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\textbf{J-CONVEXITY CONSTANTS}

\textbf{Abstract.} We introduce the \(J\)-convexity constants on Banach spaces and give some properties of the constants. We give the relations between the \(J\)-convexity constants and the \(n\)-th von Neumann-Jordan constants. Using the quantitative indices we estimate the value of \(J\)-convexity constants in Orlicz spaces.

\textbf{Keywords:} new quantitative index, \(J\)-convexity constants, \(n\)-th von Neumann-Jordan constants, Orlicz spaces.

\textbf{Mathematics Subject Classification:} 46B20, 46E30.

1. \textbf{INTRODUCTION}

Much of the significance of the concept of superreflexivity of a Banach space \(X\) is due to its numerous equivalent characterizations, see, e. g., Beauzamy [1, Part 4]. One of these characterizations is \(J(n, \varepsilon)\)-convexity. We restate the definition from [2] as follows.

\textbf{Definition 1.1.} Given \(n\) and \(0 < \varepsilon < 1\), we say that a Banach spaces \(X\) is \(J(n, \varepsilon)\)-convex if for all elements \(z_1, \ldots, z_n \in U_X = \{x \in X : \|x\| \leq 1\}\) there is

\[
\inf_{1 \leq k \leq n} \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| < n(1 - \varepsilon).
\]

\textbf{Definition 1.2.} We define the \(J\)-convexity constants, for \(n \geq 2\), by

\[
J(n, X) = \sup \left\{ \inf_{1 \leq k \leq n} \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| : z_1, \ldots, z_n \in U_X \right\},
\]

and

\[
J_n(X) = \inf \left\{ \varepsilon : 0 < \varepsilon < 1, \text{and there exists } z_1, \ldots, z_n \in U_X \text{such that} \right. \left. \inf_{1 \leq k \leq n} \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| \geq n(1 - \varepsilon) \right\}.
\]

It is known that a Banach space is superreflexive if and only if it is $J(n, \varepsilon)$-convex for some $n$ and $\varepsilon > 0$ ([3] and [4]). It is evident that:

(i) $J(n, X) \leq n$ and $0 \leq J_n(X) < 1$ for $n \geq 2$.
(ii) $X$ is superreflexive if and only if $J_n(X) > 0$ for some $n$ or, equivalently, $J(n, X) < n$ for some $n$.
(iii) $J(n, X) = n(1 - \varepsilon)$ if and only if $J_n(X) = \varepsilon$.
(iv) $J(n, X) = n$ if and only if $J_n(X) = 0$.
(v) For a Banach spaces $X$ the following conditions are equivalent:
   1) $X$ is not superreflexive,
   2) $J_n(X) = 0$ for all $n \in \mathbb{N}$,
   3) $J(n, X) = n$ for all $n \in \mathbb{N}$.

Let

$$
\Phi(u) = \int_0^{|u|} \phi(t)dt, \quad \Psi(v) = \int_0^{|v|} \psi(s)ds
$$

be a pair of complementary $N$-functions, where the right derivative $\phi$ of $\Phi$ is right-continuous and nondecreasing, $\phi(t) > 0$ whenever $t > 0$, $\phi(0) = 0$ and $\lim_{t \to \infty} \phi(t) = \infty$; the right derivative $\psi$ of $\Psi$ satisfies the same conditions as $\phi$. Assume that $(G, \Sigma, m)$ is a (Lebesgue) measure space and $L^0(G, \Sigma, m)$ is the space of $\Sigma$-measurable functions defined on $G$. The Orlicz space is defined as

$$
L^\Phi(G) = \{x \in L^0 : x \text{ is measurable in } G, \rho_\Phi(\lambda x)dt < \infty \text{ for some } \lambda > 0\},
$$

where $\rho_\Phi(x) = \int_G \Phi(x(t))dt$. The Luxemburg norm (gauge norm) and the Orlicz norm in $L^\Phi(G)$ are defined, respectively, by

$$
\|x\|_{\Phi} = \inf\{c > 0 : \rho_\Phi \left( \frac{x}{c} \right) \leq 1\}
$$

and

$$
\|x\|_\Phi = \inf_{k>0} \frac{1}{k} [1 + \rho_\Phi(kx)].
$$

As usual, we denote $L^{(\Phi)} = (L^\Phi, \| \cdot \|_{\Phi}), L^\Phi = (L^\Phi, \| \cdot \|_\Phi)$ for short.

An $N$-function $\Phi(u)$ is said to satisfy the $\Delta_2$-condition for small $u$ (for all $u$ or for large $u$), which is written as $\Phi \in \Delta_2(0)(\Phi \in \Delta_2 \text{ or } \Phi \in \Delta_2(\infty))$, if there exist $u_0 > 0$ and $c > 0$ such that $\Phi(2u) \leq c\Phi(u)$ for $0 \leq u \leq u_0$ (for all $u \geq 0$ or for $u \geq u_0$).

An $N$-function $\Phi(u)$ satisfies the $\nabla_2$-condition for small $u$ (for all $u \geq 0$ or for large $u$), which is written as $\Phi \in \nabla_2(0)$ (or $\Phi \in \nabla_2(\infty))$, if its complementary $N$-function (see [6] or [8]) $\Psi \in \Delta_2(0)$ ($\Psi \in \Delta_2 \text{ or } \Psi \in \Delta_2(\infty)$). The basic facts on Orlicz spaces can be found in [8].
New quantitative indices for an $N$-function $\Phi$ are defined by

$$
\alpha_{\Phi}(n) = \liminf_{u \to \infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)}, \quad \beta_{\Phi}(n) = \limsup_{u \to \infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)},
$$

$$
\alpha_0^{\Phi}(n) = \liminf_{u \to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)}, \quad \beta_0^{\Phi}(n) = \limsup_{u \to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)},
$$

$$
\alpha_{\Phi}(n) = \inf_{u > 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)}, \quad \beta_{\Phi}(n) = \sup_{u > 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)}.
$$

For $n = 2$, we denote these constants by $\alpha_{\Phi}, \beta_0, \alpha_0^{\Phi}, \beta_0^{\Phi}, \alpha^{\Phi}$ and $\beta_{\Phi}$ (see [8]). Clearly, $\frac{1}{n} \leq \alpha_{\Phi}(n) \leq \min\{\alpha_{\Phi}(n), \alpha_0^{\Phi}(n)\}, \max\{\beta_{\Phi}(n), \beta_0^{\Phi}(n)\} \leq \beta_{\Phi}(n) \leq 1$.

**Proposition 1.1.** ([7]) Let $\Phi$ be an $N$-function. Then:

(i) $\Phi \notin \triangle_2(\infty) \iff \beta_{\Phi}(n) = 1; \Phi \notin \nabla_2(\infty) \iff \alpha_{\Phi}(n) = \frac{1}{n}$.

(ii) $\Phi \notin \triangle_2(0) \iff \beta_0^{\Phi}(n) = 1; \Phi \notin \nabla_2(0) \iff \alpha_0^{\Phi}(n) = \frac{1}{n}$.

(iii) $\Phi \notin \triangle_2 \iff \beta_{\Phi}(n) = 1; \Phi \notin \nabla_2 \iff \alpha_{\Phi}(n) = \frac{1}{n}$.

The following results concern these new indices.

**Proposition 1.2.** ([7]) Let $\Phi$ and $\Psi$ be a pair of complementary $N$-functions and $n \geq 2$. Then:

$$
n\alpha_{\Phi}(n)\beta_{\Phi}(n) = 1 = n\alpha_{\Phi}(n)\beta_{\Phi}(n), \quad (3)
$$

$$
n\alpha_0^{\Phi}(n)\beta_0^{\Phi}(n) = 1 = n\alpha_0^{\Phi}(n)\beta_0^{\Phi}(n), \quad (4)
$$

$$
n\alpha_{\Phi}(n)\beta_{\Phi}(n) = 1 = n\alpha_{\Phi}(n)\beta_{\Phi}(n), \quad (5)
$$

2. THE RELATIONS BETWEEN $J$-CONVEXITY CONSTANTS AND VON NEUMANN-JORDAN CONSTANTS

In order to discuss the $J$-convexity constants, we need the $n$-th von Neumann-Jordan constants defined as follows.

**Definition 2.1.** We define the $n$-th von Neumann-Jordan constants, for $n \geq 2$, by

$$
C_{NJ}^{(n)}(X) = \sup \left\{ \frac{\sum_{k=1}^{n} \left\| \sum_{h=1}^{k} z_h \right\|}{\sum_{i=1}^{n} \left\| z_i \right\|^2} : z_i \in X, \sum_{i=1}^{n} \left\| z_i \right\|^2 \neq 0 \right\}.
$$

When $n = 2$, $C_{NJ}^{(2)}(X)$ is the von Neumann-Jordan constants of a Banach space $X$ (see [8]).
Theorem 2.1.

(i) For any Banach space $X$, there holds

$$J(n, X) \leq \sqrt{n} C_{NJ}^n(X). \quad (6)$$

(ii) $J(n, X) < n$ if and only if $C_{NJ}^n(X) < n$.

Proof. (i) Let $x_1, x_2, \ldots, x_n \in U(X)$. Then

$$n \min_{1 \leq i \leq n} \left\| \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \right\|^2 \leq n \left( \sum_{k=1}^{n} \left\| \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \right\| \right)^2 \leq C_{NJ}^n(X) n \sum_{i=1}^{n} \left\| x_i \right\|^2 \leq n^2 C_{NJ}^n(X). \quad (7)$$

(ii) By (i), the sufficiency is clear. Now we prove the necessity.

$$C_{NJ}^n(X) = \sup \left\{ \sum_{1 \leq k \leq n} \left( \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \right) \right\} = \left\{ \sum_{1 \leq k \leq n} \left( \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \right) \right\} = \sup \left\{ \sum_{1 \leq k \leq n} \left( \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \right) \right\} = \left\{ \sum_{1 \leq k \leq n} \left( \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \right) \right\}.$$

Since $J(n, X) < n$, there exists $0 < \delta < 1$ such that

$$\sup \left\{ \inf_{1 \leq k \leq n} \left( \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \right) : \{x_i\}^n \subset U_X \right\} < n - \delta. \quad (7)$$

Suppose $1 = \left\| x_n \right\| \geq \left\| x_i \right\| > 1 - \frac{\delta}{2(n-1)} (i = 1, 2, \ldots, n-1)$. By (7), there is

$$\inf_{1 \leq k \leq n} \left( \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \right) < n - \delta.$$

Without loss of generality, we assume that

$$\left\| x_1 - x_2 - \ldots - x_n \right\| = \inf_{1 \leq k \leq n} \left( \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \right) < n - \delta.$$
Hence

\[
\sum_{1 \leq k \leq n} \| \frac{\sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i}{n(\sum_{i=1}^{n-1} \| x_i \|^2 + 1)} \|^2 \leq \frac{(n-\delta)^2}{n[(n-1) \cdot (1 - \frac{\delta}{2(n-1)})^2 + 1]} + \frac{\sum_{2 \leq k \leq n} n(\sum_{i=1}^{n-1} \| x_i \|^2 + 1)}{n \cdot \left( \frac{\sum_{i=1}^{n-1} \| x_i \|^2 + 1}{\sum_{i=1}^{n-1} \| x_i \|^2 + 1} \right)} = \frac{(n-\delta)^2}{n[(n-1) - \delta + \frac{\delta^2}{4(n-1)}]} + n - 1 < n - \frac{\delta}{n} + n - 1 = n - \frac{\delta}{n}.
\]

If there exists \(1 \leq i \leq n-1\) such that \(\| x_i \| \leq 1 - \frac{\delta}{2(n-1)}\), we may, without loss of generality, assume that \(\| x_i \| \leq 1 - \frac{\delta}{2(n-1)}\). Then

\[
\sum_{1 \leq k \leq n} \| \frac{\sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i}{n(\sum_{i=1}^{n-1} \| x_i \|^2 + 1)} \|^2 \leq n(\| x_1 \| + \| x_2 \| + \ldots + \| x_{n-1} \| + 1)^2 = n - \frac{1}{\left( \sum_{i=1}^{n-1} \| x_i \|^2 + 1 \right)} \]

\[
= n - \left( \sum_{i=1}^{n-1} \| x_i \|^2 + 1 \right) - \left( \| x_1 \|^2 + 2\| x_1 \|(\| x_2 \| + \ldots + \| x_{n-1} \| + 1) + \right.
\]

\[
+ (\| x_2 \| + \ldots + \| x_{n-1} \| + 1)^2) / \left( \sum_{i=1}^{n-1} \| x_i \|^2 + 1 \right) \]

\[
= n - \left( 1 - \| x_1 \|^2 \right) + (\| x_1 \| - \| x_2 \|) + \ldots + (\| x_1 \| - \| x_{n-1} \|)^2 +
\]

\[
+ (n-1)(\sum_{i=2}^{n-2} \| x_i \|^2 + 1) - (\| x_2 \| + \ldots + \| x_{n-1} \| + 1)^2) / \left( \sum_{i=1}^{n-1} \| x_i \|^2 + 1 \right) \leq
\]

\[
\leq n - \frac{1 - \| x_1 \|^2 + n - \frac{\delta^2}{4(n-1)^2}}{n(n-1)} = n - \frac{\delta^2}{4n(n-1)^2}.
\]
Therefore,
\[ C^n_{NJ}^*(X) \leq \max \left\{ n - \frac{\delta}{n}, n - \frac{\delta^2}{4n(n-1)^2} \right\} < n. \]

\[ \square \]

3. LOWER BOUNDS FOR J-CONVEXITY CONSTANTS IN ORLICZ SPACES

In this section, we will give lower bounds of \( J \)-convexity for three Orlicz spaces.

**Theorem 3.1.** Let \( \Phi \) be an \( N \)-function and \( n \geq 2 \). Then:

\[ \max \left\{ \frac{1}{\alpha_{\Phi}(n)} n\beta_{\Phi}(n) \right\} \leq J(n, L^{(\Phi)}[0, 1]), \] (8)

\[ \max \left\{ \frac{1}{\alpha_{\Phi}^0(n)} n\beta_{\Phi}^0(n) \right\} \leq J(n, l^{(\Phi)}), \] (9)

\[ \max \left\{ \frac{1}{\alpha_{\Phi}(n)} n\beta_{\Phi}(n) \right\} \leq J(n, L^{(\Phi)}[0, \infty)). \] (10)

**Proof.** We only prove (8). The proofs of inequalities (9) and (10) are similar. By the definition of \( \alpha_{\Phi}(n) \), there exists \( 0 < u_k \to \infty \) such that

\[ \lim_{k \to \infty} \frac{\Phi^{-1}(u_k)}{\Phi^{-1}(nu_k)} = \alpha_{\Phi}(n). \]

So for \( \varepsilon > 0 \), there exists \( k_0 \geq 1 \) such that for any \( k \geq k_0 \), there is

\[ \frac{\Phi^{-1}(u_k)}{\Phi^{-1}(nu_k)} < \alpha_{\Phi}(n) + \varepsilon. \] (11)

Put \( u_0 = u_{k_0} > 1 \). Let \( G_i, 1 \leq i \leq n \) be non-overlapping subsets of \([0, 1]\), and \( m(G_i) = \frac{1}{nu_0}, 1 \leq i \leq n \). Define \( x_i(t) = \Phi^{-1}(nu_0)\chi_{G_i}(t), 1 \leq i \leq n \). Then \( \|x_i\|_{\Phi} = 1 \), and for any \( 1 \leq k \leq n \), there holds

\[ \| \sum_{i=1}^k x_i - \sum_{k+1}^n x_i \|_{\Phi} = \Phi^{-1}(nu_0)\|\chi_{G_1 \cup G_2 \cup \ldots \cup G_n}\|_{\Phi} = \frac{\Phi^{-1}(nu_0)}{\Phi^{-1}(u_0)} > \frac{1}{\alpha_{\Phi}(n) + \varepsilon}. \]

Therefore

\[ J(n, L^{(\Phi)}[0, 1]) > \frac{1}{\alpha_{\Phi}(n) + \varepsilon}, \]

which proves (8), because \( \varepsilon \) is arbitrary.
Now we prove that \( n\beta\Phi(n) \leq J(n, L^p[0,1]) \). By the definition of \( \beta\Phi(n) \), for any given \( \varepsilon > 0 \) we choose a \( v_0 > 1 \) such that \( \frac{\Phi^{-1}(v_0)}{\Phi^{-1}(nv_0)} > \beta\Phi(n) - \frac{\varepsilon}{n} \). We divide \([0, \frac{1}{v_0}]\) into \( n \) non-overlapping intervals \( A_1, A_2, \ldots, A_n \) of the same length. Define:

\[
x_1(t) = \Phi^{-1}(v_0)(\chi_{A_1} + \chi_{A_2} + \chi_{A_3} + \ldots + \chi_{A_n}),
\]

\[
x_2(t) = \Phi^{-1}(v_0)(-\chi_{A_1} + \chi_{A_2} + \chi_{A_3} + \ldots + \chi_{A_n}),
\]

\[
x_3(t) = \Phi^{-1}(v_0)((-\chi_{A_1} - \chi_{A_2} + \chi_{A_3} + \ldots + \chi_{A_n})),
\]

\[
\ldots \ldots
\]

\[
x_n(t) = \Phi^{-1}(v_0)(-\chi_{A_1} - \chi_{A_2} - \chi_{A_3} - \ldots - \chi_{A_{n-1}} + \chi_{A_n}).
\]

Obviously, \( \|x_i\|_{\Phi} = 1 (i = 1, 2, \ldots, n) \). For any \( 1 \leq k \leq n \),

\[
\| \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \|_{\Phi} \geq \| \Phi^{-1}(v_0)n\chi_{A_k} \|_{\Phi} = \frac{n\Phi^{-1}(v_0)}{\Phi^{-1}(nv_0)} \geq n\beta\Phi(n) - \varepsilon.
\]

The latter implies that \( J(n, L^p[0,1]) \geq n\beta\Phi(n) - \varepsilon \). This proves that \( J_n(L^p[0,1]) \geq n\beta\Phi(n) \), because \( \varepsilon \) is arbitrary.

**Corollary 3.1.**

(i) If \( \Phi \notin \triangle_2(\infty) \cap \nabla_2(\infty) \), then \( J(n, L^p[0,1]) = n \).

(ii) If \( \Phi \notin \triangle_2(0) \cap \nabla_2(0) \), then \( J(n, L^p) = n \).

(iii) If \( \Phi \notin \triangle_2 \cap \nabla_2 \), then \( J(n, L^p[0,\infty]) = n \).

**Proof.** We only prove (i). If \( \Phi \notin \triangle_2(\infty) \), then \( \beta\Phi(n) = 1 \) by Proposition 1.1. By (8), there is \( J(n, L^p[0,1]) = n \). If \( \Phi \notin \nabla_2(\infty) \), then \( \alpha\Phi(n) = \frac{1}{n} \). By (8), the same equality holds true.

**Corollary 3.2.** Let \( 1 < p < \infty \), \( L^p \in \{ L^p[0,1], L^p[0,\infty], L^p \} \). Then for \( n \geq 2 \),

\[
\max \left\{ n^{\frac{1}{p}}, n^{1-\frac{1}{p}} \right\} \leq J(n, L^p),
\]

\[
\max \left\{ n^{\frac{2}{p}-1}, n^{1-\frac{2}{p}} \right\} \leq C_{NJ}(n, L^p).
\]

**Proof.** We put \( \Phi_p(u) = |u|^p \). Then \( L^p = L^p \) and \( \| \cdot \|_{\Phi} = \| \cdot \|_p \). The result is easy to verify.

Similarly, for the Orlicz spaces with the Orlicz norm, the following theorem holds.

**Theorem 3.2.** Let \( \Phi \) be an \( N \)-function, \( n \geq 2 \) and \( \Psi \) be a complementary \( N \)-function for \( \Phi \). Then:

\[
\max \left\{ n\beta\Phi(n), \frac{1}{\alpha\Phi(n)} \right\} \leq J(n, L^p[0,1]), \quad (12)
\]

\[
\max \left\{ n\beta\Psi(n), \frac{1}{\alpha\Psi(n)} \right\} \leq J(n, L^p), \quad (13)
\]

\[
\max \left\{ n\beta\Psi(n), \frac{1}{\alpha\Psi(n)} \right\} \leq J(n, L^p[0,\infty]). \quad (14)
\]
**Remark.** (i) By Proposition 1.2, there is
\[
\max \left\{ \frac{1}{\alpha_\Phi(n)}, n\beta_\Phi(n) \right\} = \max \left\{ n\beta_\Phi(n), \frac{1}{\alpha_\Phi(n)} \right\}.
\]
That is to say that \( J(n, L^\Phi[0,1]) \) and \( J(n, L^\Phi[0,1]) \) have the same lower bounds etc.
(ii) If \( X_\Phi \) denotes one of the Orlicz spaces in Theorem 3.1 and Theorem 3.2, then
\[ \sqrt{n} \leq J(n, X_\Phi). \]
In fact, assume that \( X_\Phi = L^\Phi \), then by Theorem 3.1, there holds
\[ \sqrt{n} \leq \max \left\{ \frac{1}{\alpha_\Phi(n)}, n\beta_\Phi(n) \right\} \leq J(n, X_\Phi). \]

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*Received: December 29, 2005.*