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# ON QUASI-SIMILARITY AND $w$-HYPONORMAL OPERATORS

**Abstract.** In this paper, it is shown that a Putnam-Fuglede type commutativity theorem holds for $w$-hyponormal operators, the normal parts of quasi-similar $w$-hyponormal operators are unitarily equivalent and a $w$-hyponormal spectral operator is normal.

**Keywords:** $w$-hyponormal operators, quasi-similarity.

**Mathematics Subject Classification:** 47B20.

## 1. INTRODUCTION

Let $H$ be an infinite dimensional complex Hilbert space and let $B(H)$ denote the algebra of operators from $H$ to itself (= bounded linear transformations).

Given $A, B \in B(H)$, define $\delta_{A,B}$: $B(H) \to B(H)$ by

$$\delta_{A,B}(X) = AX - XB,$$

for some operator $X$. Hence, $X \in ker\delta_{A,B}(X \in ker\delta_{A^*,B^*})$ will denote $AX - XB = 0$ $(A^*X - XB^* = 0)$.

The classical Putnam-Fuglede Theorem [16, p. 104] says that if $A$ and $B^*$ are normal operators such that $AX = XB$ for some operator $X$, then also $A^*X = XB^*$ (if $A$ and $B^*$ are normal operators, then $ker\delta_{A,B} = ker\delta_{A^*,B^*}$).

Over the years, a number of authors have considered the problem of the extension of the classical Putnam-Fuglede Theorem to a class (or classes) of operators more general than the class of normal operators. Here the particular classes which have drawn a lot of attention are those consisting of either hyponormal or $M$-hyponormal or dominant or $k$-quasi-hyponormal operators. (See [11, 13, 15, 19, 23–26, 28] and some of the references there.)

Recently, Jeon, Tanahashi and Uchiyama ([19, 26]) have shown that Putnam-Fuglede Theorem holds true in the case of $p$-hyponormal and log-hyponormal operators.

In the first part of this note, we extend these results ([19, 26]) to the class of $w$-hyponormal operators.

For $p > 0$, recall that ([1, 2, 10, 12, 19]) an operator $A$ is said to be $p$-hyponormal if $(A^*A)^p \geq (AA^*)^p$. A $p$-hyponormal operator is called hyponormal if $p = 1$, semi-hyponormal if p $= \frac{1}{2}$. An invertible operator A is called log-hyponormal if $\log(A^*A) \geq \log(AA^*)$.

Let $A = U|A|$ be the polar decomposition of $A$; then following ([1, 2]), we define the first ***Aluthge transform*** of $A$ by $\widetilde{A} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}}$ and define the second **Aluthge transform** of $A$ by $\widetilde{\widetilde{A}} = |\widetilde{A}|^{\frac{1}{2}} \widetilde{U} |\widetilde{A}|^{\frac{1}{2}}$, where $\widetilde{A} = \widetilde{U}|\widetilde{A}|$ is the polar decomposition of $\widetilde{A}$. An operator A is said to be $w$-hyponormal if $|\widetilde{A}| \geq |A| \geq |\widetilde{A}^*|$.

It is well known that the class of $w$-hyponormal operators contains both the $p$- and log-hyponormal operators. But neither the class of $p$-hyponormal operators nor the class of log-hyponormal operators contains the other. Also, if an operator $A$ is $w$-hyponormal, then $\widetilde{A}$ is semi-hyponormal and $\widetilde{\widetilde{A}}$ is hyponormal.

If an operator $A$ is $p$-hyponormal, then $kerA \subset kerA^*$, and if $A$ is log-hyponormal, then $kerA = kerA^*$. However, if $A$ is $w$-hyponormal, then it is not known whether the kernel condition $kerA \subset kerA^*$ holds. It is well known that there exists an example of a $w$-hyponormal operator $A$ with the property that $kerA$ is not a subset of $kerA^*$. Nevertheless, there are several properties that $p$-hyponormal operators share with $w$-hyponormal operators $A$ or $w$-hyponormal operators $A$ with $kerA \subset kerA^*$ ([3, 5]).

Recall that an operator A $\in B(H)$ is said to be dominant if for each $\lambda \in \mathbf{C}$, there exists a positive number $M_\lambda$ such that

$$(A - \lambda)(A - \lambda)^* \leq M_\lambda (A - \lambda)^*(A - \lambda).$$

If the constants $M_\lambda$ are bounded by a positive operator M, then A is said to be M-hyponormal.

Clearly the following inclusions hold and are known to be proper.

$$Hyponormal \subseteq p - Hyponormal(0 < p < 1) \subseteq w - Hyponormal \subseteq Paranormal$$

and

$$Log - hyponormal \subseteq w - Hyponormal \subseteq Paranormal.$$

An operator $X \in B(H)$ is called a quasiaffinity if X is both injective and has a dense range. Two operators A and B are said to be quasi-similar if there exist quasiaffinities X and Y such that $X \in ker\delta_{A,B}$ and $Y \in ker\delta_{B,A}$.

The operator A is said to be pure if there exists no non-trivial reducing subspace N of H such that the restriction of A to N (A $|_N$) is normal and is said to have a normal direct summand if it is not pure.

Recall that every A $\in B(H)$ has a direct sum decomposition A $= A_1 \oplus A_2$, where $A_1$ and $A_2$ are normal and pure parts, respectively. Of course, in the sum decomposition, either $A_1$ or $A_2$ may be absent. Given $X \in B(H)$, let $\overline{ranX}$ and $(kerX)^\perp$ denote the closure of the range and the orthogonal complement of the kernel of $X$.

Jeon and Duggal [12] have shown that the normal parts of quasi-similar $p$-hyponormal operators are unitarily equivalent, a $p$-hyponormal operator compactly quasi-similar to an isometry is unitary and a $p$-hyponormal operator which is also a spectral operator turns out to be normal.

Jeon, Tanahashi and Uchiyama [19] proved that results similar to those in [12] hold true for the class of log-hyponormal operators.

In the second part of this paper, we carry on with these results to the more general case of $w$-hyponormal operators and show that similar results still hold true. The major tools used to show these results is the second Aluthge transform and the kernel condition $kerA \subset kerA^*$.

## 2. A PUTNAM-FUGLEDE TYPE THEOREM FOR $W$-HYPONORMAL OPERATORS

In this section, we start by proving a Putnam-Fuglede type commutativity theorem for $w$-hyponormal operators.

**Theorem 1.** *If $A$ and $B^*$ are $w$-hyponormal operators with $kerA \subset kerA^*$ and $kerB^* \subset kerB$ such that $X \in ker\delta_{A,B}$ for some operator $X$, then $X \in ker\delta_{A^*,B^*}$, $\overline{ranX}$ reduces $A$, $(kerX)^{\perp}$ reduces $B$ and $A\mid_{\overline{ranX}}$ and $B\mid_{(kerX)^{\perp}}$ are unitarily equivalent normal operators.*

To prove Theorem 1, we need the following lemma.

**Lemma 2.** ([3, Theorem 2.4]). *If $A$ is $w$-hyponormal, then $\widetilde{A}$ is semi-hyponormal and $\widetilde{\widetilde{A}}$ is hyponormal.*

*Proof of Theorem 1.* Decompose A and $B^*$ into normal and pure parts as $A = A_1 \oplus A_2$ and $B^* = B_1^* \oplus B_2^*$. Let $X = [X_{ij}]_{i,j=1}^2$ and

$$\widetilde{\widetilde{X_{22}}} = |\widetilde{A_2}|^{\frac{1}{2}}|A_2|^{\frac{1}{2}}X_{22}|B_2^*|^{\frac{1}{2}}|\widetilde{B_2^*}|^{\frac{1}{2}}.$$

Now since $X_{22} \in ker\delta_{A_2,B_2}$, it follows that $\widetilde{\widetilde{X_{22}}} \in ker\delta_{\widetilde{\widetilde{A_2}},(\widetilde{\widetilde{B_2^*}})^*}$, where $\widetilde{\widetilde{A_2}}$ and $\widetilde{\widetilde{B_2^*}}$ ( $= \widetilde{\widetilde{(B_2)^*}}$ ) are hyponormal operators by Lemma 2. Applying Putnam-Fuglede Theorem for hyponormal operators, $\widetilde{\widetilde{X_{22}}} \in ker\delta_{\widetilde{\widetilde{A_2^*}},\widetilde{\widetilde{B_2^*}}}$, $\overline{ran\widetilde{\widetilde{X_{22}}}}$ reduces $\widetilde{\widetilde{A_2}}$, $(ker\widetilde{\widetilde{X_{22}}})^{\perp}$ reduces $\widetilde{\widetilde{B_2}}$, and $\widetilde{\widetilde{A_2}}\mid_{\overline{ran\widetilde{\widetilde{X_{22}}}}}$ and $\widetilde{\widetilde{B_2^*}}\mid_{(ker\widetilde{\widetilde{X_{22}}})^{\perp}}$ are unitarily equivalent normal operators. In particular, $\widetilde{\widetilde{A_2}}$ and $\widetilde{\widetilde{B_2^*}}$ have normal direct summands. Now by an argument similar to that used in the proof of the converse of Lemma 1 of [10], it is seen that $A_2$ and $B_2^*$ have normal direct summands. Since $A_2$ and $B_2^*$ are pure, $\widetilde{\widetilde{X_{22}}} = 0$ and hence $X_{22} = 0$. Using a similar argument to the operator equations $X_{21} \in ker\delta_{A_2,B_1}$ and $X_{12} \in ker\delta_{A_1,B_2}$, we get that $X_{21} = 0 = X_{12}$. Applying Putnam-Fuglede Theorem to the operator equation $X_{11} \in ker\delta_{A_1,B_1}$, $X_{11} \in ker\delta_{A_1^*,B_1^*}$ and consequently $X \in ker\delta_{A^*,B^*}$ and the result follows. □

**Corollary 3.** *Let $X \in ker\delta_{A,B}$ for some operator $X$. If either $A$ is a pure $w$-hyponormal operator and $B^*$ is $w$-hyponormal or $A$ is $w$-hyponormal and $B^*$ is a pure $w$-hyponormal operator with $kerA \subset kerA^*$ and $kerB^* \subset kerB$, then $X = 0$.*

Theorem 1 holds in particular for $w$-hyponormal $A(B^*)$ and hyponormal $B^*(A)$. A generalisation to a dominant operator $A$ is given in the following Corollary.

**Corollary 4.** *If $X \in ker\delta_{A,B}$ for some dominant operator $A$, a $w$-hyponormal operator $B^*$ with $kerB^* \subset kerB$ and $X \in B(H)$, then $X \in ker\delta_{A^*,B^*}$, $\overline{ranX}$ reduces $A$, $(kerX)^{\perp}$ reduces $B$ and $A \mid_{\overline{ranX}}$ and $B \mid_{(kerX)^{\perp}}$ are unitarily equivalent normal operators.*

*Proof.* Decompose $A$ and $B^*$ into normal and pure parts and Let $X = [X_{ij}]_{i,j=1}^{2}$. Apply Theorem 1 of [13] to $X_{21} \in ker\delta_{A_2,B_1}$ and $A_2X_{22}|B_2^*|^{\frac{1}{2}}|\widetilde{B_2^*}|^{\frac{1}{2}} = X_{22}|B_2^*|^{\frac{1}{2}}|\widetilde{B_2^*}|^{\frac{1}{2}}(\widetilde{\widetilde{B_2^*}})^*$. Then apply Theorem 1 to $X_{12} \in ker\delta_{A_1,B_2}$ and the proof follows. □

## 3. THE NORMAL PARTS OF QUASI-SIMILAR $W$-HYPONORMAL OPERATORS

Douglas ( [9]) proved that quasi-similar normal operators are unitarily equivalent. This result was extended by Conway ( [8]), who proved that the normal parts of quasi-similar subnormal operators are unitarily equivalent. In the same paper, Conway gave an example which shows that the pure parts of quasi-similar subnormal operators are not necessarily quasi-similar. While working on the class of hyponormal operators, Clary [7] proved that quasi-similar hyponormal operators are unitarily equivalent. This result was further extended by Williams [29] to a more general class of dominant operators. Recently Jeon and Duggal ( [12]) and Jeon and others ( [19]) extended the result of Conway ( [8]) to the class of $p$-hyponormal and log-hyponormal operators, respectively. Let us recall that the classes of $p$ or log-hyponormal operators and dominant operators are independent of each other.

In this section, we extend the results of ( [12]) and ( [19]) to a more general class of $w$-hyponormal operators.

**Theorem 5.** *Let $A$ and $B$ be quasi-similar $w$-hyponormal operators with $kerA \subset kerA^*$ and $kerB \subset kerB^*$, respectively. Let $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, where $A_1$, $B_1$ and $A_2$, $B_2$ are the normal and pure parts, respectively. Then $A_1$ and $B_1$ are unitarily equivalent. Furthermore, there exist bounded operators $X^*$ and $Y^*$ with dense ranges such that $X^* \in ker\ \delta_{A_2,B_2}$ and $Y^* \in ker\ \delta_{B_2,A_2}$.*

To prove the theorem, we need the following results which we will state as Lemmas.

**Lemma 6.** ([3, Theorem 2.6]). *Let $A$ be $w$-hyponormal with $kerA \subset kerA^*$. If $\widetilde{A}$ is normal, then $\widetilde{A} = A$.*

The following lemma is well known and a proof is included for completeness.

**Lemma 7.** *Let $A$ be $w$-hyponormal with $kerA \subset kerA^*$ and $B$ be a normal operator. If there exists an operator $X \in B(H)$ with a dense range such that $X \in ker\delta_{A,B}$, then $A$ is normal.*

*Proof.* Decompose $A = A_1 \oplus A_2$ into its normal and pure parts, respectively. Let $A_2 = U_2|A_2|$, $\widetilde{A_2} = |A_2|^{\frac{1}{2}}$ U $|A_2|^{\frac{1}{2}}$ and $\widetilde{\widetilde{A_2}} = |\widetilde{A_2}|^{\frac{1}{2}}\widetilde{U}|\widetilde{A_2}|^{\frac{1}{2}}$.
$A_2$ being pure, it is injective and $|A_2|^{\frac{1}{2}}$ is a quasiaffinity. Also since $A_1$ is normal, $\widetilde{\widetilde{A}} = \widetilde{\widetilde{A_1}} \oplus \widetilde{\widetilde{A_2}} = A_1 \oplus \widetilde{\widetilde{A_2}}$.
Now if we let T $= |\widetilde{A_2}|^{\frac{1}{2}}|A_2|^{\frac{1}{2}}$, then by a simple computation, $\widetilde{\widetilde{A_2}}T = TA_2$ and $T$ is a quasiaffinity.
Also if we let $Z = I_H \oplus T$, then clearly $Z$ is also a quasiaffinity such that $\widetilde{\widetilde{A}}Z = ZA$, where $\widetilde{\widetilde{A}}$ is a hyponormal operator.
Thus $\widetilde{\widetilde{A}}ZX = ZAX = ZXB$ and by ( [24]), $\widetilde{\widetilde{A}}$ is normal. Hence by Lemma 6, we get the result. □

With these results, we are now ready to prove the Theorem.

*Proof of Theorem 5.* By the given hypotheses, there exist quasiaffinities $X$ and $Y$ say such that $X \in$ ker $\delta_{A,B}$ and $Y \in$ ker $\delta_{B,A}$. Let $X = [X_{ij}]_{i,j=1}^2$ and $Y = [Y_{ij}]_{i,j=1}^2$ with respect to decompositions of $A$ and $B$ respectively. Then by a simple matrix calculation, we obtain $X_{21} \in ker\delta_{A_2,B_1}$ and $Y_{21} \in ker\delta_{B_2,A_1}$. Next we show that $X_{21} = 0 = Y_{21}$. Let $M = \overline{ran(X_{21})}$, then the subspace $M$ is invariant under $A_2$ and by Theorem 1, $X_{21} \in ker\delta_{A_2^*,B_1^*}$ and $M$ reduces $A_2$. Let $A_2^1 = A_2 |_M$, then by Lemma 5 of [27], $A_2^1$ is $w$-hyponormal. Now define $X_{21}^1 : H \to M$ by $X_{21}^1 x = X_{21}x$ for each $x \in H$. Then $X_{21}^1$ has a dense range and satisfies the equation $X_{21}^1 \in ker\delta_{A_2^1,B_1}$. By Lemma 7, $A_2^1$ is normal, which contradicts the fact that $A_2$ is pure and hence $X_{21} = 0$. Similarly $Y_{21} = 0$. Thus $X_{11}$ and $Y_{11}$ are injective and since $X_{11} \in ker\delta_{A_1,B_1}$ and $Y_{11} \in ker\delta_{B_1,A_1}$, by Lemma 1.1 of [29], $A_1$ and $B_1$ are unitarily equivalent. Also notice that $X_{22}$ and $Y_{22}$ have dense ranges and that $X_{22} \in ker\delta_{A_2,B_2}$ and $Y_{22} \in ker\delta_{B_2,A_2}$. Hence the proof is complete. □

From the Theorem, we immediately obtain the following corollaries.

**Corollary 8.** *If $A$ and $B$ are quasi-similar $w$-hyponormal operators with $kerA \subset kerA^*$, then $\sigma_e(A) = \sigma_e(B)$.*

*Proof.* Decompose $A$ and $B$ into its normal and pure parts, respectively. Now by the Theorem and [31, Corollary 2.9], the proof follows. □

**Remark**. As an application, we are able to answer Clary [7] and Conway's [8] famous question affirmatively.

**Corollary 9.** *If a $w$-hyponormal operator $A$ with $kerA \subset kerA^*$ is quasi-similar to a normal operator $B$, then $A$ and $B$ are unitarily equivalent normal operators.*

**Corollary 10.** *If $A$ and $B$ are quasi-similar $w$-hyponormal operators with $kerA \subset kerA^*$, then $B$ is pure whenever $A$ is pure.*

In Theorem 1, if X is a quasiaffinity, then $\overline{ranX} = H$ and $(kerX)^\perp = H$. Accordingly, the following corollary is true.

**Corollary 11.** *If $A$ and $B^*$ are $w$-hyponormal operators with $kerA \subset kerA^*$ and $kerB^* \subset kerB$ such that $X \in ker\delta_{A,B}$ for some quasiaffinity operator $X$, then $A$ and $B$ are unitarily equivalent normal operators.*

**Remark**. First, observe that from Corollary 11, we immediately recapture Corollary 9.

Also in Theorem 1, if $X$ is a quasiaffinity, then $X \in ker\delta_{A,B}$ implies $X \in ker\delta_{A^*,B^*}$, where $A$ is $w$-hyponormal operator with $kerA \subset kerA^*$ and $B$ is a normal operator.

However, in the sequel, we wish to give an alternative proof (as a generalisation of Corollary 9) of this result.

**Theorem 12.** *Let $A$ be $w$-hyponormal with $kerA \subset kerA^*$ and $B$ be a normal operator. If there exists a quasiaffinity $X \in B(H)$ such that $X \in ker\delta_{A,B}$, then $X \in ker\delta_{A^*,B^*}$.*

We will need the following results.

**Lemma 13.** ([20]). *If $B$ is a normal operator on $H$, then*

$$\bigcap(B-\lambda)H = \{0\} \quad for \quad \lambda \in \mathbf{C}.$$

**Lemma 14.** ([21]). *Let $A,\ B \in B(H)$ be such that $0 \leq B \leq T(A-\lambda)(A-\lambda)^*$ for each $\lambda \in \mathbf{C}$, where $T$ is a positive real number. Then for every $x \in B^{\frac{1}{2}}H$, there exists a bounded function $f : \mathbf{C} \longrightarrow H$ such that $(A-\lambda)f(\lambda) \equiv x$.*

*Proof of Theorem 12.* Decompose $A = A_1 \oplus A_2$ into normal and pure parts, respectively. By Lemma 2, $\widetilde{\widetilde{A}}$ is hyponormal and using Lemma 7, there exists a quasiaffinity $W$ such that $\widetilde{\widetilde{A}}W = WA$. Hence

$$\widetilde{\widetilde{A}}WX = WAX = WXB$$

and $\widetilde{\widetilde{A}}Z = ZB$, where $Z = WX$ and $Z$ is a quasiaffinity. Thus $Z^*\widetilde{\widetilde{A}}^* = B^*Z^*$, where $Z^*$ is also a quasiaffinity.
Now let

$$x \in (\widetilde{\widetilde{A}}^*\widetilde{\widetilde{A}} - \widetilde{\widetilde{A}}\widetilde{\widetilde{A}}^*)^{\frac{1}{2}}H,$$

then by Lemma 14, there exists a bounded function $f : \mathbf{C} \longrightarrow H$ such that $(\widetilde{\widetilde{A}}^* - \lambda)f(\lambda) \equiv x$, for each $\lambda \in \mathbf{C}$.
Hence

$$Z^*x = Z^*(\widetilde{\widetilde{A}}^* - \lambda)f(\lambda) = (B^*-\lambda)f(\lambda)Z^* \in \mathrm{ran}(B^*-\lambda),$$

for each $\lambda \in \mathbf{C}$. By Lemma 13, $Z^*x = 0$ and $x = 0$. Hence by Lemma 6 and Putnam-Fuglede Theorem, the result follows. □

## 4. $W$-HYPONORMAL OPERATORS QUASI-SIMILAR TO SPECTRAL OPERATORS

A spectral operator (in the sense of Dunford) is an operator with a countable additive resolution of the identity defined on the Borel sets of the complex plane. If A is spectral, then it has the canonical decomposition A = R + S, where R and S are its scalar and radical parts, respectively. For details of spectral operators, see [14, 15, 17, 18, 22].

In [22] (or [15]), it is shown that an $M$-hyponormal spectral operator is normal. However, Example 2 of [24] shows that a dominant operator may be quasinilpotent and hence spectral, without necessarily being normal.

Recently, Jean and Duggal ([12]) have shown that a $p$-hyponormal spectral operator is normal.

In this section, we try to extend this result to a more general case of $w$-hyponormal operators.

The following result is Lemma 2 of [15] (see also [22]).

**Lemma 15.** *Let $A$ be an $M$-hyponormal operator and suppose there exists an operator $X$ with a dense range and a spectral operator $B$ such that $X \in ker\delta_{A,B}$. Then there exist a positive operator $P$, a normal operator $N$ and a quasinilpotent operator $Q$ such that $(A - N)P = PQ$ and $AN = NA$.*

From the Lemma, the following Corollary ([15, Corollary 4]) is immediate.

**Corollary Q**. *If $X \in ker\delta_{A,B}$, where $B$ is spectral, $A$ is hyponormal and $X$ has a dense range, then $A$ is normal, $B$ is a scalar operator and is similar to $A$.*

The following result is an extension of Corollary Q above.

**Theorem 16.** *Let $A$ be $w$-hyponormal with $kerA \subset kerA^*$ and $B$ be a spectral operator. If there exists an operator $X \in B(H)$ with a dense range such that $X \in ker\delta_{A,B}$, then $A$ is normal, $B$ is a scalar operator and is similar to $A$.*

*Proof.* Decompose $A = A_1 \oplus A_2$ into normal and pure parts, respectively. Then, as in the proof of Lemma 7, there exists a quasiaffinity $Z$ such that $\tilde{\tilde{A}}Z = ZA$, where $\tilde{\tilde{A}}$ is a hyponormal operator.

Thus $\tilde{\tilde{A}}ZX = ZAX = ZXB$ and, by Corollary Q $\tilde{\tilde{A}}$ is normal. Hence by Lemma 6, $A$ is normal, $B$ is a scalar and is similar to $A$. □

**Corollary 17.** *Let $A$ be $w$-hyponormal with $kerA \subset kerA^*$ and $B$ be a spectral operator. If there exists a quasiaffinity $X \in B(H)$ such that $X \in ker\delta_{A,B}$, then $A$ is normal, $B$ is a scalar operator and is similar to $A$.*

**Corollary 18.** ([12, Theorem 11], [19, Theorem 12]). *Let $A$ be log or $p$-hyponormal and $B$ be a spectral operator. If there exists a quasiaffinity $X \in B(H)$ such that $X \in ker\delta_{A,B}$, then $A$ is normal, $B$ is a scalar operator and is similar to $A$.*

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