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ON ARC-COLORING OF DIGRAPHS

Abstract. In the paper we deal with the problem of the arc-colouring of some classes of digraphs (tournaments, complete digraphs and products of digraphs).

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Let us consider a digraph $D = (V(D), A(D))$ without loops and multiple arcs.

One may study two kinds of arc-colorings of digraphs. Let us recall them. The coloring of a digraph D is *of the first type* if no two arcs $(uv), (vw)$ are colored with the same color (see [1]). In other words, it is a function:

$$c : A(D) \longrightarrow \{c_1, \ldots, c_n\},$$

such that for any $(uv), (wz) \in A(D)$

$$\text{if } c(uv) = c(wz), \text{ then } v \neq w.$$

It is also natural to consider the following coloring of arcs of a digraph (called a coloring *of the second type* – see [7]) such that no pair of arcs $(uv), (wv)$ and no pair of arcs $(vu), (vw)$ is colored with the same color. In other words, it is a function

$$c : A(D) \longrightarrow \{c_1, \ldots, c_n\},$$

such that for any $(u_1v), (u_2z) \in A(D)$, $u_1 \neq u_2$

$$\text{if } c(u_1v) = c(u_2z) \text{ then } v \neq z$$

and for any $(vu_1), (zu_2) \in A(D)$, $u_1 \neq u_2$

$$\text{if } c(vu_1) = c(zu_2) \text{ then } v \neq z.$$

Similarly as in the definition of the chromatic index of graphs we define the *chromatic index* of a digraph D *of the first type* (respectively, *of the second type*) to be the minimal number of colors necessary for a coloring D of the first (respectively, second) type. We denote it by $\overrightarrow{\chi_1}'(D)$ (respectively, $\overrightarrow{\chi_2}'(D)$). The problems of calculating the chromatic indices of the first (respectively, second) type were considered in [4] and [6] (respectively, [7]).

We may also consider a *pseudototal coloring of a digraph D of the first type (respectively, second type)* (cf. [8]) to be a function $c : A(D) \cup V(D) \mapsto \{c_1, \ldots, c_n\}$ such that $c_{|A(D)}$ is an arc-coloring of D of the first type (respectively, of the second type) and $c(v) \neq c(vw)$, $c(w) \neq c(vw)$ for any $v, w \in V(D)$ such that $(vw) \in A(D)$.

The minimal number of colors necessary for a pseudototal coloring of a digraph D is called the *pseudototal chromatic index of a digraph D of the first type (respectively, of the second type)* and is denoted by $\overrightarrow{\chi_1}^p(D)$ (respectively, $\overrightarrow{\chi_2}^p(D)$).

Actually, while studying the problem of the pseudototal coloring of the digraph, we examine the arc-coloring of the digraph – we verify whether for any vertex all colors used for the minimal arc-coloring of the digraph are used for the coloring of arcs adjacent to the vertex. It turns out that the pseudototal coloring of digraphs of the second type delivers us no new phenomena: we always need an additional color for the pseudototal coloring of the digraph. In the other case the situation is different (see [8]).

Below we shall deal with special types of digraphs such as complete symmetric digraphs K_n^* and tournaments.

The complete symmetric digraph K_n^* is a digraph spanned on n vertices and defined as follows:

$$V(K_n^*) := \{v_1, \ldots, v_n\},\ A(K_n^*) := \{(v_i v_j) : i \neq j, 1 \leq i, j \leq n\}.$$

A digraph T is called *a tournament* if the following holds:

$$\text{for any } u, v \in V(T),\ u \neq v\ \big((uv) \in A(T) \ \text{ iff } \ (vu) \notin A(T)\big).$$

A digraph TT_n is called *a transitive tournament* if

$$V(TT_n) = \{v_1, \ldots, v_n\},\ A(TT_n) = \{(v_i v_j) : 1 \leq i < j \leq n\}.$$

For a digraph D we also denote the following underlying graph $\tilde{G}(D)$:

$$V(\tilde{G}(D)) := V(D),$$
$$\{u, v\} \in E(\tilde{G}(D)) \ \text{ if and only if } \ (uv) \in A(D) \text{ or } (vu) \in A(D).$$

For a digraph D, $v \in V(D)$, we define the sets of adjacent vertices as follows:

$$N^+(v) := \{w : (vw) \in A(D)\},$$
$$N^-(v) := \{w : (wv) \in A(D)\},$$
$$N(v) := N^+(v) \cup N^-(v).$$

With the sets defined as above we associate also some numbers:

$$d_D^+(v) := |N^+(v)|,\ d_D^-(v) := |N^-(v)|,\ d_D'(v) := |N(v)|.$$

We put

$$\triangle^+(D) := \max\{d_D^+(v), v \in D\},$$
$$\triangle^-(D) := \max\{d_D^-(v), v \in D\},$$
$$\triangle'(D) := \max\{d_D'(v), v \in D\}.$$

Let us recall the following results on arc-colorings of digraphs.

Theorem 1 (see [7]**).** *For any digraph D the following formula holds:*

$$\overrightarrow{\chi_2}'(D) = \triangle(G(D)) = \max\{\triangle^+(D), \triangle^-(D)\}.$$

Theorem 2 (see [4]**).** *The following equality holds:*

$$\overrightarrow{\chi_1}'(TT_n) = \lceil \log n \rceil,\ n \in \mathbb{N}.$$

To be able to formulate more general results we need to define some sequence:

$$a_{-1} := -1,\ a_0 := 0,\ a_k := \binom{k}{\lfloor \frac{k}{2} \rfloor} = \binom{k}{\lceil \frac{k}{2} \rceil},\ k = 1, 2, \ldots.$$

It is clear that the sequence (a_k) is a strictly increasing sequence of integers. Therefore, for any $N \in \mathbb{N}$ there is exactly one $k \in \mathbb{N}$ such that $a_{k-1} < N \leq a_k$. We define the function g on $\mathbb{N}$ as follows $g(N) := k$, where k is the unique integer such that $a_{k-1} < N \leq a_k$.

Now we can recall the next result

Theorem 3 (see [4, 6]**).** *For any digraph D the following inequalities hold:*

$$\lceil \log \chi(\tilde{G}(D)) \rceil \leq \overrightarrow{\chi_1}'(D) \leq g(\chi(\tilde{G}(D))).$$

Moreover, if the digraph D is symmetric (i.e. $(uv) \in A(D)$ iff $(vu) \in A(D)$) then the following equality holds:

$$\overrightarrow{\chi_1}'(D) = g(\chi(\tilde{G}(D))),$$

where $\chi(G)$ denotes the chromatic number of the graph G.

Theorem 3 delivers us the explicit formula for the chromatic index of the first type for complete digraphs K_n^*.

Corollary 4. *For any $n \geq 2$, the following equality holds:*

$$\overrightarrow{\chi_1}'(K_n^*) = g(n).$$

Let us begin with the study of some problems related to the arc-coloring.

Complete digraphs and tournaments spanned on the same set of vertices are some kind of extremal digraphs in the class of all digraphs for which the graph $\tilde{G}(D)$ is the same. Therefore, a natural question arises how much chromatic indices of two different digraphs (two different tournaments) with the same digraph $\tilde{G}(D)$ can differ from each other.

In view of Theorems 2 and 3

$$\min\{\overrightarrow{\chi_1}'(T)\colon\ T \text{ is a tournament}, |T| = n\} = \lceil \log n \rceil$$

and the minimum is attained for transitive tournaments.

Let us remark that taking $k \geq 2$ such that $g(k) = n+1$ and applying elementary transformations, we get:

$$2^{\lfloor \frac{n}{2} \rfloor} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} < k \leq \binom{n+1}{\lceil \frac{n+1}{2} \rceil} \leq 2^n, \quad k \geq 2. \tag{1}$$

Then we get

$$\lceil \log k \rceil \leq g(k) \leq 2\lceil \log k \rceil + 1. \tag{2}$$

Therefore, it is easy to see that for any tournament T such that $V(T) = k$, $(k \geq 2)$ the following inequalities hold (compare Theorem 3):

$$\lceil \log k \rceil \leq \overrightarrow{\chi_1}'(T) \leq 2\lceil \log k \rceil + 1. \tag{3}$$

A closer analysis of the mutual relation between $g(k)$ and $\log k$ leads us to a better asymptotic estimate than that in the second inequality in (2).

Let us consider namely the inequality $a_{2n} < k \leq a_{2n+2}$. In this case $g(k) = 2n+1$ or $2n + 2$. On the other hand, applying the Taylor expansion we get

$$\log a_{2n} = \log \frac{(2n)!}{(n!)^2} = n + \log \frac{1 \cdot 3 \cdots (2n-1)}{n!} = n + \sum_{j=1}^{n} \log(2 - \frac{1}{j}) =$$

$$n + \sum_{j=1}^{n} (\log 2 - \frac{1}{2j \ln 2} + O(\frac{1}{j^2})) = 2n - \frac{1}{2 \ln 2} \sum_{j=1}^{n} \frac{1}{j} + \sum_{j=1}^{n} O(\frac{1}{j^2}) \geq$$

$$\geq 2n - \frac{1}{2 \ln 2} \sum_{j=1}^{n} \frac{1}{j} - C \sum_{j=1}^{n} \frac{1}{j^2} \geq 2n - \frac{1}{2 \ln 2} \sum_{j=1}^{n} \frac{1}{j} + M_1,$$

where $C > 0$ and M_1 are some constants.

Analoguously we get

$$\log a_{2n+2} = 2(n+1) - \frac{1}{2\ln 2}\sum_{j=1}^{n+1}\frac{1}{j} + \sum_{j=1}^{n+1} O(\frac{1}{j^2}) \le$$

$$\le 2(n+1) - \frac{1}{2\ln 2}\sum_{j=1}^{n+1}\frac{1}{j} + C\sum_{j=1}^{n+1}\frac{1}{j^2} \le 2(n+1) - \frac{1}{2\ln 2}\sum_{j=1}^{n+1}\frac{1}{j} + M_2,$$

where M_2 is some constant.

Therefore, there is

$$\frac{1}{2\ln 2}\sum_{j=1}^{n+1}\frac{1}{j} - M_2 - 1 \le g(k) - \log k \le \frac{1}{2\ln 2}\sum_{j=1}^{n}\frac{1}{j} - M_1 + 2.$$

In particular,

$$\lim_{k\to\infty}(g(k) - \lceil \log k\rceil) = +\infty. \tag{4}$$

On the other hand, the convergence of the sequence $(\sum_{j=1}^{n}\frac{1}{j} - \ln n)_{n=1}^{\infty}$ and the inequality $n \le \log k$ (use (1)) imply

$$0 \le \frac{g(k)}{\log k} - 1 \le \frac{\frac{1}{2\ln 2}\sum_{j=1}^{n}\frac{1}{j} - M_1 + 1}{\log k} \le \frac{\frac{1}{2\ln 2}\ln(\log k) + M_3}{\log k},$$

where $M_3 \in \mathbb{R}$ is some constant. Since the last expression tends to 0 (as $k \to \infty$), we get (compare Corollary 7 in [4])

$$\lim_{k\to\infty}\frac{g(k)}{\lceil \log k\rceil} = 1. \tag{5}$$

In particular, we get the following inequalities

$$\lceil \log k\rceil \le g(k) \le c_k\lceil \log k\rceil, \tag{6}$$

where $c_k > 1$, $c_k \to 1$.

Up to now we have only known examples of tournaments for which the equality $\overrightarrow{\chi_1}'(T) = \lceil \log V(T)\rceil$ holds. Below we show the existence of a tournament with c_k larger than one.

Proposition 5. Let us fix $n = 2^k$, $k \ge 1$. Let T be defined as follows:

$$V(T) := \mathbb{Z}_n,\ A(T) := \{(ij) : i < j,\ (ij) \ne (0\ (n-1))\} \cup \{((n-1)\ 0)\}.$$

Then $\overrightarrow{\chi_1}'(T) = k + 1 = \lceil \log n\rceil + 1$ for $n \ge 2$.

Proof. Actually, one obtains the digraph T from TT_n changing the direction of one arc only. Since the digraph TT_n is colorable with k colors, the digraph T is colorable with $k+1$ colors. To complete the proof it is sufficient to show that the digraph

T is not colorable with k colors. Suppose the opposite. Then with any vertex v we may associate $C(v)$ – a set of colors used for coloring vertices $(vw) \in A(T)$. Since each pair of different vertices is joined with some arc, the sets are pairwise different. Therefore, the set $\{C(v) : v \in V(T)\}$ consists of $n = 2^k$ different subsets of the sets of colors (consisting of k elements). Therefore, for one of $v \in V(T)$ there is $C(v) = \emptyset$, a contradiction. □

Proposition 5 suggests the following problems.

Problem 6. For any $k \geq 3$, find a tournament T (if it exists) such that $V(T) = k$ and $\overrightarrow{\chi_1}'(T) = \lceil \log k \rceil + 1$.

Find $b_k := \max\{\overrightarrow{\chi_1}'(T) : \ T \text{ is a tournament}, V(T) = k\}$ (it belongs to the set $\{\lceil \log k \rceil, \lceil \log k \rceil + 1, \dots g(k)\}$).

In connection with the last question, let us remark that the following property holds:

Any number from $\{\lceil \log k \rceil, \dots, b_k\}$ is a chromatic index of some tournament with k vertices.

It follows from the following property of tournaments. Changing the direction of exactly one arc in the tournament (so the procedure leaves the digraph to be still a tournament) the chromatic index of a new digraph changes by at most one. Then it is sufficient to notice that starting with the transitive tournament TT_k, after a finite number of such changes, we may easily arrive at any other tournament with the same set of vertices.

Remark 7. For a digraph D (with $\chi(\tilde{G}(D)) \geq 3$) the following inequalities hold (use Theorem 3, (2) and the inequality $\chi(G) \leq \triangle(G) + 1$):

$$\overrightarrow{\chi_1}'(D) \leq g(\chi(\tilde{G}(D))) \leq 2\lceil \log \chi(\tilde{G}(D)) \rceil + 1 \leq 2\lceil \log(\triangle(\tilde{G}(D)) + 1) \rceil + 1 = \\ = 2\lceil \log(\triangle'(D) + 1) \rceil + 1.$$

Below we shall recall some characterization of digraphs with small chromatic index. But before we formulate this result, we have to introduce some notation.

For a given digraph D, define some graph $\hat{G}(D)$. First we define $\hat{V}(D) \subset V(D)$ to be the set of vertices $v \in V(D)$ such that

$$d_D^+(v) \cdot d_D^-(v) > 0.$$

Then we define $\hat{G}(D)$ as follows:

$$V(\hat{G}(D)) := \hat{V}(D),$$
$$E(\hat{G}(D)) := \{\{u, v\} \colon \ (uv) \in A(D) \text{ or } (vu) \in A(D) \text{ and } u, v \in \hat{V}(D)\}.$$

Theorem 8 (see [8])**.** *For any digraph D the inequality $\overrightarrow{\chi_1}'(D) \leq 2$ is equivalent to the inequality $\chi(\hat{G}(D)) \leq 2$.*

The condition from Theorem 8 is equivalent to the nonexistence of odd cycles in $\hat{G}(D)$. It turns out that the result of Theorem 8 may be expressed with help of a notion of generalized directed cycles.

A digraph D is called *a generalized directed cycle* if the graph $\hat{G}(D)$ is a cycle. A generalized directed cycle is called *odd* if the cycle $\hat{G}(D)$ is odd.

The structure of generalized directed cycles is much more complicated than the structure of oriented cycles. In particular, for a fixed natural number, there are non-isomorphic generalized directed cycles with the same number of vertices (see Fig. 1).

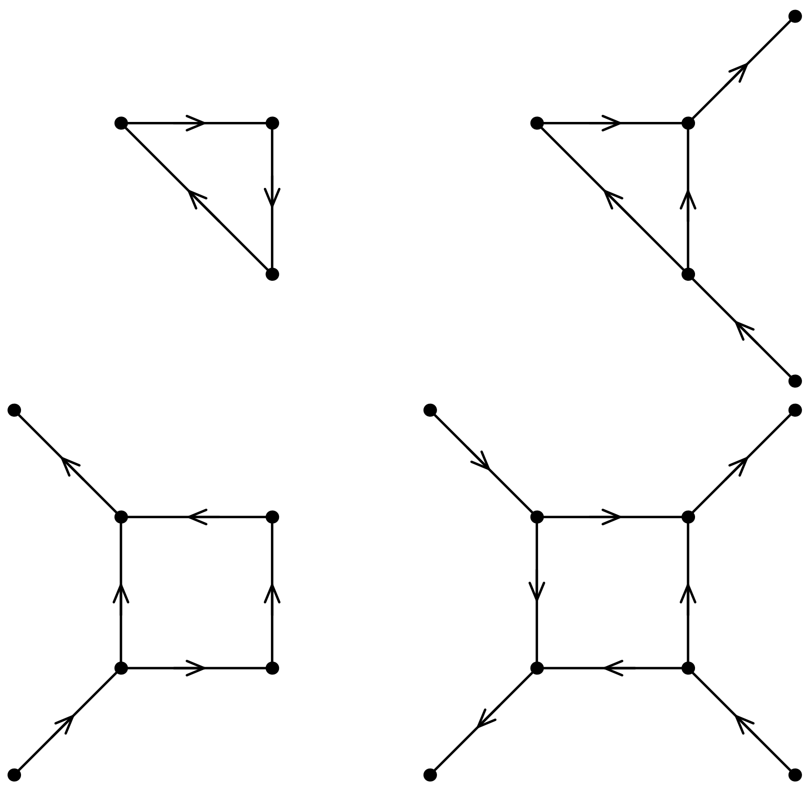

Fig. 1

The result from Theorem 8 may now be expressed as follows:

Corollary 9. *For a digraph D the inequality $\overrightarrow{\chi_1}'(D) \geq 3$ is equivalent to the existence in D of an odd generalized directed cycle.*

Below we show some sharper version of Theorem 5 from [8], which gives us the complete characterization of the arc-coloring of the first type with the minimal number of colors of complete symmetric digraphs. Namely, we show the following result.

Theorem 10. *For any $n \in \mathbb{N}$, the following equality holds*

$$\overrightarrow{\chi_1}^p(K_n^*) = g(n) + 1.$$

The above result was proven in [8] in the special case of $k = a_n$ for some n.

Before we start the proof we need one more notation. Let c denote an arc coloring of D. Assume that the arcs of the digraph D are colored with n colors (say $c_1, \ldots, c_n$). For any vertex $v \in V(D)$, let us define the following sets:

$$\begin{aligned} C_+(v) &:= \{c_j : \text{ there is a } (vw) \in A(D) \text{ such that } c(vw) = c_j\}, \\ C_-(v) &:= \{c_j : \text{ there is a } (wv) \in A(D) \text{ such hat } c(wv) = c_j\}, \\ C_0(v) &:= \{c_1, \ldots, c_n\} \setminus (C_+(v) \cup C_-(v)). \end{aligned}$$

It follows from the definition of the arc-coloring that

$$C_+(v) \cap C_-(v) = \emptyset \text{ for any } v \in V(D).$$

Proof of Theorem 10. Let $a_{n-1} < k \leq a_n$. To prove the theorem, it is sufficient to show that there is no arc coloring of K_k^* with n colors such that for any $v \in V(K_k^*)$ we have $C_0(v) \neq \emptyset$, where $C_0(v)$ is defined as above.

Suppose that such a coloring exists. Then we define a family of sets $\{C_+(v) : v \in V(K_k^*)\}$ such that $C_+(v) \subset \{c_1, \ldots, c_n\}$ for any $v \in K_k^*$ (certainly $0 < |C_+(v)| < n$). Additionally, with any $v \in V(K_k^*)$ we associate a color $c(v) \notin C_+(v)$ satisfying the following condition:

$$\text{for any } v, w \in V(K_k^*), \quad \text{if} \quad C_+(v) \subset C_+(w) \cup \{c(w)\}, \text{ then } v = w.$$

In fact, for a coloring with desired properties, let $c(v)$ denote any element from $C_0(v)$ (we make this choice for any $v \in V(K_k^*)$). Assume that $C_+(v) \subset C_+(w) \cup \{c(w)\}$ for some $v \neq w$, $v, w \in V(K_k^*)$. We know that for any $v, w \in V(K_k^*)$, $v \neq w$ there is a $c_j \in C_+(v) \cap C_-(w) \subset (C_+(w) \cup \{c(w)\}) \cap C_-(w) = \{c(w)\}$; so $c_j = c(w) \in C_0(w)$, and therefore $c_j \notin C_-(w)$ – contradiction. □

Consequently, to finish the proof it is sufficient to make use of the following lemma, which will be proven later.

Lemma 11. *Let X be a given set of n elements. Let us consider a family $\mathcal{M} \subset \{(A, a) : \emptyset \neq A \subset X,\ a \in X \setminus A\}$ satisfying the following property:*

$$\text{If } (A, a), (B, b) \in \mathcal{M} \text{ and } A \subset B \cup \{b\}, \text{ then } A = B \text{ and } a = b. \tag{7}$$

Then $|\mathcal{M}| \leq a_{n-1}$.

In fact, it is sufficient to put $\mathcal{M} := \{C_+(v), c_+(v)) :\ v \in V(K_n^*)\}$.

To prove Lemma 11 we need the following result of B. Bollobás (see [2])

Lemma 12 (see [2]). *Let $A_1, \dots, A_m$ and $B_1, \dots, B_m$ be subsets of a set consisting of n elements such that for any $i, j = 1, \dots, m$ $A_i \cap B_j = \emptyset$ iff $i = j$. Then*

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

Proof of Lemma 11. Let us write $\mathcal{M} = \{(A_j, a_j) : j = 1, \dots, m\}$. Our aim is to show that $m \leq a_{n-1}$. For any $i \in \{1, \dots, m\}$ define $B_i := X \setminus (A_i \cup \{a_i\})$. Then the assumptions of Lemma 12 for $A_1, \dots, A_m$, $B_1, \dots, B_m$ are satisfied, which implies, in view of Lemma 12, that

$$m \leq \max\left\{\binom{n-1}{|A_i|} : i = 1, 2, \dots, m\right\} \leq a_{n-1}. \qquad \square$$

Now let us try to establish some results concerning the chromatic indices of products of digraphs. At first let us recall the definition of the products of graphs and digraphs.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We define the graph $G_1 \times G_2$ as follows:

$$V(G_1 \times G_2) := V(G_1) \times V(G_2),$$
$$E(G_1 \times G_2) := \{\{(u_1, u_2), (v_1, v_2)\} : \{u_1, v_1\} \in E(G_1), \{u_2, v_2\} \in E(G_2)\}.$$

It is easy to see that

$$\chi(G_1 \times G_2) \leq \min\{\chi(G_1), \chi(G_2)\}. \tag{8}$$

For small chromatic numbers inequality (8) becomes the equality.

Theorem 13 (see [3]). *If* $\min\{\chi(G_1), \chi(G_2)\} \leq 4$ *then*

$$\chi(G_1 \times G_2) = \min\{\chi(G_1), \chi(G_2)\}.$$

S. T. Hedetniemi conjectured in [5] that Theorem 13 remains true for all graphs.

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs. Following [6] we define a digraph $D_1 \times D_2$ as follows

$$V(D_1 \times D_2) := V(D_1) \times V(D_2),$$
$$A(D_1 \times D_2) = \{((u_1, u_2)(v_1, v_2)) : \ (u_1 v_1) \in A(D_1), (u_2 v_2) \in A(D_2)\}.$$

Similarly as in the case of products of graphs, we get the following estimate.

Theorem 14. *For any digraphs D_1 and D_2 the following inequality holds:*

$$\overrightarrow{\chi_1}'(D_1 \times D_2) \leq \min\left\{\overrightarrow{\chi_1}'(D_1), \overrightarrow{\chi_1}'(D_2)\right\}. \tag{9}$$

Proof. Let k denote the right-hand side of inequality (9). Assume that the minimum is attained for the digraph D_1. Let c denote the arc-coloring of D_1 with k colors. We define now the arc coloring of $D_1 \times D_2$ with k colors as follows:

$$\text{if } \ ((u_1,u_2)(v_1,v_2)) \in A(D_1 \times D_2), \quad \text{then } \ \tilde{c}\big(((u_1,u_2)(v_1,v_2))\big) := c(u_1v_1). \qquad \square$$

We now prove the following

Theorem 15. *If* $\min\{\overrightarrow{\chi_1}'(D_1), \overrightarrow{\chi_1}'(D_2)\} \leq 3$ *then*

$$\overrightarrow{\chi_1}'(D_1 \times D_2) = \min\{\overrightarrow{\chi_1}'(D_1), \overrightarrow{\chi_1}'(D_2)\}.$$

Proof. Let us denote the minimum by k. It follows from Remark 10 in [8] and (9) that the theorem is valid if $k = 1$ or $k = 2$. It is sufficient to show that if $\overrightarrow{\chi_1}'(D_1) = \overrightarrow{\chi_1}'(D_2) = 3$ then $\overrightarrow{\chi_1}'(D_1 \times D_2) = 3$.

Let us remark that from the definition it follows that

$$\hat{V}(D_1 \times D_2) = \hat{V}(D_1) \times \hat{V}(D_2).$$

Moreover, the property

$$\{(u_1,u_2),(v_1,v_2)\} \in \hat{E}(D_1 \times D_2)$$

is equivalent to:

$$\{u_1,v_1\} \in \hat{E}(D_1) \quad \text{and} \quad \{u_2,v_2\} \in \hat{E}(D_2).$$

Actually, $\{(u_1,u_2),(v_1,v_2)\} \in \hat{E}(D_1 \times D_2)$ is equivalent to $((u_1,u_2)(v_1,v_2)) \in A(D_1 \times D_2)$ or $((v_1,v_2)(u_1,u_2)) \in A(D_1 \times D_2)$, so consequently, also to $((u_1v_1) \in A(D_1)$ and $(u_2v_2) \in A(D_2))$ or $((v_1u_1) \in A(D_1)$ and $(v_2u_2) \in A(D_2))$, which gives the desired equivalence.

In other words,

$$\hat{G}(D_1 \times D_2) = \hat{G}(D_1) \times \hat{G}(D_2)$$

(up to an isomorphism).

Now Theorem 13 and Theorem 8 complete the proof. $\square$

Problem 16. Do there exist digraphs D_1, D_2 such that $\overrightarrow{\chi_1}'(D_1) = \overrightarrow{\chi_1}'(D_2) = 4$ and $\overrightarrow{\chi_1}'(D_1 \times D_2) = 3$? On the other hand, note that in case $D_1 = D_2$ it is evident that the equality $\overrightarrow{\chi_1}'(D_1 \times D_2) = \overrightarrow{\chi_1}'(D_1)$ holds.

For a better understanding of the chromatic index of the product of digraphs, let us define

$$h(r) := \min\{\overrightarrow{\chi_1}'(D_1 \times D_2) : \ \overrightarrow{\chi_1}'(D_1) = \overrightarrow{\chi_1}'(D_2) = r\}$$

and

$$f(r) := \min\{\chi(G_1 \times G_2) : \ \chi(G_1) = \chi(G_2) = r\}.$$

Then the following result holds

Theorem 17. *Divergence* $f(r) \to \infty$ *is equivalent to the divergence* $h(r) \to \infty$ *(as* $r \to \infty$*).*

Proof. Let us remark that $\tilde{G}(D_1 \times D_2) = \tilde{G}(D_1) \times \tilde{G}(D_2)$ (up to an isomorphism). It is sufficient to make use of Theorem 3 and formula (2) (because one can see that divergence of $\chi(\tilde{G}(D))$ to infinity is equivalent to the divergence of $\overrightarrow{\chi_1}'(D)$ to infinity). $\square$

Remark 18. As far as the arc-coloring of the products of digraphs of the second type is concerned the situation is clear. Namely, in view of Theorem 1 we have:

$$\begin{aligned}\overrightarrow{\chi_2}'(D_1 \times D_2) = \max\{\triangle^+(D_1 \times D_2), \triangle^-(D_1 \times D_2)\} = \\ \max\{\triangle^+(D_1)\triangle^+(D_2), \triangle^-(D_1)\triangle^-(D_2)\} \leq \\ \max\{\triangle^+(D_1), \triangle^-(D_1)\}\max\{\triangle^+(D_2), \triangle^-(D_2)\} = \overrightarrow{\chi_2}'(D_1)\overrightarrow{\chi_2}'(D_2).\end{aligned}$$

Acknwoledgments

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