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ON LIPSCHITZIAN OPERATORS OF SUBSTITUTION GENERATED BY SET-VALUED FUNCTIONS

Abstract. We consider the Nemytskii operator, i.e., the operator of substitution, defined by $(N\phi)(x) := G(x, \phi(x))$, where G is a given multifunction. It is shown that if N maps a Hölder space H_α into H_β and N fulfils the Lipschitz condition then

$$G(x, y) = A(x, y) + B(x), \tag{1}$$

where $A(x, \cdot)$ is linear and $A(\cdot, y), B \in H_\beta$. Moreover, some conditions are given under which the Nemytskii operator generated by (1) maps H_α into H_β and is Lipschitzian.

Keywords: Nemytskii operator, Hölder functions, set-valued functions, Jensen equation.

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In 1982 J. Matkowski showed (cf. [3]) that a composition operator mapping the function space $\mathrm{Lip}(I, \mathbb{R})$ $(I = [0, 1])$ into itself is Lipschitzian with respect to the Lipschitzian norm if and only if its generator has the form

$$g(x, y) = a(x)y + b(x), \quad x \in I, y \in \mathbb{R}, \tag{2}$$

for some $a, b \in \mathrm{Lip}(I, \mathbb{R})$. This result was extended to a lot of spaces by J. Matkowski and others (cf. [4]). Let $\mathrm{Lip}^r(I, \mathbb{R}), r \in (0, 1]$, denote the space of all functions $\phi\colon I \to \mathbb{R}$ which satisfy the Hölder condition with the constant r. Suppose that $N\colon \mathrm{Lip}^r(I, \mathbb{R}) \to \mathrm{Lip}^s(I, \mathbb{R})$ $(s \in (0, 1])$. A. Matkowska showed (cf. [2]) that, in the case of $s \leq r$, the operator N is Lipschitzian if and only if its generator g has form (2) for some $a, b \in \mathrm{Lip}^r(I, \mathbb{R})$. In the case of $r < s$, the operator N is a Lipschitz map if and only if there is $b \in \mathrm{Lip}^s(I, \mathbb{R})$ such that

$$g(x, y) = b(x), \quad x \in I, y \in \mathbb{R}.$$

Set-valued versions of Matkowski's results were investigated in papers [9, 10] and others. The main goal of this paper is to examine a Nemytskii operator acting from one Hölder space into another and generated by a set-valued function.

1.

If Z is a real normed space then by $cc(Z)$ we denote the space of all non-empty, compact and convex subsets of Z. Let d denote the Hausdorff metric on the set $cc(Z)$. Moreover, by $n(Z), b(Z)$ we denote the family of non-empty and non-empty, bounded subsets of Z, respectively. If $A \in b(Z)$, then let us define $\|A\|$ as follows: $\|A\| := \sup\{\|z\| : z \in A\}$.

Now assume that Y, Z are vector spaces and C is a convex cone in Y (a subset C of a real vector space is said to be a convex cone if $C + C \subseteq C$, $\lambda C \subseteq C$ for $\lambda \geq 0$). A set-valued function $F\colon C \to n(Z)$ is said to be *superadditive* if the condition $F(y_1) + F(y_2) \subseteq F(y_1 + y_2)$ holds for $y_1, y_2 \in C$. A set-valued function $F\colon C \to n(Z)$ is said to be $\mathbb{Q}_+$-homogenous if the equality $F(\lambda y) = \lambda F(y)$ holds for $\lambda \in \mathbb{Q}_+, y \in C$. Now, let Y, Z be real normed spaces and let C be a convex cone in Y. A set-valued function $F\colon C \to n(Z)$ is called *lower semicontinuous* at $y_0 \in C$ if for every open set V in Z such that $F(y_0) \cap V \neq \emptyset$ there exist a neighbourhood U of zero in Y such that $F(y) \cap V \neq \emptyset$ for $y \in (y_0 + U) \cap C$. A set-valued function $F\colon C \to n(Z)$ is called lower semicontinuous if it is lower semicontinuous at every point of C.

Lemma 1. [6, Lemma 2]. *Let Z be a real normed space. If A, B and C are non-empty, compact and convex subsets of Z, then $d(A + B, A + C) = d(B, C)$.*

The next lemma is an easy consequence of Lemma 1.

Lemma 2. *Let Z be a real normed space. If A, B, C, D are non-empty, compact and convex subsets of Z, then $d(A + C, B + D) \leq d(A, B) + d(C, D)$.*

Lemma 3. [5, Theorem 5.6, p. 64]. *Let Y be a vector space and let Z be a Hausdorff topological vector space. Moreover, let C be a convex cone in Y. A set-valued function F defined on C, with non-empty and compact values in Z, satisfies the Jensen equation*

$$F\big(\frac{1}{2}(y_1 + y_2)\big) = \frac{1}{2}(F(y_1) + F(y_2)), \quad y_1, y_2 \in C$$

if and only if there exist an additive set-valued function A, defined on C with non-empty, compact and convex values in Z and a non-empty, compact and convex subset B of Z such that $F(y) = A(y) + B$, $y \in C$.

Lemma 4. [8, Lemma 4]. *Let Y and Z be real normed spaces and let C be a convex cone in Y. Suppose that $(F_j : j \in J)$ is a family of superadditive, lower semicontinuous and Q_+-homogeneous set-valued functions $F_j\colon C \to n(Z)$. If C is of the second category in C (C is endowed with the metric induced from Y)and $\bigcup_{j \in J} F_j(y) \in b(Z)$ for $y \in C$, then there exists a constant $M, 0 < M < +\infty$, such that*

$$\sup_{j \in J} \|F_j(y)\| \leq M\|y\|, \quad y \in C.$$

Remark. If Y is an infinite-dimensional linear topological space which is a countable union of finite-dimensional subspaces, then Y is of the first category (cf. [7, p. 52]).

An $\alpha\colon [0,1] \to [0,1]$ is said to be a Hölder function [1, p.182], if $\alpha(t) > 0$ for $t \in (0,1]$, $\alpha(0) = 0 = \lim_{t\to 0} \alpha(t), \alpha(1) = 1$, and moreover, α and α^*, where

$$\alpha^*(t) := \begin{cases} t/\alpha(t) & \text{for } t \in (0,1], \\ 0 & \text{for } t = 0 \end{cases}$$

are increasing.

For two Hölder functions α and β, we write

$$\alpha \preceq \beta \quad \text{if} \quad \alpha(t) = O(\beta(t)) \quad \text{as} \quad t \to 0.$$

Let α be a Hölder function and $(\mathcal{M}, d_{\mathcal{M}})$ be a metric space. We define the Hölder space $H_\alpha(I, \mathcal{M})$, where $I = [0,1]$, as a set of all continuous functions $\phi\colon I \to \mathcal{M}$ for which

$$h_\alpha(\phi) := \sup_{s\in(0,1]} \frac{\omega(\phi, s)}{\alpha(s)} < +\infty,$$

where

$$\omega(\phi, s) := \sup\{d_{\mathcal{M}}(\phi(x_1), \phi(x_2)) : x_1, x_2 \in I, |x_1 - x_2| \le s\}. \tag{3}$$

For a non-empty subset $\mathcal{C} \subseteq \mathcal{M}$, by $H_\alpha(I, \mathcal{C})$ we denote the set of all functions $\phi \in H_\alpha(I, \mathcal{M})$ such that $\phi(I) \subseteq \mathcal{C}$.

If a set $\mathcal{M}$ is endowed with the structure of a real normed space, then $H_\alpha(I, \mathcal{M})$ is also endowed with that structure; the linear operations are defined in the usual way and the norm is given by the formula

$$\|\phi\|_\alpha := \|\phi(0)\| + h_\alpha(\phi).$$

Let now Z be a real normed space and let d be the Hausdorff metric on the set $cc(Z)$. On the space $H_\alpha(I, cc(Z))$, the metric may be defined by

$$d_\alpha(F, \overline{F}) := d(F(0), \overline{F}(0)) + \sup_{s\in(0,1]} \frac{\omega(F, \overline{F}, s)}{\alpha(s)}, \quad F, \overline{F} \in H_\alpha(I, cc(Z)),$$

where

$$\omega(F, \overline{F}, s) := \sup\{d(F(x_1) + \overline{F}(x_2), F(x_2) + \overline{F}(x_1)) : x_1, x_2 \in I, |x_1 - x_2| \le s\}.$$

First we shall verify that $d_\alpha(F, \overline{F})$ is finite for $F, \overline{F} \in H_\alpha(I, cc(Z))$ (it is obvious that $d_\alpha(F, \overline{F})$ is nonnegative). Let us take $s \in (0,1]$ and $x_1, x_2 \in I$ such that $|x_1 - x_2| \le s$. By Lemma 1, there is

$$\begin{aligned} &d(F(x_1) + \overline{F}(x_2), F(x_2) + \overline{F}(x_1)) \le \\ &\le d(F(x_1) + \overline{F}(x_2), F(x_2) + \overline{F}(x_2)) + d(F(x_2) + \overline{F}(x_2), F(x_2) + \overline{F}(x_1)) = \\ &= d(F(x_1), F(x_2)) + d(\overline{F}(x_1), \overline{F}(x_2)) \le \omega(F, s) + \omega(\overline{F}, s); \end{aligned}$$

here $\omega(F,s)$ is given by formula (3), where $d_{\mathcal{M}}$ is replaced by the Hausdorff metric d on $cc(Z)$. Therefore

$$\omega(F,\overline{F},s) \le \omega(F,s) + \omega(\overline{F},s).$$

Hence

$$\omega(F,\overline{F},s)/\alpha(s) \le \omega(F,s)/\alpha(s) + \omega(\overline{F},s)/\alpha(s) \le h_\alpha(F) + h_\alpha(\overline{F}).$$

It implies that $d_\alpha(F,\overline{F})$ is finite. The triangle inequality may be obtained in the following way. Let us take $s \in (0,1]$ and $x_1, x_2 \in I$, such that $|x_1 - x_2| \le s$. Then

$$\begin{aligned}
d(F(x_1)+\overline{F}(x_2), F(x_2)+\overline{F}(x_1)) &= d(F(x_1)+\overline{F}(x_2)+\overline{\overline{F}}(x_2), F(x_2)+\overline{F}(x_1)+\overline{\overline{F}}(x_2)) \le \\
&\le d(F(x_1)+\overline{F}(x_2)+\overline{\overline{F}}(x_2), F(x_2)+\overline{F}(x_2)+\overline{\overline{F}}(x_1)) + \\
&\quad + d(F(x_2)+\overline{F}(x_2)+\overline{\overline{F}}(x_1), F(x_2)+\overline{F}(x_1)+\overline{\overline{F}}(x_2)) = \\
&= d(F(x_1)+\overline{\overline{F}}(x_2), F(x_2)+\overline{\overline{F}}(x_1)) + d(\overline{F}(x_2)+\overline{\overline{F}}(x_1), \overline{F}(x_1)+\overline{\overline{F}}(x_2)) \le \\
&\le \omega(F,\overline{\overline{F}},s) + \omega(\overline{\overline{F}},\overline{F},s).
\end{aligned}$$

Hence

$$\omega(F,\overline{F},s) \le \omega(F,\overline{\overline{F}},s) + \omega(\overline{\overline{F}},\overline{F},s).$$

Therefore,

$$\begin{aligned}
d_\alpha(F,\overline{F}) &= d(F(0),\overline{F}(0)) + \sup_{s\in(0,1]} \frac{\omega(F,\overline{F},s)}{\alpha(s)} \le \\
&\le d(F(0),\overline{\overline{F}}(0)) + \sup_{s\in(0,1]} \frac{\omega(F,\overline{\overline{F}},s)}{\alpha(s)} + d(\overline{\overline{F}}(0),\overline{F}(0)) + \sup_{s\in(0,1]} \frac{\omega(\overline{\overline{F}},\overline{F},s)}{\alpha(s)} = \\
&= d_\alpha(F,\overline{\overline{F}}) + d_\alpha(\overline{\overline{F}},\overline{F}),
\end{aligned}$$

which means that d_α satisfies the triangle inequality.

If E, E' are arbitrary non-empty sets, by $\mathcal{F}(E,E')$ we denote the set of all functions $f\colon E \to E'$. Every function $g\colon I \times E \to E'$ generates the so-called *Nemytskii operator* $N\colon \mathcal{F}(I,E) \to \mathcal{F}(I,E')$, defined by the formula

$$(N\phi)(x) := g(x,\phi(x)), \quad \phi \in \mathcal{F}(I,E), \quad x \in I.$$

Let Y, Z be real normed spaces, and let C be a convex cone in Y, of the second category in C. Consider the set

$$\mathcal{L}(C, cc(Z)) := \{A\colon C \to cc(Z) : A \text{ is additive and continuous}\}.$$

The formula

$$d_{\mathcal{L}}(A,B) := \sup_{y\in C\setminus\{0\}} \frac{d(Ay,By)}{\|y\|} \tag{4}$$

yields a metric in $\mathcal{L}(C, cc(Z))$ (cf. [9] and [10]).

2.

Theorem 1. Let Y, Z be real normed spaces, C be a convex cone in Y and let α and β be Hölder functions.

a) Assume that the Nemytskii operator N generated by $G\colon I \times C \to cc(Z)$ satisfies the following conditions:

1) $N\colon H_\alpha(I, C) \to H_\beta(I, cc(Z))$,

2) there exists $L \geq 0$ such that

$$d_\beta(N\phi, N\overline{\phi}) \leq L\|\phi - \overline{\phi}\|_\alpha, \quad \phi, \overline{\phi} \in H_\alpha(I, C). \tag{5}$$

Then there exist functions $A\colon I \times C \to cc(Z), B\colon I \to cc(Z)$ such that $B, A(\cdot, y)$ belongs to the space $H_\beta(I, cc(Z))$ for every $y \in C$, the function $A(x, \cdot)$ belongs to the space $\mathcal{L}(C, cc(Z))$ for every $x \in I$ and

$$G(x, y) = A(x, y) + B(x), \quad x \in I, y \in C.$$

Moreover, if C is of the second category in C, then the function $I \ni x \mapsto A(x, \cdot) \in \mathcal{L}(C, cc(Z))$ satisfies the Hölder condition

$$d_{\mathcal{L}}(A(x_1, \cdot), A(x_2, \cdot)) \leq L\beta(|x_1 - x_2|), \quad x_1, x_2 \in I,$$

where $d_{\mathcal{L}}$ is given by (4).

b) Assume that the condition $\alpha \preceq \beta$ does not hold. Then the operator N satisfies conditions 1) and 2) if and only if the function G is of the form

$$G(x, y) = B(x), \quad x \in I, y \in C,$$

where B belongs to the space $H_\beta(I, cc(Z))$. In this case the operator N is a constant function.

Proof. *a*) First we shall prove that the inequality

$$d(G(x, y), G(x, \overline{y})) \leq L\|y - \overline{y}\|, \quad x \in I, y, \overline{y} \in C \tag{6}$$

holds. Let us fix $x \in I, y, \overline{y} \in C$. Now define $\phi, \overline{\phi}\colon I \to C$ as follows: $\phi(t) = y, \overline{\phi}(t) = \overline{y}, t \in I$. It is obvious that $\phi, \overline{\phi} \in H_\alpha(I, C)$. From the definition of the metric d_β, we get

$$d(N\phi(0), N\overline{\phi}(0)) + \omega(N\phi, N\overline{\phi}, 1)/\beta(1) \leq d_\beta(N\phi, N\overline{\phi}).$$

Hence

$$d(G(0, y), G(0, \overline{y})) + d(G(x, y) + G(0, \overline{y}), G(x, \overline{y}) + G(0, y)) \leq d_\beta(N\phi, N\overline{\phi}). \tag{7}$$

Moreover,

$$\begin{aligned} d(G(x, y), G(x, \overline{y})) &= d(G(x, y) + G(0, \overline{y}), G(x, \overline{y}) + G(0, \overline{y})) \leq \\ \leq d(G(x, y) + G(0, \overline{y}), G(x, \overline{y}) + G(0, y)) &+ d(G(x, \overline{y}) + G(0, y), G(x, \overline{y}) + G(0, \overline{y})) = \\ &= d(G(0, y), G(0, \overline{y})) + d(G(x, y) + G(0, \overline{y}), G(0, y) + G(x, \overline{y})); \end{aligned}$$

according to (5) and (7), we hence get

$$d(G(x,y),G(x,\overline{y})) \le d_\beta(N\phi, N\overline{\phi}) \le L\|\phi-\overline{\phi}\|_\alpha = L\|y-y\|,$$

which completes the proof of inequality (6). Now, let us take $x_1, x_2 \in I$ such that $0 \le x_1 < x_2 \le 1$ and let $y_1, y_2 \in C$. Consider the function $\phi\colon I \to Y$ defined by

$$\phi(t) = \begin{cases} y_1 & \text{for } t \in [0,x_1],\\ y_1 + \frac{t-x_1}{x_2-x_1}(y_2-y_1) & \text{for } t \in [x_1,x_2],\\ y_2 & \text{for } t \in [x_2,1]. \end{cases} \tag{8}$$

It is obvious that $\phi(I) \subseteq C$. Moreover, ϕ is continuous. We shall prove that $\phi \in H_\alpha(I,C)$. It is easily seen that the following equalities hold:

$$\omega(\phi,s) = \|y_2-y_1\| \quad \text{for} \quad s \ge x_2-x_1,$$

$$\omega(\phi,s) = \frac{s}{x_2-x_1}\|y_2-y_1\| \quad \text{for} \quad s \ge 0, s \le x_2-x_1$$

($\omega(\phi,s)$ is given by formula (3), where the metric $d_{\mathcal{M}}$ is induced by the norm $\|\cdot\|$ in Y). Since α is increasing, there is

$$\sup_{s\in(0,1]} \frac{\omega(\phi,s)}{\alpha(s)} = \frac{\|y_2-y_1\|}{\alpha(x_2-x_1)} < +\infty.$$

Hence $\phi \in H_\alpha(I,C)$ and $\|\phi\|_\alpha = \|y_1\| + \|y_2-y_1\|/\alpha(x_2-x_1)$. Let $\overline{y}_1, \overline{y}_2 \in C$ and let us define a function $\overline{\phi}\colon I \to Y$ by putting $\overline{y}_1, \overline{y}_2$ instead of y_1, y_2, respectively, in definition (8). Obviously, $\overline{\phi} \in H_\alpha(I,C)$. Let us note that

$$(\phi-\overline{\phi})(t) = \begin{cases} y_1-\overline{y}_1 & \text{for } t \in [0,x_1],\\ y_1-\overline{y}_1 + \frac{t-x_1}{x_2-x_1}[(y_2-\overline{y}_2)-(y_1-\overline{y}_1)] & \text{for } t \in [x_1,x_2],\\ y_2-\overline{y}_2 & \text{for } t \in [x_2,1]. \end{cases} \tag{9}$$

It implies that $\phi-\overline{\phi} \in H_\alpha(I,Y)$ and

$$\|\phi-\overline{\phi}\|_\alpha = \|y_1-\overline{y}_1\| + \|y_2-\overline{y}_2-(y_1-\overline{y}_1)\|/\alpha(x_2-x_1). \tag{10}$$

Now let $u, v \in C$. Putting $y_1 = \overline{y}_2 = \frac{1}{2}(u+v) \in C, \overline{y}_1 = u \in C, y_2 = v \in C$ into definitions of the functions ϕ and $\overline{\phi}$, we get

$$\|\phi-\overline{\phi}\|_\alpha = 2^{-1}\|v-u\|.$$

Let $r = x_2 - x_1$; there follows that

$$\omega(N\phi, N\overline{\phi}, r)/\beta(r) \le \sup_{s\in(0,1]} \frac{\omega(N\phi, N\overline{\phi}, s)}{\beta(s)} \le d_\beta(N\phi, N\overline{\phi}).$$

Therefore, from (5) we get

$$\omega(N\phi, N\overline{\phi}, r) \leq 2^{-1}L\|v-u\|\beta(r).$$

Hence

$$d(N\phi(x_1) + N\overline{\phi}(x_2), N\phi(x_2) + N\overline{\phi}(x_1)) \leq 2^{-1}L\|v-u\|\beta(r),$$

i.e.,

$$d\Big(G\big(x_1, \frac{u+v}{2}\big) + G\big(x_2, \frac{u+v}{2}\big), G\big(x_2, v\big) + G(x_1, u)\Big) \leq L\Big\|\frac{v-u}{2}\Big\|\beta(r).$$

Taking $x \in I$ and letting $x_1, x_2 \to x$ we obtain (since $\lim_{r\to 0}\beta(r) = 0$ and $G(\cdot, y)$ is continuous for $y \in C$)

$$d\Big(2G(x, \frac{u+v}{2}), G(x,v) + G(x,u)\Big) = 0.$$

Thus

$$G\Big(x, \frac{u+v}{2}\Big) = \frac{1}{2}(G(x,v) + G(x,u)).$$

By virtue of Lemma 3, there exist functions $A\colon I \times C \to cc(Z)$ and $B\colon I \to cc(Z)$, where $A(x,\cdot)$ is additive for $x \in I$, such that

$$G(x,y) = A(x,y) + B(x).$$

Let $x \in I$ and $y, \overline{y} \in C$. By (6),

$$d(A(x,y), A(x,\overline{y})) = d(A(x,y) + B(x), A(x,\overline{y}) + B(x)) =$$
$$= d(G(x,y), G(x,\overline{y})) \leq L\|y - \overline{y}\|.$$

Thus the function $A(x,\cdot), x \in I$ is continuous. To prove that $B \in H_\beta(I, cc(Z))$, note that $A(x,\cdot)$ is additive

$$G(x,0) = A(x,0) + B(x) = \{0\} + B(x) = B(x),$$

and $G(\cdot, y) \in H_\beta(I, cc(Z))$ for every $y \in C$, in particular for $y = 0$.

We shall now prove that for every $y \in C$ the function $A(\cdot, y)$ belongs to the set $H_\beta(I, cc(Z))$. Let $x_1, x_2 \in I$ and $y \in C$. There is

$$d(A(x_1,y), A(x_2,y)) = d(A(x_1,y) + B(x_1), A(x_2,y) + B(x_1)) \leq$$
$$\leq d(A(x_1,y) + B(x_1), A(x_2,y) + B(x_2)) + d(A(x_2,y) + B(x_2), A(x_2,y) + B(x_1)) =$$
$$= d(G(x_1,y), G(x_2,y)) + d(B(x_1), B(x_2)).$$

Since $G(\cdot, y)$ and B are continuous, so is $A(\cdot, y)$. Let $y \in C, s \in (0,1]$ and let us take $x_1, x_2 \in I$ such that $|x_1 - x_2| \leq s$.
Then

$$d(A(x_1,y), A(x_2,y)) \leq d(G(x_1,y), G(x_2,y)) + d(B(x_1), B(x_2)) \leq$$
$$\leq \omega(G(\cdot,y), s) + \omega(B, s).$$

Therefore,

$$\omega(A(\cdot,y))/\beta(s) \le h_\beta(G(\cdot,y)) + h_\beta(B).$$

Thus the function $A(\cdot,y)$ belongs to the space $H_\beta(I,cc(Z))$.

We shall now prove that the function $I \ni x \mapsto A(x,\cdot) \in \mathcal{L}(C,cc(Z))$ satisfies the Hölder condition. Let us take $x_1,x_2 \in I$, such that $x_1 < x_2$, and let $y_1,y_2,\overline{y}_1,\overline{y}_2 \in C$. Moreover, let us define ϕ and $\overline{\phi}$ as previously. There is

$$d(N\phi(x_1) + N\overline{\phi}(x_2), N\phi(x_2) + N\overline{\phi}(x_1)) \le L\|\phi - \overline{\phi}\|_\alpha \beta(x_2 - x_1). \tag{11}$$

Let now $y,\overline{y} \in C$. Putting $y_1 = \overline{y}_2 = y + \overline{y} \in C$, $\overline{y}_1 = \overline{y}$, $y_2 = 2y + \overline{y} \in C$ into (10) and (11), we get

$$d(G(x_1,y+\overline{y}) + G(x_2,y+\overline{y}), G(x_2,2y+\overline{y}) + G(x_1,\overline{y})) \le L\|y\|\beta(x_2 - x_1).$$

Hence

$$d(A(x_1,y+\overline{y})+B(x_1)+A(x_2,y+\overline{y})+B(x_2), A(x_2,2y+\overline{y})+B(x_2)+A(x_1,\overline{y})+B(x_1)) \le$$
$$\le L\|y\|\beta(x_2 - x_1).$$

Thus

$$d(A(x_1,y),A(x_2,y)) \le L\|y\|\beta(x_2 - x_1).$$

Therefore,

$$d_{\mathcal{L}}(A(x_1,\cdot),A(x_2,\cdot)) = \sup_{y\in C\setminus\{0\}} \frac{d(A(x_1,y),A(x_2,y))}{\|y\|} \le L\beta(x_2 - x_1).$$

Obviously that inequality is also true in the case of $x_1 \ge x_2$, which completes the proof of part *a*).

b) Assume that N satisfies conditions 1) and 2). From (5) and (10) we get

$$\frac{d(G(x_1,y_1) + G(x_2,\overline{y}_2), G(x_2,y_2) + G(x_1,\overline{y}_1))}{\beta(x_2 - x_1)} \le$$
$$\le L\Big[\|y_1 - \overline{y}_1\| + \frac{\|y_2 - \overline{y}_2 - (y_1 - \overline{y}_1)\|}{\alpha(x_2 - x_1)}\Big].$$

Putting $y_1 = \overline{y}_1$ in the above inequality, we obtain

$$d(G(x_2,\overline{y}_2),G(x_2,y_2)) \le L(\beta/\alpha)(x_2 - x_1)\|y_2 - \overline{y}_2\| \tag{12}$$

If the condition $\alpha \preceq \beta$ does not hold, then it is easy to see that there exists a sequence $(t_n), t_n \in (0,1], t_n \to 0$, such that $(\beta/\alpha)(t_n) \to 0$. Now let us take $x_1 \in [0,1)$ and let $x_2^{(n)} := x_1 + t_n$ (the condition $x_2^{(n)} \in [0,1]$ holds for almost all n). There is $x_2^{(n)} \to x_1$; from the continuity of $G(\cdot,y), y \in C$ and from inequality (12) we get $G(x_1,\overline{y}_2) = G(x_1,y_2)$. Hence

$$G(x,y) = G(x,0) = B(x), \quad x \in [0,1), y \in C.$$

If $x = 1$, then we may take $x_2 = 1$ and $x_1^{(n)} := 1 - t_n$. Then $x_1^{(n)} \to 1$ and from (12) we get $G(1, \overline{y}_2) = G(1, y_2)$, which completes the proof of the equality

$$G(x, y) = B(x), \quad x \in I, y \in C.$$

Conversely, if we assume, that the above equality holds, then it is easy to observe that N is a constant function and satisfies the Lipschitz condition. □

3.

Theorem 2. *Let Y be a real Banach space, Z be a real normed space, C be a convex cone in Y, satisfying equality $Y = C \cup (-C)$, α and β be Hölder functions and let $\alpha \preceq \beta$. Assume that $A\colon I \times C \to cc(Z), B\colon I \to cc(Z)$ are such functions that $A(\cdot, y), B$ belong to the space $H_\beta(I, cc(Z))$ for $y \in C$ and $A(x, \cdot)$ belongs to the space $\mathcal{L}(C, cc(Z))$ for $x \in I$. Moreover, let the function $I \ni x \mapsto A(x, \cdot) \in \mathcal{L}(C, cc(Z))$ satisfy the Hölder condition*

$$d_{\mathcal{L}}(A(x_1, \cdot), A(x_2, \cdot)) \le L\beta(|x_1 - x_2|), \quad x_1, x_2 \in I,$$

where $d_{\mathcal{L}}$ is given by (4).
If we define the function $G\colon I \times C \to cc(Z)$ in the following way:

$$G(x, y) = A(x, y) + B(x), \quad x \in I, y \in C,$$

then the Nemytski operator N generated by G maps the set $H_\alpha(I, C)$ into the space $H_\beta(I, cc(Z))$ and satisfies the Lipschitz condition, i.e., there exists a constant $L' \ge 0$ such that

$$d_\beta(N\phi, N\overline{\phi}) \le L' \|\phi - \overline{\phi}\|_\alpha, \quad \phi, \overline{\phi} \in H_\alpha(I, C).$$

Proof. First we shall prove that the following formula holds:

$$\bigcup_{x \in I} A(x, y) \in b(Z), \tag{13}$$

for an arbitrary $y \in C$. Let $x \in I, y \in C$; there is

$$\|A(x, y)\| = d(A(x, y), \{0\}) \le d(A(x, y), A(0, y)) + d(A(0, y), \{0\}).$$

Moreover,

$$d(A(x, y), A(0, y)) \le \omega(A(\cdot, y), 1) \le h_\beta(A(\cdot, y)).$$

Hence

$$\|A(x, y)\| \le h_\beta(A(\cdot, y)) + d(A(0, y), \{0\}).$$

Thus (13) holds. Moreover, $\{A(x, \cdot)\}_{x \in I}$ is a family of additive and continuous functions. By Lemma 4, there exists a constant $M, 0 < M < +\infty$, such that

$$d(A(x, y), \{0\}) = \|A(x, y)\| \le M\|y\|, \quad x \in I, y \in C. \tag{14}$$

Let us take $x \in I, y_1, y_2 \in C$ and let $y_2 - y_1 \in C$. According to (14), we get

$$d(A(x,y_2),A(x,y_1)) = d(A(x,y_2-y_1)+A(x,y_1),A(x,y_1)+\{0\}) =$$
$$= d(A(x,y_2-y_1),\{0\}) \leq M\|y_2-y_1\|.$$

In the case of $y_1 - y_2 \in C$, we can also get the inequality

$$d(A(x,y_2),A(x,y_1)) \leq M\|y_2-y_1\|. \tag{15}$$

Thus inequality (15) holds for every $x \in I$ and $y_1, y_2 \in C$.
Since the function $I \ni x \mapsto A(x,\cdot) \in \mathcal{L}(C, cc(Z))$ satisfies the Hölder condition, then

$$d(A(x_1,y),A(x_2,y)) \leq L\|y\|\beta(|x_1-x_2|), \quad x_1,x_2 \in I, y \in C. \tag{16}$$

We shall now prove that N maps the set $H_\alpha(I,C)$ into the space $H_\beta(I,cc(Z))$. Let $\phi \in H_\alpha(I,C)$ and $x_1, x_2 \in I$. According to (15) and (16), we get

$$\begin{aligned} d(N\phi(x_1),N\phi(x_2)) &= d(g(x_1,\phi(x_1)),g(x_2,\phi(x_2))) = \\ &= d(A(x_1,\phi(x_1))+B(x_1),A(x_2,\phi(x_2))+B(x_2)) \leq \\ &\leq d(A(x_1,\phi(x_1)),A(x_2,\phi(x_2)))+d(B(x_1),B(x_2)) \leq \\ &\leq d(A(x_1,\phi(x_1)),A(x_2,\phi(x_1)))+ \\ &\quad + d(A(x_2,\phi(x_1)),A(x_2,\phi(x_2)))+d(B(x_1),B(x_2)) \leq \\ &\leq L\|\phi(x_1)\|\beta(|x_1-x_2|)+M\|\phi(x_1)-\phi(x_2))\|+ \\ &\quad + d(B(x_1),B(x_2)). \end{aligned}$$

Thus $N\phi$ is continuous, since ϕ and B are continuous. Now let $s \in (0,1]$ and let us take $x_1, x_2 \in I$ such that $|x_1 - x_2| \leq s$. It is easy to check that $\|\phi(x)\| \leq \|\phi\|_\alpha$, for every $x \in I$ and for every Hölder function α. Accordingly,

$$d(N\phi(x_1),N\phi(x_2)) \leq$$
$$\leq L\|\phi(x_1)\|\beta(|x_1-x_2|)+M\|\phi(x_1)-\phi(x_2))\|+d(B(x_1),B(x_2)) \leq$$
$$\leq L\|\phi\|_\alpha\beta(s)+M\omega(\phi,s)+\omega(B,s).$$

Hence

$$\begin{aligned} \frac{\omega(N\phi,s)}{\beta(s)} &\leq L\|\phi\|_\alpha + M\frac{\omega(\phi,s)}{\alpha(s)}\frac{\alpha(s)}{\beta(s)}+\frac{\omega(B,s)}{\beta(s)} \leq \\ &\leq L\|\phi\|_\alpha + MM'h_\alpha(\phi)+h_\beta(B), \end{aligned}$$

where $M' > 1$ is a constant such that $\alpha(s)/\beta(s) \leq M'$ for $s \in (0,1]$. Therefore, $N\phi \in H_\beta(I,cc(Z))$.

Now we shall prove that N is Lipschitzian. Let us take $x_1, x_2 \in I, y_1, y_2, y_3, y_4 \in C$ and let $y_2 - y_3 \in C$. There is

$$d(A(x_1,y_1)+A(x_2,y_2),A(x_2,y_3)+A(x_1,y_4)) \leq$$
$$\leq d(A(x_1,y_1+y_2),A(x_1,y_3+y_4))+d(A(x_2,y_2-y_3),A(x_1,y_2-y_3)) \leq$$
$$\leq M\|y_1+y_2-y_3-y_4\|+L\beta(|x_1-x_2|)\|y_2-y_3\|. \tag{17}$$

We can also get inequality (17) in the case of $y_3 - y_2 \in C$. Now let $\phi, \overline{\phi} \in H_\alpha(I, C)$ and $s \in (0,1]$. From the definition, there follows

$$\omega(N\phi, N\overline{\phi}, s) = \sup_{x_1, x_2 \in I, |x_1 - x_2| \leq s} d(N\phi(x_1) + N\overline{\phi}(x_2), N\phi(x_2) + N\overline{\phi}(x_1)).$$

Now let us take $x_1, x_2 \in I$ such that $|x_1 - x_2| \leq s$; using inequality (17), we get

$$\begin{aligned} d(N\phi(x_1) &+ N\overline{\phi}(x_2), N\phi(x_2) + N\overline{\phi}(x_1)) = \\ &= d(A(x_1, \phi(x_1)) + A(x_2, \overline{\phi}(x_2)), A(x_2, \phi(x_2)) + A(x_1, \overline{\phi}(x_1))) \leq \\ &\leq M\|(\phi - \overline{\phi})(x_1) - (\phi - \overline{\phi})(x_2)\| + L\beta(|x_1 - x_2|)\|(\phi - \overline{\phi})(x_2)\| \leq \\ &\leq M\omega(\phi - \overline{\phi}, s) + L\beta(s)\|\phi - \overline{\phi}\|_\alpha. \end{aligned}$$

Hence

$$\frac{\omega(N\phi, N\overline{\phi}, s)}{\beta(s)} \leq M \frac{\omega(\phi - \overline{\phi}, s)}{\alpha(s)} \frac{\alpha(s)}{\beta(s)} + L\|\phi - \overline{\phi}\|_\alpha,$$

which implies that

$$\sup_{s \in (0,1]} \frac{\omega(N\phi, N\overline{\phi}, s)}{\beta(s)} \leq MM' h_\alpha(\phi - \overline{\phi}) + L\|\phi - \overline{\phi}\|_\alpha. \tag{18}$$

From inequality (15), we get

$$d(N\phi(0), N\overline{\phi}(0)) = d(A(0, \phi(0)), A(0, \overline{\phi}(0))) \leq M\|\phi(0) - \overline{\phi}(0)\|.$$

Now using inequality (18), we obtain

$$\begin{aligned} d_\beta(N\phi, N\overline{\phi}) &= d(N\phi(0), N\overline{\phi}(0)) + \sup_{s \in (0,1]} \frac{\omega(N\phi, N\overline{\phi}, s)}{\beta(s)} \leq \\ &\leq M\|\phi(0) - \overline{\phi}(0)\| + MM' h_\alpha(\phi - \overline{\phi}) + L\|\phi - \overline{\phi}\|_\alpha \leq \\ &\leq MM' \left[\|\phi(0) - \overline{\phi}(0)\| + h_\alpha(\phi - \overline{\phi})\right] + L\|\phi - \overline{\phi}\|_\alpha = \\ &= MM'\|\phi - \overline{\phi}\|_\alpha + L\|\phi - \overline{\phi}\|_\alpha = \\ &= [MM' + L]\|\phi - \overline{\phi}\|_\alpha, \end{aligned}$$

and we may take $L' = MM' + L (\geq 0)$. □

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