
C_{21} &= -(\lambda + c\partial/\partial y)u - u_\tau - f_1 f_2^*, \\
C_{22} &= -f_2 f_2^* + u\bar{u}, \\
C_{23} &= (\lambda + c\partial/\partial y) f_2^* - f_{2,x}^* + u f_1^*, \\
C_{31} &= -(\lambda + c\partial/\partial y) f_1 - f_{1,x} - f_{1,\tau}, \\
C_{32} &= -(\lambda + c\partial/\partial y) f_2 - f_{2,x} + \bar{u} f_1, \\
C_{33} &= (\lambda + c\partial/\partial y)^2 + w - f_2 f_2^*,
\end{aligned}$$

to which one can effectively apply the standard inverse spectral transform method [9, 13].

The results obtained above can also be used to construct a wide class of integrable (3+1)-dimensional nonlinear dynamical systems with triple Lax-type linearizations [17].

5. CONCLUSION

Several regular Lie-algebraic approaches [7, 17, 19, 22, 27] to constructing Lax-type integrable multi-dimensional (mainly 2+1) nonlinear dynamical systems on functional manifolds and their supersymmetric generalizations are well known. In this paper we have developed a new method for introducing another variable into Lax-type integrable (2+1)-dimensional dynamical systems arising as flows on dual spaces to the centrally extended matrix Lie-algebra of integral-differential operators. It is based on the natural constructed hierarchy of additional invariants [18, 20]. The resulting integrable (3+1)-dimensional dynamical systems obtained by means of this method possess infinite sequences of conservation laws and related triple Lax-type linearizations. Owing to the latter property, their soliton type solutions can be found by means of either the standard inverse spectral transform method [9, 13] or Darboux-Backlund transformations [11, 21, 24].

The structure of the constructed Lie-Backlund transformation (23), being a key point of the devised method, strongly depends on an *ad*-invariant scalar product chosen for an operator Lie algebra $\hat{\mathcal{G}}$ and on a suitable Lie algebra decomposition (see [4, 18]). Since there exist other possibilities of choosing the corresponding *ad*-invariant scalar products on $\hat{\mathcal{G}}$, such naturally decompositions give rise to other Backlund transformations.

In an another paper this method will be developed for some special centrally extended Lie algebras of super-integral-differential operators [10, 15].

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