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Yarema A. Prykarpatsky, Anatoliy M. Samoilenko

THE STUDY OF DELSARTE-LIONS TYPE BINARY TRANSFORMATIONS, THEIR DIFFERENTIAL-GEOMETRIC AND OPERATOR STRUCTURE WITH APPLICATIONS. PART 2

Abstract. The Gelfand-Levitan integral equations for Delsarte-Lions type transformations in multidimension are studied. The corresponding spectral and analytical properties of Delsarte-Lions transformed operators are analyzed by means of the differential-geometric and topological tools. An approach for constructing Delsarte-Lions type transmutation operators for multidimensional differential expressions is devised.

Keywords: Delsarte transmutation operators, generalized de Rham-Hodge differential complex, Delsarte-Lions type transformations, Gelfand-Levitan-Marchenko type integral equations, mulktidimensional differential operator pencils.

Mathematics Subject Classification: Primary 34A30, 34B05, Secondary 34B15.

1. INTRODUCTION: GENERALIZED DE RHAM-HODGE COMPLEXES AND THEIR PROPERTIES

1.1. We begin with recalling some differential-geometric properties of Delsarte-Lions type transformations that were discussed in Part 1 for differential operator expressions acting in a multidimensional functional space $\mathcal{H} = L_1(\mathrm{T};H)$, where $\mathrm{T} = \mathbb{R}^2$ and $H := L_2(\mathbb{R}^2;\mathbb{C}^2)$. They appear to have a deep relationship with classical generalized de Rham-Hodge theory [3–6, 27] devised in the middle of the past century for a set of commuting differential operators defined, in general, on a smooth compact m -dimensional metric space M. Concerning our problem of describing the differential-geometric and spectral structure of Delsarte-Lions type transmutations acting in $\mathcal{H}$, following [30] we preliminarily consider some backgrounds of a generalized de Rham-Hodge differential complex theory devised for studying these transformations of differential operators. Consider a smooth metric space M being a suitably compactified form of the space $\mathbb{R}^m$, $m \in \mathbb{Z}_+$. Then on $M_{\mathrm{T}} := \mathrm{T} \times M$ one can define the standard Grassmann algebra $\Lambda(M_{\mathrm{T}};\mathcal{H})$ of differential forms on $\mathrm{T}\times M$ and consider an I.V. Skrypnik's [3–6] generalized external anti-differentiatial operator $d_{\mathcal{L}} : \Lambda(M_{\mathrm{T}};\mathcal{H}) \to \Lambda(M_{\mathrm{T}};\mathcal{H})$ acting as follows: for any $\beta^{(k)} \in \Lambda^k(M_{\mathrm{T}};\mathcal{H})$, $k = \overline{0,m}$,

$$d_{\mathcal{L}}\beta^{(k)} := \sum_{j=1}^{2} dt_j \wedge \mathrm{L}_j(t;x|\partial)\beta^{(k)} + \sum_{i=1}^{m} dx_i \wedge \mathrm{A}_i(t;x;\partial)\beta^{(k)} \in \Lambda^{k+1}(M_{\mathrm{T}};\mathcal{H}), \quad (1.1)$$

where $\mathrm{A}_i \in C^2(\mathrm{T};\mathcal{L}(H))$, $i = \overline{1,m}$, are some differential operator mappings and

$$\mathrm{L}_j(t;x|\partial) := \partial/\partial t_j - L_j(t;x|\partial), \quad (1.2)$$

$j = \overline{1,2}$, are suitably defined linear differential operators in $\mathcal{H}$, commuting with each other, that is

$$[\mathrm{L}_1, \mathrm{L}_2] = 0, \quad [\mathrm{A}_k, \mathrm{A}_i] = 0 \quad \text{and} \quad [\mathrm{L}_j, \mathrm{A}_i] = 0 \quad (1.3)$$

for all $j = \overline{1,2}$ and $i,k = \overline{1,m}$. We will put, in general, that differential expressions

$$L_j(t;x|\partial) := \sum_{|\alpha|=0}^{n_j(L)} a_\alpha^{(j)}(t;x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}, \quad (1.4)$$

with coefficients $a_\alpha^{(j)} \in C^1(\mathrm{T};C^\infty(M;End\mathbb{C}^N))$, $|\alpha| = \overline{0,n_j(L)}$ $n_j^\alpha \in \mathbb{Z}_+$, $j = \overline{0,1}$, define some closed normal operators in the Hilbert space H for ant $t \in \mathrm{T}$. It is easy to observe that the generalization of $d_{\mathcal{L}}$ defined by (1.1) is a generalization of the usual external anti-differentiation

$$d = \sum_{j=1}^{m} dx_j \wedge \frac{\partial}{\partial x_j} + \sum_{s=1}^{2} dt_s \wedge \frac{\partial}{\partial t_s} \quad (1.5)$$

for which, evidently, commutation conditions

$$[\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}] = 0,\ [\frac{\partial}{\partial t_s}, \frac{\partial}{\partial t_l}] = 0[\frac{\partial}{\partial x_j}, \frac{\partial}{\partial t_s}] = 0 \quad (1.6)$$

hold for all $j,k = \overline{1,m}$ and $s,l = \overline{1,2}$. If now in (1.5) we substitute $\partial/\partial x_j \longrightarrow \mathrm{A}_j$, $\partial/\partial t_s \longrightarrow \mathrm{L}_s$, $j = \overline{1,m}$, $s = \overline{1,2}$, we get the anti-differentiation

$$d_{\mathcal{A}} := \sum_{j=1}^{m} dx_j \wedge \mathrm{A}_j(t;x|\partial) + \sum_{j=1}^{2} dt_s \wedge \mathrm{L}_s(t;x|\partial), \quad (1.7)$$

where the differential expressions $\mathrm{A}_j, \mathrm{L}_S : \mathcal{H} \longrightarrow \mathcal{H}$ for all $j,k = \overline{1,m}$ and $s,l = \overline{1,2}$, satisfy the commutation conditions $[\mathrm{A}_j, \mathrm{A}_k] = 0$, $[\mathrm{L}_s, \mathrm{L}_s] = 0$, $[\mathrm{A}_j, \mathrm{L}_s] = 0$, then operation (1.7) defines an external generalized anti-differential operator on $\Lambda(M_{\mathrm{T}};\mathcal{H})$, with respect to which the co-chain sequence

$$\mathcal{H} \longrightarrow \Lambda^0(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d_{\mathcal{A}}} \Lambda^1(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d_{\mathcal{A}}} \ldots \xrightarrow{d_{\mathcal{A}}} \Lambda^{\mathrm{m}+2}(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d_{\mathcal{A}}} 0 \quad (1.8)$$

is evidently a cohomological complex, that is $d_{\mathcal{A}} d_{\mathcal{A}} \equiv 0$. As anti-differential (1.1) is a particular case of (1.7), we conclude that the corresponding co-chain sequence (1.8) is a cohomological complex too.

1.2. Below we will follow the ideas developed in [3, 4, 35]. A differential form $\beta \in \Lambda(M_{\mathrm{T}};\mathcal{H})$ will be called $d_{\mathcal{A}}$-closed if $d_{\mathcal{A}}\beta = 0$ and a form $\gamma \in \Lambda(M_{\mathrm{T}};\mathcal{H})$ will be called exact or $d_{\mathcal{A}}$-homological to zero if there exists on M_{T} such a form $\omega \in \Lambda(M_{\mathrm{T}};\mathcal{H})$ that $\gamma = d_{\mathcal{A}}\omega$.

Consider now the standard [8, 34, 35, 48] algebraic Hodge star-operation

$$* : \Lambda^{\mathrm{k}}(M_{\mathrm{T}};\mathcal{H}) \longrightarrow \Lambda^{\mathrm{m+2-k}}(M_{\mathrm{T}};\mathcal{H}) \tag{1.9}$$

$k = \overline{0, m+2}$, defined as follows: if $\beta \in \Lambda^{\mathrm{k}}(M_{\mathrm{T}};\mathcal{H})$, then the form $*\beta \in \Lambda^{\mathrm{m+2-k}}(M_{\mathrm{T}};\mathcal{H})$ is such that:

— $\langle \alpha, *\beta\rangle_{(m+2-k)} := \langle\langle \alpha, *\beta\rangle_{\mathbb{C}^N}\rangle_{m+2-k} = \langle\langle \alpha, \wedge\beta\rangle_{\mathbb{C}^N}, d\mu_g\rangle_{m+2}$ for any $\alpha \in \Lambda^{\mathrm{m+2-k}}(M_{\mathrm{T}};\mathcal{H})$, where $d\mu_g$ is an invariant measure on the metric space M_{T} with positive definite Riemannian metrics $g : T(M_{\mathrm{T}}) \times \mathrm{T}(M_{\mathrm{T}}) \to \mathbb{C}$, the scalar product

$$\langle \sigma_1^{(1)} \wedge \sigma_2^{(1)} \wedge \ldots \wedge \sigma_k^{(1)}, \gamma_1^{(1)} \wedge \gamma_2^{(1)} \wedge \ldots \wedge \gamma_k^{(1)}\rangle_k := \det\{\langle \sigma_i^{(1)}, \gamma_j^{(1)}\rangle_1 : i, j = \overline{1, k}\};$$

$\langle \sigma_i^{(1)}, \gamma_j^{(1)}\rangle_1 := \langle \hat{g}^{-1}\sigma_i^{(1)}, \hat{g}^{-1}\gamma_i^{(1)}\rangle_g$, 1-forms $\sigma_i^{(1)}, \gamma_j^{(1)} \in \Lambda^1(M_{\mathrm{T}})$, $i, j = \overline{1, k}$, and $\hat{g} : \mathrm{T}(M_{\mathrm{T}}) \to \mathrm{T}^*(M_{\mathrm{T}})$ is the canonical isomorphism, generated by the metrics $\langle \cdot, \cdot\rangle_g$ on $\mathrm{T}(M_{\mathrm{T}})$;

— $(m-k+2)$-dimensional volume $|*\beta|$ of form $*\beta$ equals k-dimensional volume $|\beta|$ of the form β;

— the $(m+2)$-dimensional measure $\langle \beta, \wedge * \beta\rangle \geq 0$ under the fixed orientation on M_{T}.

Further, on the space $\Lambda(M_{\mathrm{T}};\mathcal{H})$ define the following natural scalar product: for any $\beta, \gamma \in \Lambda^{\mathrm{k}}(M_{\mathrm{T}};\mathcal{H})$, $k = \overline{0, m}$:

$$(\beta, \gamma) := \int_{M_{\mathrm{T}}} \langle \beta, *\gamma\rangle_{\mathbb{C}^N}. \tag{1.10}$$

Subject to scalar product (1.10), one can naturally construct the corresponding Hilbert space

$$\mathcal{H}_\Lambda(M_{\mathrm{T}}) := \mathop{\oplus}_{k=0}^{m+2} \mathcal{H}_\Lambda^k(M_{\mathrm{T}}) \tag{1.11}$$

suitable for our further consideration. Notice that the Hodge star-operation satisfies the following easily verified property: for any $\beta, \gamma \in \mathcal{H}_\Lambda^k(M_{\mathrm{T}})$, $k = \overline{0, m}$,

$$(\beta, \gamma) = (*\beta, *\gamma), \tag{1.12}$$

that is the Hodge operation $* : \mathcal{H}_\Lambda(M_{\mathrm{T}}) \to \mathcal{H}_\Lambda(M_{\mathrm{T}})$ is unitary and its standard adjoint with respect to scalar product (1.10) operation satisfies the condition $(*)' = (*)^{-1}$.

Denote by $d_{\mathcal{L}}^{'}$ the formally adjoint expression to weak differential operation (1.1). By means of the operations $d'_{\mathcal{L}}$ and $d_{\mathcal{L}}$ in the $\mathcal{H}_{\Lambda}(M_{\mathrm{T}})$ one can naturally define [3, 8, 34, 35, 37] a generalized Laplace-Hodge operator $\Delta_{\mathcal{L}} : \mathcal{H}_1(M_{\mathrm{T}}) \longrightarrow \mathcal{H}_1(M_{\mathrm{T}})$ as

$$\Delta_{\mathcal{L}} = d'_{\mathcal{L}} d_{\mathcal{L}} + d'_{\mathcal{L}} d_{\mathcal{L}}. \tag{1.13}$$

Take a form $\beta \in \mathcal{H}_{\Lambda}(M_{\mathrm{T}})$ satisfying the equality

$$\Delta_{\mathcal{L}} \beta = 0. \tag{1.14}$$

Such a form is called [35] harmonic. One can also verify that a harmonic form $\beta \in \mathcal{H}_{\Lambda}(M_{\mathrm{T}})$ satisfies simultaneously the following two adjoint conditions:

$$d'_{\mathcal{L}} \beta = 0, \quad d_{\mathcal{L}} \beta = 0 \tag{1.15}$$

easily stemming form (1.13) and (1.14).

Lemma 1.1. *The following differential operators in* $\mathcal{H}_{\Lambda}(M_{\mathrm{T}})$

$$d^*_{\mathcal{L}} := * d'_{\mathcal{L}} (*)^{-1} \tag{1.16}$$

defines also a new external anti-differential operations in $\mathcal{H}_1(M_{\mathrm{T}})$, *subject to which the respectively dual to* (1.8) *co-chain sequence*

$$\mathcal{H} \longrightarrow \Lambda^0(M_{\mathrm{T}}; \mathcal{H}) \xrightarrow{d^*_{\mathcal{L}}} \Lambda^1(M_{\mathrm{T}}; \mathcal{H}) \xrightarrow{d^*_{\mathcal{L}}} \dots \xrightarrow{d^*_{\mathcal{L}}} \Lambda^{m+2}(M_{\mathrm{T}}; \mathcal{H}) \xrightarrow{d^*_{\mathcal{L}}} 0 \tag{1.17}$$

is a cohomological complex.

Proof. Really, the statement holds true owing to the property $d^*_{\mathcal{L}} d^*_{\mathcal{L}} = 0$, following from definition (1.16). □

1.3. Denote further by $\mathcal{H}^k_{\Lambda(\mathrm{L})}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$, the cohomology groups of $d_{\mathcal{L}}$-closed and by $\mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$, $k = \overline{0, m+2}$, the cohomology groups of $d_{\mathcal{L}^*}$-closed differential forms, respectively, and by $\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$, the abelian groups of harmonic differential forms from the Hilbert sub-spaces $\mathcal{H}^k_{\Lambda}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$. Before formulating next results, define the standard Hilbert-Schmidt rigged chain [12, 13] of positive and negative Hilbert spaces of differential forms

$$\mathcal{H}^k_{\Lambda,+}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda,-}(M_{\mathrm{T}}), \tag{1.18}$$

the corresponding hereditary rigged chains of harmonic forms:

$$\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M_{\mathrm{T}}) \tag{1.19}$$

and cohomology groups:

$$\begin{aligned} \mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}}) &\subset \mathcal{H}^k_{\Lambda(\mathcal{L})}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}}), \\ \mathcal{H}^k_{\Lambda(\mathcal{L}^*),+}(M_{\mathrm{T}}) &\subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*),-}(M_{\mathrm{T}}) \end{aligned} \tag{1.20}$$

for any $k = \overline{0, m+2}$. Assume also that generalized Laplace-Hodge operator (1.13) is reduced upon the space $\mathcal{H}^0_\Lambda(M)$. Now by reasoning similar to those in [8, 35] one can formulate a slightly generalized [3–5, 35] de Rham-Hodge theorem.

The groups of harmonic forms $\mathcal{H}^k_{\Lambda,-}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$, are, respectively, isomorphic to the homology groups $(H_k(M_{\mathrm{T}};\mathbb{C}))^{|\Sigma|}$, $k = \overline{0, m+2}$, where $H_k(M_{\mathrm{T}};\mathbb{C})$ is the k-th cohomology group of the manifold M_{T} with complex coefficients, the set $\Sigma \subset \mathbb{C}^p$, $p \in \mathbb{Z}_+$, is a set of suitable "spectral" parameters marking the linear space of independent $d^*_{\mathcal{L}}$-closed 0-form from $\mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}})$ and, moreover, the following direct sum decompositions

$$\begin{aligned}\mathcal{H}^k_{\Lambda,-}(M_{\mathrm{T}}) &= \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M_{\mathrm{T}}) \oplus \Delta_{\mathcal{L}}\mathcal{H}^k_{\Lambda,-}(M_{\mathrm{T}}) = \\ &= \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M_{\mathrm{T}}) \oplus d_{\mathcal{L}}\mathcal{H}^{k-1}_{\Lambda,-}(M_{\mathrm{T}}) \oplus d'_{\mathcal{L}}\mathcal{H}^{k+1}_{\Lambda,-}(M_{\mathrm{T}})\end{aligned}$$

hold for any $k = \overline{0, m+2}$.

Another result of this type was earlier formulated in [3–6] and reads as the following generalized de Rham-Hodge theorem.

Theorem 1.2. *The generalized cohomology groups* $\mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$, *are, respectively, isomorphic to the cohomology groups* $(\mathrm{H}^k(M_{\mathrm{T}};\mathbb{C}))^{|\Sigma|}$, $k = \overline{0, m+2}$.

Proof. A proof of this theorem is based on some special sequence [3–6] of differential Lagrange type identities. □

Define the following closed subspace

$$\mathcal{H}^*_0 := \{\varphi^{(0)}(\eta) \in \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_{\mathrm{T}}) : d^*_{\mathcal{L}}\varphi^{(0)}(\eta) = 0, \varphi^{(0)}(\eta)|_\Gamma, \eta \in \Sigma\} \tag{1.21}$$

for some smooth $(m+1)$-dimensional hypersurface $\Gamma \subset M_{\mathrm{T}}$ and $\Sigma \subset (\sigma(L) \cap \bar{\sigma}(L)) \times \Sigma_\sigma \subset \mathbb{C}^p$, where $\mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_{\mathrm{T}})$ is, as above, a suitable Hilbert-Schmidt rigged [12, 13] zero-order cohomology group Hilbert space from the co-chain given by (1.20), $\sigma(L)$ and $\sigma(L^*)$ are, respectively, mutual generalized spectra of the sets of differential operators L and L^* in H at $t = 0 \in \mathrm{T}$. Thereby, the dimension $\dim\mathcal{H}^*_0 = card$ $\Sigma := |\Sigma|$ is assumed to be known. The next lemma earlier stated by I.V. Skrypnik [3–6] is utmost importance meaning for a proof of Theorem (1.2).

Lemma 1.3. *There exists a set of differential* $(k+1)$*-forms* $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2}$, *and a set of* k*-forms* $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2}$, *parametrized by the set* $\Sigma \ni \eta$, *being semilinear in* $(\varphi^{(0)}(\eta), \psi^{(k)}) \in \mathcal{H}^*_0 \times \mathcal{H}^k_{\Lambda,-}(M_{\mathrm{T}})$, *such that*

$$Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] = dZ^k[\phi^{(0)}(\eta), \psi^{(k)}] \tag{1.22}$$

for all $k = \overline{0, m+2}$ *and* $\eta \in \Sigma$.

Proof. A proof is based on the following Lagrange type identity generalizing identity (3.3) of Part 1 and holding for any pair $(\varphi^0(\eta), \psi(k)) \in \mathcal{H}_0^* \times \mathcal{H}_{\Lambda,-}^k(M_{\mathrm{T}})$:

$$\begin{aligned} 0 &= \langle d_{\mathcal{L}}^* \phi^{(0)}(\eta), \wedge * (\psi^{(k)} \wedge \overline{\gamma})\rangle = \langle * d'_{\mathcal{L}}(*)^{-1}\varphi^{(0)}(\eta), *(\psi^{(k)} \wedge \overline{\gamma})\rangle = \\ &= \langle * d'_{\mathcal{L}}(*)^{-1}\phi^{(0)}(x), \wedge\psi^{(k)} \wedge \overline{\gamma})\rangle = \\ &= \langle (*)^{-1}\varphi^{(0)}(\eta), \wedge d_{\mathcal{L}}\psi^{(k)} \wedge \overline{\gamma}\rangle + Z^{(k+1)}[\psi^{(0)}(\eta), \wedge d_{\mathcal{L}}\psi(k)] \wedge \overline{\gamma} = \\ &= \langle (*)_{-1}\varphi^{(0)}(\eta), \wedge d_{\mathcal{L}}\psi^{(k)} \wedge \overline{\gamma}\rangle + dZ^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \wedge \overline{\gamma}, \end{aligned}$$

where $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2}$, and $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2}$, are some semilinear differential forms on M_{T} parametrized by a parameter $\lambda \in \Sigma$, and $\overline{\gamma} \in \Lambda^{m+1-k}(M_{\mathrm{T}};\mathbb{C})$ is an arbitrary constant $(m+1-k)$-form. Thereby, the semilinear differential $(k+1)$-forms $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M_{\mathrm{T}};\mathbb{C})$ and k-forms $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2}$, $\lambda \in \Sigma$, constructed above are exactly those searched for in the Lemma. □

1.4. Now based on Lemma (1.3), one can construct the cohomology group isomorphism claimed in Theorem 1.2 formulated above. Namely, following [3–5], let us take some singular simplicial [34,35] complex $\mathcal{K}(M_{\mathrm{T}})$ of the metric space M_{T} and introduce a set of linear mappings $B_\lambda^{(k)} : \mathcal{H}_{\Lambda,-}^k M_{\mathrm{T}} \longrightarrow C_k(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2,}$ $\lambda \in \Sigma$, where $C_k(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2,}$ are free abelian groups over the field $\mathbb{C}$ generated by all k-chains of singular simplexes $S^{(k)} \subset M_{\mathrm{T}}$, $k = \overline{0, m+2}$, respectiuvely, from the simplicial complex $\mathcal{K}(M_{\mathrm{T}})$, as follows:

$$B_\lambda^{(k)}(\psi^{(k)}) := \sum_{S^{(k)} \in C_k(M_{\mathrm{T}};\mathbb{C}))} S^{(k)} \int_{S^{(k)}} Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \tag{1.23}$$

with $\psi^{(k)} \in \mathcal{H}_{\Lambda,-}^k(M_{\mathrm{T}})$, $k = \overline{0, m+2}$. The following theorem proved fearlier in [3–5] with use of mappings (1.23) holds.

Theorem 1.4. *The set of operators* (1.23) *parametrized by* $\lambda \in \Sigma$ *realizes the cohomology group isomorphism referred to in Theorem 1.2.*

Proof. One can get a proof of this theorem passing over, in (1.23), to the corresponding cohomology $\mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_{\mathrm{T}})$ and homology $\mathrm{H}_k(M_{\mathrm{T}};\mathbb{C})$ groups of M_{T} for every $k = \overline{0, m+2}$. Taking an element $\psi^{(k)} := \psi^{(k)}(\mu) \in \mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_{\mathrm{T}})$, $k = \overline{0, m+2}$, solving the equation $d_{\mathcal{L}}\psi^{(k)}(\mu) = 0$ with $\mu \in \Sigma_k$ being some set of the related "spectral" parameters marking elements of the subspace $\mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_{\mathrm{T}})$, from (1.23) and (1.22), one can easily conclude that $dZ^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] = 0$ for all $(\lambda, \mu) \in \Sigma \times \Sigma_k$, $k = \overline{0, m+2}$. Owing to the Poincare lemma [33–35], this, in particular, means that there exist differential $(k-1)$-forms $\Omega^{(k-1)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu] \in \Lambda^{k-1}(M;\mathbb{C})$, $k = \overline{0, m+2}$, such that

$$Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] = d\Omega^{(k-1)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] \tag{1.24}$$

for all pairs $(\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}})$ parametrized by $(\lambda,\mu) \in \Sigma \times \Sigma_k$, $k = \overline{0, m+2}$. As a result of passing to the homology groups $\mathrm{H}_k(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2}$, on the right-hand side of (1.23), owing to the standard Stokes theorem [33–35] , it can be shown that the mappings

$$B_\lambda^{(k)} : \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}}) \longrightarrow H_k(M_{\mathrm{T}};\mathbb{C}) \tag{1.25}$$

are isomorphisms for every $k = \overline{0, m+2}$ and $\lambda \in \Sigma$. Making further use of the Poincare duality [8, 34, 35] between the homology groups $H_k(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2}$, and the cohomology groups $H^k(M;\mathbb{C})$, $k = \overline{0, m+2}$, respectively, one finally obtains the statement claimed of Theorem 1.4. □

2. THE STRUCTURE OF DELSARTE-LIONS TYPE OPERATORS IN MULTIDIMENSION

2.1. Now take into account that our differential operators $\mathrm{L}_j : \mathcal{H} \longrightarrow \mathcal{H}$, $j = \overline{1, 2}$, are of special form (1.2). Also assume that differential expressions (1.4) are normal closed operators defined on dense subspace $D(\mathcal{L}) \subset L_2(M;\mathbb{C}^N)$.

Then due to Theorem 1.4 one can find such a pair $(\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}})$ parametrized by elements $(\lambda,\mu) \in \Sigma \times \Sigma_k$, for which the equivalent equality

$$B_\lambda^{(m)}(\psi^{(0)}(\mu)dx) = S^{(m)}_{(t;x)} \int_{\partial S^{(m)}_{(t;x)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \tag{2.1}$$

holds, where $S^{(m)}_{(t;x)} \in C_m(M_{\mathrm{T}};\mathbb{C})$ is some arbitrary but fixed element parametrized by a chosen point $(t;x) \in \partial S^{(m)}_{(t;x)}$. Consider the following integral expressions

$$\Omega_{(t;x)}(\lambda,\mu) := \int_{\sigma^{(m-1)}_{(t;x)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \tag{2.2}$$
$$\Omega_{(t_0;x_0)}(\lambda,\mu) := \int_{\sigma^{(m-1)}_{(t_0;x_0)}} \Omega^{(m-1)}[\phi^{(0)}(\lambda), psi_{(0)}(\mu)dx],$$

with a point $(t_0;x_0) \in M_{\mathrm{T}} \cap \partial S^{(m)}_{(t_0;x_0)}$ being fixed, the boundaries $\sigma^{(m-1)}_{(t;x)} := \partial S^{(m)}_{t;x}$, $\sigma^{(m-1)}_{(t_0;x_0)} := \partial S^{(m)}_{t_0;x_0}$ being assumed to be homological to each other as $(t;x_0) \longrightarrow (t;x) \in M_{\mathrm{T}}$, $(\lambda,\mu) \in \Sigma \times \Sigma_k$, and interpret them as the kernels [12, 13] of the corresponding invertible integral operators of Hilbert-Schmidt type $\Omega_{(t;x)}, \Omega_{(t_0;x_0)} : L_2^\rho(\Sigma;\mathbb{C}) \longrightarrow L_2^\rho(\Sigma;\mathbb{C})$, where ρ is some finite Borel measure on the parameter set Σ. Now define the invertible operators expressions

$$\Omega_\pm : \psi^{(0)}(\mu) \longrightarrow \tilde{\psi}^{(0)}(\mu) \tag{2.3}$$

for $\psi^{(0)}(\mu)dx \in \mathcal{H}^m_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}})$ and some $\tilde{\psi}^{(0)}(\mu)dx \in \mathcal{H}^m_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}})$, $\mu \in \Sigma$, where, by definition, for any $\eta \in \Sigma$

$$\begin{aligned} \tilde{\psi}^{(0)}(\eta) &:= \psi^{(0)}(\eta)\cdot\Omega^{-1}_{(t;x)}\cdot\Omega_{(t_0;x_0)} = \\ &= \int_\Sigma d\rho(\mu)\int_\Sigma d\rho(\xi)\psi^{(0)}(\mu)\Omega^{-1}_{(t;x)}(\mu,\xi)\Omega_{(t_0;x_0)}(\xi,\eta), \end{aligned} \tag{2.4}$$

being motivated by expression (2.1). Namely, consider the following diagram

$$\begin{array}{ccc} \mathcal{H}^m_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}}) & \overset{\Omega_\pm}{\rightarrow} & \mathcal{H}^m_{\Lambda(\tilde{\mathcal{L}}),-}(M_{\mathrm{T}}), \\ B^{(m)}_\lambda \downarrow & \swarrow \tilde{B}^{(m)}_\lambda & \\ H_m(M_{\mathrm{T}};\mathbb{C}) & & \end{array} \tag{2.5}$$

which is assumed to be commutative for some another co-chain complex

$$\mathcal{H} \longrightarrow \Lambda^0(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} \Lambda^1(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} \dots \xrightarrow{d_{\tilde{\mathcal{L}}}} \Lambda^{m+2}(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} 0. \tag{2.6}$$

Here, by definition, the generalized "anti-differentiation" is

$$d_{\tilde{\mathcal{L}}}: = \sum_{j=1}^2 dt_j \wedge \tilde{\mathrm{L}}_j(t;x|\partial) \tag{2.7}$$

and

$$\begin{aligned} \tilde{\mathrm{L}}_j &= \partial/\partial t_j - \tilde{L}_j(t;x|\partial), \\ \tilde{L}_j(t;x|\partial) &:= \sum_{|\alpha|=0}^{n_j(\tilde{L})} \tilde{a}^{(j)}_\alpha(t;x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}, \end{aligned} \tag{2.8}$$

where coefficients $\tilde{a}^{(j)}_\alpha \in C^1(\mathrm{T}; C^\infty(M;\mathrm{End}\mathbb{C}^N)$, $|\alpha| = \overline{0, n_j(\tilde{L})}$, $n_j(\tilde{L}) := n_j(L) \in \mathbb{Z}_+$, $j = \overline{1,2}$. The corresponding isomorphisms $\tilde{B}^{(m)}_\lambda : \mathcal{H}^m_{\Lambda(\tilde{\mathcal{L}}),+}(M_{\mathrm{T}}) \longrightarrow H_m(M_{\mathrm{T}};\mathbb{C})$, $\lambda \in \Sigma$, act, by definition, as follows:

$$\tilde{B}^{(m)}_\lambda(\tilde{\psi}^{(0)}(\mu)dx) = S^{(m)}_{(t;x)}\int_{\partial S^{(m)}_{(t;x)}} \tilde{\Omega}^{(m-1)}[\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)dx] \tag{2.9}$$

where $\tilde{\varphi}^{(0)}(\lambda) \in \tilde{\mathcal{H}}^*_0 \subset \mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}^*),-}(M_{\mathrm{T}})$, $\lambda \in (\sigma(\tilde{L}) \cap \bar{\sigma}(\tilde{L}^*)) \times \Sigma_\sigma$,

$$\tilde{\mathcal{H}}^*_0 := \{\tilde{\varphi}^{(0)}(\lambda) \in \mathcal{H}^m_{\Lambda(\mathcal{L}^*),-}(M_{\mathrm{T}}) : d^*_{\tilde{\mathcal{L}}}\tilde{\varphi}^{(0)}(x) = 0, \tilde{\varphi}^{(0)}(\lambda)|_{\tilde{\Gamma}} = 0, \lambda \in \Sigma\} \tag{2.10}$$

for a hypersurface $\tilde{\Gamma} \subset M_{\mathrm{T}}$. Respectively, one defines the following closed subspace

$$\tilde{\mathcal{H}}_0 := \{\tilde{\psi}^{(0)}(\mu) \in \mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}^*),-}(M_{\mathrm{T}}) : d^*_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\lambda) = 0, \tilde{\psi}^{(0)}(\mu)|_{\tilde{\Gamma}} = 0, \mu \in \Sigma\} \tag{2.11}$$

for the hyperspace $\tilde{\Gamma} \subset M_{\mathrm{T}}$, introduced above.

Suppose now that elements (2.4) belong to closed subspace (2.11), that is

$$d_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\mu) = 0. \tag{2.12}$$

Similarly to (2.11), define a closed subspace $\mathcal{H}_0^* \subset \mathcal{H}^m_{\Lambda(\mathcal{L}^*),-}(M_{\mathrm{T}})$ as follows:

$$\mathcal{H}_0 := \{\psi^{(0)}(\lambda) \in \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_{\mathrm{T}}) : d_{\mathcal{L}}\psi^{(0)}(\lambda) = 0, \psi^{(0)}(\lambda)|_\Gamma = 0, \lambda \in \Sigma\} \tag{2.13}$$

for all $\mu \in \Sigma$. Then owing to the commutativity of diagram (2.5), there exist the corresponding two invertible mappings

$$\mathbf{\Omega}_\pm : \mathcal{H}_0 \to \tilde{\mathcal{H}}_0, \tag{2.14}$$

depending on the ways they were extended onto the entire Hilbert space $\mathcal{H}^m_{\Lambda,-}(M_{\mathrm{T}})$. Now extend operators (2.14) onto the entire Hilbert space $\mathcal{H}^m_{\Lambda,-}(M_{\mathrm{T}})$ by means of the standard method [26, 43] of variation of constants, taking into account that for kernels $\Omega_{(t;x)}(\lambda,\mu), \Omega_{(t_0;x_0)}(\lambda,\mu) \in L_2^{(p)}(\Sigma;\mathbb{C}) \otimes L_2^{(p)}(\Sigma;\mathbb{C})$, $\lambda,\mu \in \Sigma$, one can write down the following relationships:

$$\begin{aligned}
\Omega_{(t;x)}(\lambda,\mu) - \Omega_{(t_0;x_0)}(\lambda,\mu) &= \int_{\partial S^{(m)}_{(t;x)}} \Omega^{(m-1)}[\varphi^{(0)}(x), \psi^{(0)}(\mu)dx] - \\
&\quad - \int_{\partial S^{(m)}_{(t_0;x_0)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] = \\
&= \int_{S^{(m)}_\pm(\sigma^{(m-1)}_{(t;x)}, \sigma^{(m-1)}_{(t_0;x_0)})} d\Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] = \\
&= \int_{S^{(m)}_\pm(\sigma^{(m-1)}_{(t;x)}, \sigma^{(m-1)}_{(t_0;x_0)})} Z^{(m)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx],
\end{aligned} \tag{2.15}$$

where, by definition, m-dimensional open surfaces $S^{(m)}_\pm(\sigma^{(m-1)}_{(t;x)}, \sigma^{(m-1)}_{(t_0;x_0)}) \subset M_{\mathrm{T}}$ are spanned smoothly without self-intersection between two homological cycles $\sigma^{(m-1)}_{(t;x)} = \partial S^{(m)}_{(t;x)}$ and $\sigma^{(m-1)}_{(t_0;x_0)} = \partial S^{(m)}_{(t_0;x_0)} \in C_{m-1}(M_{\mathrm{T}};\mathbb{C})$ in such a way that the boundary $\partial(S^{(m)}_+(\sigma^{(m-1)}_{(t_0;x_0)}, \sigma^{(m-1)}_{(t_0;x_0)}) \cup S^{(m)}_-(\sigma^{(m-1)}_{(t;x)}, \sigma^{(m-1)}_{(t_0;x_0)})) = \varnothing$. Making use of relationship (2.15), one can thereby easily find the following integral operator expressions in $\mathcal{H}_-$:

$$\mathbf{\Omega}_\pm = \mathbf{1} - \int_\Sigma d\rho(\eta)\tilde{\psi}^{(0)}(\xi)\Omega^{-1}_{(t_0;x_0)}(\xi,\eta) \int_{S^{(m)}_\pm(\sigma^{(m-1)}_{(t;x)}, \sigma^{(m-1)}_{(t_0;x_0)})} Z_{(m)}[\varphi^{(0)}(\eta), (\cdot)dx] \tag{2.16}$$

defined for fixed pairs $(\varphi^{(0)}(\xi), \psi^0(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}^{(0)}(\xi), \tilde{\psi}^0(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda,\mu \in \Sigma$, being bounded invertible operators of Volterra type [23] on the whole Hilbert space $\mathcal{H}$. Thereby, we have proved the following theorem.

Theorem 2.1. *Let mappings* (2.14) *be given by Volterrian operator expressions* (2.16)*, where the semi-linear forms* $Z_{(m)}[\cdot,\cdot]$ *are defined by means of relationships* (1)*. Then diagram* (2.5) *commutes.*

Moreover, for the differential operators $\tilde{\mathrm{L}}_j : \mathcal{H} \longrightarrow \mathcal{H},\ \ j = \overline{1,2}$, one can easily get the following expressions

$$\tilde{\mathrm{L}}_j = \mathbf{\Omega}_{\pm} \mathrm{L}_j \mathbf{\Omega}_{\pm}^{-1} \tag{2.17}$$

for $j = \overline{1,2}$, where the left-hand sides of (2.17) do nott depend on the sings "$\pm$" of the right-hand sides. Thereby, Volterrian integral operators (2.16) are the Delsarte-Lions transmutation operators, mapping a given set $\mathcal{L}$ of differential operators into a new set $\tilde{\mathcal{L}}$ of differential operators transformed via Delsarte expressions (2.17).

2.2. Suppose now that none of the differential operators $L_j(t;x|\partial)$, $j = \overline{1,2}$, considered above depends on the variable $t \in \mathrm{T} \subset \mathbb{R}^2_+$. Then, evidently, one can take

$$\begin{aligned}
\mathcal{H}_0 &:= \{\psi^{(0)}_\mu(\xi) \in L_{2.-}(M;\mathbb{C}^N) : L_j \psi^{(0)}_\mu(\xi) = \mu_j \psi^{(0)}_\mu(\xi), \\
&\quad j = \overline{1,2}, \psi^{(0)}_\mu(\xi)|_{\tilde{\Gamma}} = 0, \mu = (\mu_1,\mu_2) \in \sigma(\tilde{L}) \cap \overline{\sigma}(L^*), \xi \in \Sigma_\sigma\}, \\
\tilde{\mathcal{H}}_0 &:= \{\tilde{\psi}^{(0)}_\mu(\xi) \in L_{2.-}(M;\mathbb{C}^N) : \tilde{L}_j \tilde{\psi}^{(0)}_\mu(\xi) = \mu_j \tilde{\psi}^{(0)}_\mu(\xi), \\
&\quad j = \overline{1,2}, \tilde{\psi}^{(0)}_\mu(\xi)|_{\tilde{\Gamma}} = 0, \mu = (\mu_1,\mu_2) \in \sigma(\tilde{L}) \cap \overline{\sigma}(L^*), \xi \in \Sigma_\sigma\}, \\
\mathcal{H}^*_0 &:= \{\varphi^{(0)}_\lambda(\eta) \in L_{2.-}(M;\mathbb{C}^N) : L^*_j \varphi^{(0)}_\lambda(\eta) = \bar{\lambda}_j \varphi^{(0)}_\lambda(\eta), j = \overline{1,2}, \\
\varphi^{(0)}_\lambda(\eta)|_{\tilde{\Gamma}} &= 0, \lambda = (\lambda_1,\lambda_2) \in \sigma(\tilde{L}) \cap \overline{\sigma}(L^*), \eta \in \Sigma_\sigma\}, \\
\tilde{\mathcal{H}}^*_0 &:= \{\tilde{\varphi}^{(0)}_\lambda(\eta) \in L_{2.-}(M;\mathbb{C}^N) : \tilde{L}^*_j \tilde{\varphi}^{(0)}_\lambda(\eta) = \bar{\lambda}_j \varphi^{(0)}_\lambda(\eta), \\
&\quad j = \overline{1,2}, \tilde{\varphi}^{(0)}_\lambda(\eta)|_{\tilde{\Gamma}} = 0, \lambda = (\lambda_1,\lambda_2) \in \sigma(\tilde{L}) \cap \overline{\sigma}(L^*), \eta \in \Sigma_\sigma\},
\end{aligned} \tag{2.18}$$

and construct the corresponding Delsarte-Lions transmutation operators

$$\begin{aligned}
\mathbf{\Omega}_{\pm} = 1 - \int_{\sigma(\tilde{L})\cap\overline{\sigma}(L^*)} d\rho_\sigma(\lambda) \int_{\Sigma_\sigma \times \Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) d\rho_{\Sigma_\sigma}(\eta) \times \\
\times \int_{S^{(m)}_{\pm} \sigma^{(m-1)}_{(t_0;x_0)}, \sigma^{(m-1)}_{(t_0;x_0)}} dx \tilde{\psi}^{(0)}_\lambda(\xi) \Omega^{-1}_{x_0}(\lambda;\xi;\eta) \bar{\varphi}^{(0),\intercal}_\lambda(\eta)(\cdot)
\end{aligned} \tag{2.19}$$

acting already in the Hilbert space $L_{2,+}(M;\mathbb{C}^N)$, where for any $(\lambda;\xi,\eta) \in (\sigma(\tilde{L}) \cap \overline{\sigma}(L^*) \times \Sigma^2_\sigma$, kernels

$$\Omega_{(x_0)}(\lambda;\xi,\eta) := \int_{\sigma^{(m-1)}_{x_0}} \Omega^{(m-1)}[\varphi^{(0)}_\lambda(\xi), \psi^{(0)}_\lambda(\eta) dx] \tag{2.20}$$

for $(\xi,\eta) \in \Sigma^2_\sigma$ and every $\lambda \in \sigma(\tilde{L})\ \cap \overline{\sigma}(L^*)$, belong to $L^{(\rho)}_2(\Sigma_\sigma;\mathbb{C}) \otimes L^{(\rho)}_2(\Sigma_\sigma;\mathbb{C})$. Moreover, as $\partial\mathbf{\Omega}_{\pm}/\partial t_j = 0$, $j = \overline{1,2}$, one gets easily the set of differential expressions

$$\tilde{L}_j(x|\partial) := \mathbf{\Omega}_{\pm} L_j(x|\partial) \mathbf{\Omega}_{\pm}^{-1} \tag{2.21}$$

$j = \overline{1,2}$, also commuting with each other.

Volterrian operators (2.19) possess some additional properties Namely, define the following Fredholm type integral operator in H

$$\mathbf{\Omega} := \mathbf{\Omega}_{+}^{-1} \mathbf{\Omega}_{-}, \tag{2.22}$$

which can be written in the form

$$\mathbf{\Omega}{=}\mathbf{1}{+}\Phi(\mathbf{\Omega}), \tag{2.23}$$

where the operator $\Phi(\mathbf{\Omega}) \in \mathcal{B}_\infty(H)$. Moreover, owing to relationships (2.20) one easily concluds that the following commutator conditions

$$[\mathbf{\Omega}, L_j] = 0 \tag{2.24}$$

hold for $j = \overline{1,2}$.

Denote now by $\hat{\Phi}(\mathbf{\Omega}) \in H_- \otimes H_-$ and $\hat{K}_+(\mathbf{\Omega})$, $\hat{K}_-(\mathbf{\Omega}) \in H_- \otimes H_-$ the kernels corresponding [12,13] to operators $\mathbf{\Phi} \in \mathcal{B}_\infty(H)$ and $\mathbf{\Omega}_\pm \in \mathcal{B}_\infty(H)$. Then, owing to the fact that $\sup K_+(\mathbf{\Omega}) \cap \sup K_-(\mathbf{\Omega}) = \oslash$, from (2.22) and (2.23), one gets the well known Gelfand-Levitan-Marchenko linear integral equation

$$\hat{K}_+(\mathbf{\Omega}) + \Phi(\hat{\mathbf{\Omega}}) + \hat{K}_+(\mathbf{\Omega})_+ \cdot \hat{\Phi}(\mathbf{\Omega}) = \hat{K}_-(\mathbf{\Omega}), \tag{2.25}$$

allowing to find the kernel $\hat{K}_+(\mathbf{\Omega}) \in H_- \otimes H_-$ for $(t;x) \in \sup K_+(\mathbf{\Omega}_+)$ factorizing Fredholmian operator (2.22). Conditions (2.24) can be rewritten suitably as follows:

$$(L_{j,ext} \otimes \mathbf{1})\hat{\Phi}(\mathbf{\Omega}) = (1 \otimes L^*_{j,ext})\hat{\Phi}(\mathbf{\Omega}), \tag{2.26}$$

where $L_{j,ext} \in \mathcal{L}(H_-)$, $j = \overline{1,2}$, and their adjoint $L^*_{j,ext} \in \mathcal{L}(H_-)$, $j = \overline{1,2}$, are the respective extensions [12,26] of the differential operators L_j and $L^*_j \in \mathcal{L}(H)$, $j = \overline{1,2}$.

Concerning relationships (2.21), one can write down [12, 26] kernel conditions similar to (2.26):

$$(\tilde{L}_{j,ext} \otimes \mathbf{1})\hat{K}_\pm(\mathbf{\Omega}) = (\mathbf{1} \otimes L^*_{j,ext})\hat{K}_\pm(\mathbf{\Omega}), \tag{2.27}$$

where, as above, $\tilde{L}_{j,ext} \in \mathcal{L}(H_-)$, $j = \overline{1,2}$, are the respective rigging extensions of the differential operators $\tilde{L}_j \in \mathcal{L}(H)$, $j = \overline{1,2}$.

2.3. Proceed now to analyzing the question of the general differential expression structure of transformed operator expression (2.17). It is evident that conditions (2.25) and (2.26) on the kernels $\hat{K}_\pm(\mathbf{\Omega}) \in \mathcal{H}_- \otimes \mathcal{H}_-$ of Delsarte-Lions transmutation operators are necessary for operator expression (2.17) to exist and be differential. Consider the question whether these conditions are sufficient.

For studying this question, let us consider Volterrian operators (2.19) with kernels satisfying the conditions (2.25) and (2.26), assuming that suitable oriented surfaces $S^{(m)}_\pm(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)}}) \in C_m(M_{\mathrm{T}};\mathbb{C})$ can be given as follows:

$$\begin{aligned} &S^{(m)}_+(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)}}) = \{(t';x') \in M_{\mathrm{T}} : t' = P(t;x|x') \in \mathrm{T}\} \\ &S^{(m)}_-(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)}}) = \{(t';x') \in M_{\mathrm{T}} : t' = P(t;x|x') \in \mathrm{T}\backslash[t_0,t]\}, \end{aligned} \tag{2.28}$$

where a mapping $P \in C^\infty(M_{\mathrm{T}} \times M;\mathrm{T})$ is piecewise smooth and such that the boundaries $\partial S^{(m)}_\pm(\sigma^{(m-1)}_{(t;x)}, \sigma^{(m-1)}_{(t_0;x_0)}) = \pm(\sigma^{(m-1)}_{(t;x)} - \sigma^{(m-1)}_{(t_0;x_0)})$ with cycles $\sigma^{(m-1)}_{(t;x)}$ and $\sigma^{(m-1)}_{(t_0;x_0)} \in \mathcal{K}(M_{\mathrm{T}})$ are homological to each other for any choice of points $(t_0;x_0)$ and

$(t;x) \in M_{\mathrm{T}}$. Then by means of some simple but cumbersome calculations, based on considerations from [41] and [9], one can seeb that the resulting expressions on the left-hand side of

$$\tilde{\mathrm{L}} = \mathrm{L} + [\mathrm{K}_{\pm}(\mathbf{\Omega}), \mathrm{L}] \cdot \mathbf{\Omega}_{\pm}^{-1} \tag{2.29}$$

are differential ones exactly equal to each other, if the expression for an operator $\mathrm{L} \in \mathcal{L}(\mathcal{H})$ was differential too.

Concerning the inverse operators $\mathbf{\Omega}_{\pm}^{-1} \in \mathcal{B}(\mathcal{H})$ appearing in (2.29), one can notice here that, owing to the functional symmetry between closed subspaces $\mathcal{H}_0$ and $\tilde{\mathcal{H}}_0 \subset \tilde{\mathcal{H}}_-$, defining relationships (2.14) and (2.4) are reversible, that is, there exist inverse operator mappings $\mathbf{\Omega}_{\pm}^{-1} : \tilde{\mathcal{H}}_0 \to \mathcal{H}_0$, such that

$$\mathbf{\Omega}_{\pm}^{-1} : \tilde{\psi}^{(0)}(\lambda) \longrightarrow \psi^{(0)}(\lambda) := \tilde{\psi}^{(0)}(\lambda) \cdot \tilde{\Omega}_{(t;x)}^{-1} \tilde{\Omega}_{(t;x)} \tag{2.30}$$

for some suitable kernels $\tilde{\Omega}_{(t;x)}(\lambda,\mu)$ and $\tilde{\Omega}_{(t_0;x_0)}(\lambda,\mu) \in L_2^{(\rho)}(\Sigma;\mathbb{C}) \otimes L_2^{(\rho)}(\Sigma;\mathbb{C})$, related naturally to the transformed differential expression $\tilde{\mathrm{L}} \in \mathcal{L}(\mathcal{H})$. Thereby, owing to expressions (2.30), one can write down the following integral operator expressions similar to (2.19):

$$\begin{aligned} \mathbf{\Omega}_{\pm}^{-1} = \mathbf{1} - \int_{\Sigma} d\rho(\xi) \int_{\Sigma} d\rho(\eta) \psi^{(0)}(\xi) \tilde{\Omega}_{t_0;x_0}^{-1}(\xi,\eta) \times \\ \times \int_{S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)})} \tilde{Z}^{(m)}[\tilde{\varphi}^{(0)}(\eta), (\cdot)dx], \end{aligned} \tag{2.31}$$

defined for fixed pairs $(\tilde{\varphi}^{(0)}(\xi), \tilde{\psi}^{(0)}(\eta)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$ and $(\varphi^{(0)}(\xi), \psi^{(0)}(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\xi, \eta \in \Sigma$, and being bounded invertible operators of Volterra type on the entire Hilbert space $\mathcal{H}$. In particular, the compatibility conditions $\mathbf{\Omega}_{\pm}\mathbf{\Omega}_{\pm}^{-1} = \mathbf{1} = \mathbf{\Omega}_{\pm}^{-1}\mathbf{\Omega}_{\pm}$ must be fulfilled identically on $\mathcal{H}$, involving some restrictions identifying both measures ρ and Σ and suitable asymptotic conditions on the coefficient functions of the differential expression $\mathrm{L} \in \mathcal{L}$. Restrictions of this type were already mentioned before in [42,44,45], where in particular the relationships with the local and nonlocal Riemann problems were discussed. identically measures ρ and Σ and possible asymptotic conditions of coefficient functions of the differential expression $\mathrm{L} \in \mathcal{L}$. Such kinds of restrictions were already mentioned before in [42, 44, 45, 48], where the relationships with the local and nonlocal Riemann problems were discussed.

2.4. Within the framework of the general construction presented above one can give a natural interpretation of so called Backlund transformations for coefficient function of a given differential operator expression $\mathrm{L} \in \mathcal{L}(\mathcal{H})$. Namely, following the symbolic considerations in [47], we reinterpret the approach devised there for constructing the Backlund transformations making use of the techniques based on the theory of Delsarte transmutation operators. Let us define two Delsarte-Lions transformed differential operator expressions

$$\mathrm{L}_1 = \mathbf{\Omega}_{1,\pm} \mathrm{L} \mathbf{\Omega}_{1,\pm}^{-1}, \qquad \mathrm{L}_2 = \mathbf{\Omega}_{2,\pm} \mathrm{L} \mathbf{\Omega}_{2,\pm}^{-1}, \tag{2.32}$$

where $\mathbf{\Omega}_{1,+}, \mathbf{\Omega}_{2,-} \in \mathcal{B}(\mathcal{H})$ are some Delsarte transmutation Volterrian operators in $\mathcal{H}$ with non-intersecting Borel spectral measures ρ_1 and ρ_2 on Σ, but such that the following conditions

$$\mathbf{\Omega}_{1,+}^{-1}\mathbf{\Omega}_{1,-} = \mathbf{\Omega} = \mathbf{\Omega}_{2,+}^{-1}\mathbf{\Omega}_{2,-}. \tag{2.33}$$

Making now use of conditions (2.32) and relationships (2.33), one easily finds that the operator $\mathrm{B} := \mathbf{\Omega}_{2,-}\mathbf{\Omega}_{1,+}^{-1} \in \mathcal{B}(\mathcal{H})$ satisfies the following operator equations:

$$\mathrm{L}_2\mathrm{B} = \mathrm{B}\mathrm{L}_1, \quad \mathbf{\Omega}_{2,\pm}\mathrm{B} = \mathrm{B}\mathbf{\Omega}_{1,\pm}, \tag{2.34}$$

which motivate the next definition.

Definition 2.2. *An invertible symbolic mapping* $\mathrm{B} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}\ (\mathcal{H})$ *will be called a Delsarte-Backlund transformation of an operator* $\mathrm{L}_1 \in \mathcal{L}(\mathcal{H})$ *if there holds the condition*

$$[\mathrm{QB}, \mathrm{L}_1] = 0 \tag{2.35}$$

for some linear differential expression $\mathrm{Q} \in \mathcal{L}(\mathcal{H})$.

Condition (2.35) can be realized as follows. Take any differential expression $\mathrm{q} \in \mathcal{L}(\mathrm{H})$ satisfying the symbolic equation

$$[\mathrm{qB}, \mathrm{L}] = 0 \tag{2.36}$$

Then, making use of the transformations like those in (2.32), from (2.33) one finds that

$$[\mathrm{QB}, \mathrm{L}_1] = 0, \tag{2.37}$$

where, owing to (2.34),

$$\mathrm{QB} := \mathbf{\Omega}_{1,+}q\mathrm{B}\mathbf{\Omega}_{1,+}^{-1} = \mathbf{\Omega}_{1,+}\mathrm{q}\mathbf{\Omega}_{2,+}^{-1}\mathrm{B}. \tag{2.38}$$

Therefore, the expression $\mathrm{Q} = \mathbf{\Omega}_{1,+}q\mathbf{\Omega}_{2,+}^{-1}$ appears to be differential too, owing to condition (2.34).

The above consideration related to the symbolic mapping $\mathrm{B} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ gives rise to an effective tool for constructing self-Backlund transformations for coefficients of given differential operator expressions $\mathrm{L}_1, \mathrm{L}_2 \in \mathcal{L}(\mathcal{H})$ having many applications in soliton theory.

2.5. Return now to studying the structure of Delsarte-Lions transformations for a polynomial differential operator pencil

$$\mathrm{L}(\lambda; x|\partial) := \sum_{j=0}^{n(L)} L_j(x|\partial)\lambda^j, \tag{2.39}$$

where $r(L) \in \mathbb{Z}_+$ and $\lambda \in \mathbb{C}$ is a complex-valued parameter.We need to find the Delsarte-Lions transformations $\mathbf{\Omega}_{\lambda,\pm} \in \mathcal{B}(\mathcal{H})$, $\lambda \in \mathbb{C}$, corresponding to (2.39)such that, for some polynomial differential operators pencil $\tilde{\mathrm{L}}(\lambda; x|\partial) \in \mathcal{L}(\mathcal{H})$, the following Delsarte-Lions [2] condition

$$\tilde{\mathrm{L}}\mathbf{\Omega}_{\lambda,\pm} = \mathbf{\Omega}_{\lambda,\pm}\mathrm{L} \tag{2.40}$$

holds for almost all $\lambda \in \mathbb{C}$. In order to find such transformations $\mathbf{\Omega}_{\lambda\pm} \in \mathcal{B}(\mathcal{H})$, let us consider a differential operator $\mathrm{L}_\tau(x|\partial) \in \mathcal{L}(\mathcal{H}_\tau)$, depending on $\tau \in \mathbb{R}$, where

$$\mathrm{L}_\tau(x|\partial) := \sum_{j=0}^{n(L)} L_j(x|\partial)\partial^j/\partial\tau^j, \tag{2.41}$$

acting in the functional space $\mathcal{H}_\tau = C^{q(L)}(\mathbb{R};\mathcal{H})$ for some $q(L) \in \mathbb{Z}_+$. Then one can easily construct the corresponding Delsarte-Lions transformations $\mathbf{\Omega}_{\tau,\pm} \in \mathcal{B}(\mathcal{H}_\tau)$ of Volterra type for some differential operator expression

$$\tilde{\mathrm{L}}_\tau(x|\partial) := \sum_{j=0}^{n(L)} \tilde{L}_j(x|\partial)\partial^j/\partial\tau^j, \tag{2.42}$$

for which the following Delsarte-Lions [2] transmutation conditions

$$\tilde{\mathrm{L}}_\tau\mathbf{\Omega}_{\tau,\pm} = \mathbf{\Omega}_{\tau,\pm}\mathrm{L}_\tau \tag{2.43}$$

hold in $\mathcal{H}_\tau$. Thus, making use of the results obtained above, one can write down:

$$\begin{aligned} \mathbf{\Omega}_{\tau,\pm} = \mathbf{1} - \int_\Sigma d\rho_\Sigma(\xi)\int_\Sigma d\rho_\Sigma(\eta)\tilde{\psi}_\tau^{(0)}(\lambda;\xi)\Omega^{-1}_{(\tau_0;x_0)}(\lambda;\xi,\eta)\times \\ \times\int_{S_\pm^{(m)}(\sigma^{(m-1)}_{(\tau;x)},\sigma^{(m-1)}_{(\tau_0;x_0)})} Z^{(m)}[\tilde{\varphi_\tau^{(0)}}(\lambda;\eta),(\cdot)dx], \end{aligned} \tag{2.44}$$

defined by means of the following closed subspaces $\mathcal{H}_{\tau,0} \subset \mathcal{H}_{\tau,-}$ and $\mathcal{H}^*_{\tau,0} \subset \mathcal{H}^*_{\tau,-}$:

$$\begin{aligned} \mathcal{H}_{\tau,0} :&= \{\psi_\tau^{(0)}(\lambda;\xi) \in \mathcal{H}_{\tau,-} : \mathrm{L}_\tau\psi_\tau^{(0)}(\lambda;\xi) = 0, \psi_\tau^{(0)}(\lambda;\xi)|_{\tau=0} = \\ &= \psi^{(0)}(\lambda;\xi) \in \mathcal{H}, \mathrm{L}\psi^{(0)}(\lambda;\xi) = 0, \psi^{(0)}(\lambda;\xi)|_\Gamma = 0, \lambda \in \mathbb{C}, \xi \in \Sigma\}, \\ \mathcal{H}^*_{\tau,0} :&= \{\varphi_\tau^{(0)}(\lambda;\eta) \in \mathcal{H}^*_{\tau,-} : \mathrm{L}_\tau\varphi_\tau^{(0)}(\lambda;\eta) = 0, \varphi_\tau^{(0)}(\lambda;\eta)|_{\tau=0} = \\ &= \varphi^{(0)}(\lambda;\eta) \in \mathcal{H}^*, \mathrm{L}\varphi^{(0)}(\lambda;\eta) = 0, \varphi^{(0)}(\lambda;\eta)|_\Gamma = 0, \lambda \in \mathbb{C}, \eta \in \Sigma\}. \end{aligned} \tag{2.45}$$

Based on representations (2.28) and related results obtained in [21, 48] one can formulate the following important lemma.

Lemma 2.3. *Let differential expression* (2.42) *be Delsarte-Lions transformed by means of Volterrian transmutation operators* (2.44), *satisfying conditions* (2.43). *Then it will remain to be differential iff the surfaces* $S_\pm^{(m)}(\sigma^{(m-1)}_{(\tau;x)},\sigma^{(m-1)}_{(\tau_0;x_0)})$ *are generated by suitably chosen standard simplicial polyhedra* $\sigma^{(m-1)}_{(\tau;x)}$ *and* $\sigma^{(m-1)}_{(\tau_0;x_0)} \in \mathbb{R}^m \times \mathbb{R}$.

Recalling now that our operators $L_j \in \mathcal{L}(\mathcal{H})$, $j = \overline{0, r(L)}$, do not depend on the parameter $\tau \in \mathbb{R}$, from (2.44) one can easily derive:

$$\mathbf{\Omega}_{\pm} = \mathbf{1} - \int_{\Sigma} d\rho_{\Sigma}(\xi) \int_{\Sigma} d\rho_{\Sigma}(\eta) \tilde{\psi}^{(0)}(\lambda;\xi) \Omega^{-1}_{(x_0)}(\lambda;\xi,\eta) \times$$
$$\times \int_{S^{(m)}_{\pm}(\sigma^{(m-1)}_{(x)}, \sigma^{(m-1)}_{(x_0)})} Z^{(m)}_0[\varphi^{(0)}(\lambda;\eta), (\cdot)dx],$$

where we have put $\sigma^{(m-1)}_x := \sigma^{(m-1)}_{(\tau_0;x)}$, $\sigma^{(m-1)}_{x_0} := \sigma^{(m-1)}_{(\tau_0;x_0)} \in C_{m-1}(\mathbb{R}^m;\mathbb{C})$ and

$$Z^{(m)}_0[\varphi^{(0)}(\lambda;\eta), \psi^{(0)}dx] := Z^{(m)}[\varphi^{(0)}_\tau(\lambda;\eta), \psi^{(0)}_\tau dx]|_{d\tau=0}. \tag{2.46}$$

The closed subspaces $\mathcal{H}_0 \in \mathcal{H}_-$ and $\mathcal{H}^*_0 \in \mathcal{H}^*_-$ corresponding to (2) are given as

follows:

$$\mathcal{H}_0 := \{\psi^{(0)}(\lambda;\xi) \in \mathcal{H}_- : \mathrm{L}\psi^{(0)}(\lambda;\xi) = 0, \psi^{(0)}(\lambda;\xi)|_\Gamma = 0, \lambda \in \mathbb{C}, \xi \in \Sigma\}, \tag{2.47}$$
$$\mathcal{H}^*_{\tau,0} := \{\varphi^{(0)}(\lambda;\eta) \in \mathcal{H}^*_- : \mathrm{L}\varphi^{(0)}(\lambda;\eta) = 0, \varphi^{(0)}(\lambda;\eta)|_\Gamma = 0, \lambda \in \mathbb{C}, \eta \in \Sigma\}.$$

As a consequence, the following theorem is true.

Theorem 2.4. *Let Volterrian operator expressions* (2) *be properly defined for almost all $\lambda \in \mathbb{C}$, meromorphic functions with respect to the parameter $\lambda \in \mathbb{C}$. Then Delsarte-Lions transformed differenrial expresions* (2.39) *remain differential and polynomial with respect to the parameter $\lambda \in \mathbb{C}$ too.*

Thereby, making use of expressions (2) one can construct the Delsarte-Lions transformed linear differential pencil $\tilde{\mathrm{L}} \in \mathcal{L}(\mathcal{H})$, whose coefficients are related to those of the pencil $\mathrm{L} \in \mathcal{L}(\mathcal{H})$ via some Backlund type relationships useful for applications (see [26, 28, 43, 48]) in soliton theory.

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REFERENCES

[1] J. Delsarte, *Sur certaines transformations fonctionelles relative aux equations lineaires aux derives partielles du second ordre,* C.R. Acad. Sci. Paris, 1938, v. 206, p. 178–182.

[2] J. Delsarte, J. Lions, *Transmutations d'operateurs differentielles dans le domain complex,* Comment. Math. Helv., 1957, v. 52, p. 113–128.

[3] I.V. Skrypnik, *Periods of A-closed forms,* Proceedings of the USSR Academy of Sciences, 1965, v. 160, N4, p. 772–773 (in Russian).

[4] I.V. Skrypnik, *A harmonique fields with peculiarities,* Ukr. Math. Journal, 1965, v.17, N4, p. 130–133 (in Russian).

[5] I.V. Skrypnik, *The generalized De Rham theorem,* Proceed. of UkrSSR Acad. of Sci., 1965,N1, p. 18–19. (in Ukrainian).

[6] I.V. Skrypnik, *A harmonic forms on a compact Riemannian space,* Proceed. of UkrSSR Acad. of Sci., N2, p.174–175 (in Ukrainian).

[7] Y.B. Lopatynski, *On harmonic fields on Riemannian manifolds,* Ukr. Math. Journal, 1950, v.2, p. 56–60 (in Russian).

[8] S.S. Chern, *Complex manifolds,* Chicago University Publ., USA, 1956.

[9] L.D. Faddeev, *Quantum inverse scattering problem. II.* in Modern problems of mathematics, M: VINITI Publ., 1974, v. 3, p. 93–180 (in Russian).

[10] S.P. Novikov (Editor), *Theory of solitons.* Moscow, Nauka Publ., 1980 (in Russian).

[11] Levitan B.M. *Sturm-Liouvilee inverse problems.* Moscow Nauka, 1984 (in Russian).

[12] Yu.M. Berezansky *Eigenfunctions expansions related with selfadjoint operators,* Kiev, Nauk. Dumka Publ., 1965 (in Russian).

[13] F.A. Berezin, M.A. Shubin *Schrodinger equation,* Moscow, the Moscow University Publisher, 1983 (in Russian).

[14] A.L. Bukhgeim, *Volterra equations and inverse problems,* Moscow, Nauka, 1983 (in Russian).

[15] V.A. Marchenko, *Spectral theory of Sturm-Liouville operators,* Kiev, Nauk. Dumka Publ., 1972 (in Russian).

[16] L.D. Faddeev, L.A. Takhtadjyan, *Hamiltonian approach to soliton theory,* Moscow, Nauka, 1986 (in Russian).

[17] B.M. Levitan, I.S. Sargsian, *Sturm-Liouville and Dirac operators,* Moscow, Nauka Publ., 1988. (in Russian).

[18] V.B. Matveev, M.I. Salle, *Darboux-Backlund transformations and applications,* NY, Springer, 1993.

[19] P.I. Holod, A.U. Klimyk, *Theory of symmetry,* Moscow, Factorial, 2002 (in Russian).

[20] I.M. Krichever, *Algebro-geometric methods in theory of nonlinear equations,* Russian Mathem. Surveys, 1977, v. 32, N6, p. 183–208 (in Russian).

[21] L.P. Nizhnik, *Inverse scattering problems for hyperbolic equations.* Kiev, Nauk. Dumka Publ., 1991 (in Russian).

[22] L.P. Nizhnik, M.D. Pochynaiko, *The integration of a spatially two-dimensional Schrodinger equation by the inverse problem method,* Func. Anal. and Appl., 1982, v. 16, N1, p. 80–82 (in Russian).

[23] I.C. Gokhberg, M.G. Krein, *Theory of Volterra operators in Hilbert spaces and its applications,* Moscow, Nauka, 1967 (in Russian).

[24] Ya.V. Mykytiuk, *Factorization of Fredholmian operators. Mathematical Studii,* Proceedings of Lviv Mathematical Society, 2003, v. 20, N2, p. 185–199 (in Ukrainian).

[25] A.M. Samoilenko, Y.A. Prykarpatsky, V.G. Samoylenko, *The structure of Darboux-type binary transformations and their applications in soliton theory,* Ukr. Mat. Zhurnal, 2003, v. 55, N12, p. 1704–1723 (in Ukrainian).

[26] A.K. Prykarpatsky, A.M. Samoilenko, Y.A. Prykarpatsky, *The multi-dimensional Delsarte transmutation operators, their differential-geometric structure and applications.* Part.1. Opuscula Mathematica, 2003, v. 23, p. 71–80 / arXiv:math-ph/0403054 v1 29 Mar 2004/.

[27] J. Golenia, Y.A. Prykarpatsky, A.M. Samoilenko, A.K. Prykarpatsky, *The general differential-geometric structure of multidimensional Delsarte transmutation operators in parametric functional spaces and their applications in soliton theory. Part 2.* Opuscula Mathematica, 2004, N 24 /arXiv: math-ph/0403056 v 1 29 Mar 2004/.

[28] A.M. Samoilenko, Y.A. Prykarpatsky, *Algebraic-analytic aspects of completely integrable dynamical systems and their perturbations,* Kyiv, NAS, Inst. Mathem. Publisher, v. 41, 2002 (in Ukrainian).

[29] Y.A. Prykarpatsky, A.M. Samoilenko, A.K. Prykarpatsky, V.Hr. Samoylenko, *The Delsarte-Darboux type binary transformations and their differenetial-geometric and operator staructure.* arXiv: math-ph/0403055 v 1 29 Mar 2004.

[30] Y.A. Prykarpatsky, A.M. Samoilenko, A.K. Prykarpatsky, *De Rham-Hodge theory. A survey of the spectral and differential geometric aspects of the De Rham-Hodge theory related with Delsarte transmutation operators in multidimension and its applications to spectral and soliton problems. Part 1.* //lanl-arXiv:math-ph/0406062 v 1, 8 April 2004.

[31] J.C.C. Nimmo, *Darboux tarnsformations from reductions of the KP-hierarchy,* Preprint of the Dept. of Mathem. at the University of Glasgow, November 8, 2002, 11 p.

[32] A.K. Prykarpatsky, I.V. Mykytiuk, *Algebraic integrability of nonlinear dynamical systems on manifolds: classical and quantum aspects,* Kluwer Acad. Publishers, the Netherlands, 1998.

[33] C. Godbillon, *Geometrie differentielle et mechanique analytique,* Paris, Hermann, 1969.

[34] R. Teleman, *Elemente de topologie si varietati diferentiabile,* Bucuresti Publ., Romania, 1964

[35] G. De Rham, *Varietes differentielles,* Hermann, Paris, 1955.

[36] G. De Rham, *Sur la theorie des formes differentielles harmoniques,* Ann. Univ. Grenoble, 1946, 22, p. 135–152.

[37] F. Warner, *Foundations of differential manifolds and Lie groups,* Academic Press, NY, 1971.

[38] M.A. Shubin, *Pseudo-differential operators and spectral theory,* Moscow, Nauka Publ., 1978 (in Russian).

[39] N. Danford, J.T. Schwartz, *Linear operators, v.2,* InterSci. Publ., NY, 1963.

[40] B.N. Datta, D.R. Sarkissian *Feedback control in distributed parameter gyroscopic systems: a solution of the partial eigenvalue assignment problem,* Mechanical Systems and Signal Processing, 2002, v. 16, N. 1, p. 3–17.

[41] I.M. Gelfand, G.E. Shilov, *Generalized functions and actions upon them,* Second edition. Moscow, Nauka Publisher, 1959 (in Russian).

[42] V.E. Zakharov, A.B. Shabat, *A scheme of integration of nonlinear equations of mathematical physics via the inverse scattering problem,* Part 1. Func. Anal. and it Appl., 1974, v. 8, N3, p. 43–53; Part 2. 1979, v. 13, N3, p. 13–32 (in Russian).

[43] M.D. Pochynaiko, Yu. Sydorenko, *Integrating some (2+1)-dimensional integrable systems by methods of inverse scattering problem and binary Darboux transformations,* Matematychni Studii, 2003, N20, p. 119–132.

[44] B.G. Konopelchenko, *On the integrable equations and degenerate dispersiopn laws in multidimensional soaces,* J. Phys. A: Math. and Gen., 1983, v. 16, p. L311–L316.

[45] V.E. Zakharov, *Integrable systems in multidimensional spaces,* Lect. Notes in Phys., 1982, v. 153, p. 190–216.

[46] V.E. Zakharov, S.V. Manakov, *On a generalization of the inverse scattering problem,* Theoret. Mathem. Physics, 1976, v. 27, N3, p. 283–287.

[47] D. Levi , L. Pilloni, P.M. Santini, *Backlund transformations for nonlinear evolution equations in (2+1)-dimensions,* Phys. Lett, 1981, v. 81A, N8, p. 419–423.

[48] O. Hentosh, M. Prytula, A. Prykarpatsky, *Differential-geometric and Lie-algebraic foundations of studying integrable nonlinear dynamical systems on functional manifolds,* Lviv Naational University Publ., Lviv, Second edition, 2006 (in Ukrainian).

Y.A. Prykarpatsky
yarchyk@wms.mat.agh.edu.pl
yarchyk@gmail.com

AGH University of Science and Technology,
Faculty of Applied Mathematics,
al. Mickiewicza 30, 30-059 Cracow, Poland

The Institute of Mathematics at the NAS,
Kiev 01601, Ukraine

A.M. Samoilenko
sam@imath.kiev.ua

The Institute of Mathematics at the NAS,
Kiev 01601, Ukraine

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