Tomasz S. Zabawa

STABILITY OF SOLUTIONS OF INFINITE SYSTEMS OF NONLINEAR DIFFERENTIAL-FUNCTIONAL EQUATIONS OF PARABOLIC TYPE

Abstract. A parabolic initial boundary value problem and an associated elliptic Dirichlet problem for an infinite weakly coupled system of semilinear differential-functional equations are considered. It is shown that the solutions of the parabolic problem is asymptotically stable and the limit of the solution of the parabolic problem as $t \to \infty$ is the solution of the associated elliptic problem. The result is based on the monotone methods.

Keywords: stability of solutions, infinite systems, parabolic equations, elliptic equations, semilinear differential-functional equations, monotone iterative method.

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Let S be an infinite set. Let $G \subset \mathbb{R}^m$ be an open bounded domain with $C^{2+\alpha}$ boundary ($\alpha \in (0,1)$) and $D := (0,T) \times G$, where $T \leq \infty$. Let $\bar{D} := [0,T) \times \bar{G}$.

We consider a boundary initial value problem for an infinite weakly coupled system of semilinear autonomous differential-functional parabolic equations of the form:

$$\frac{\partial u^i(t,x)}{\partial t} - \mathcal{L}^i[u^i](t,x) = f^i(x, u(t,x), u(t,\cdot)) \quad \text{for} \quad t > 0,\ x \in G,\ i \in S, \tag{1}$$

$$u^i(t,x) = h^i(x) \quad \text{for} \quad t > 0,\ x \in \partial G,\ i \in S, \tag{2}$$

$$u^i(0,x) = h^i(x) \quad \text{for} \quad x \in \bar{G},\ i \in S \tag{3}$$

and the associated Dirichlet problem for an infinite weakly coupled system of semilinear differential-functional elliptic equations of the form:

$$-\mathcal{L}^i[u^i](x) = f^i(x, u(x), u(\cdot)) \quad \text{for } x \in G,\ i \in S, \tag{4}$$

$$u^i(x) = h^i(x) \quad \text{for } x \in \partial G,\ i \in S, \tag{5}$$

where

$$\mathcal{L}^i[u^i](t,x) := \sum_{j,k=1}^{m} a^i_{jk}(x) u^i_{x_j x_k}(t,x) + \sum_{j=1}^{m} b^i_j(x) u^i_{x_j}(t,x)$$

are strongly uniformly elliptic in $\bar{G}$,

$$f^i\colon\ \bar{G} \times \mathcal{B}(S) \times C_S(\bar{G}) \ni (x,y,z) \mapsto f^i(x,y,z) \in \mathbb{R}$$

for every $i \in S$. The notation $f(x, u(x), u(\cdot))$ means that the dependence of f on the second variable is a function-type dependence and on the third variable is a functional-type dependence. In (1), $f(x, u(t,x), u(t,\cdot))$ means that the dependence on the third variable is a functional-type dependence with respect to x, but a function-type dependence with respect to t.

The following results extend and generalize the results of D. H. Sattinger [7], H. Amann [3] and S. Brzychczy [4].

We use the following notation. Let $\mathcal{B}(S)$ be the Banach space of all bounded functions $w\colon\ S \to \mathbb{R}$, $w(i) = w^i$ $(i \in S)$ with the norm $\|w\|_{\mathcal{B}(S)} := \sup_{i\in S} |w^i|$. $\mathcal{B}(S)$ is endowed with a partial order $w \le \tilde{w}$ defined as $w^i \le \tilde{w}^i$ for every $i \in S$. Elements of $\mathcal{B}(S)$ will be denoted by $(w^i)_{i\in S}$, too. Let $C(\bar{G})$ be the space of all continuous functions $v : \bar{G} \to \mathbb{R}$ with the norm $\|v\|_{C(\bar{G})} := \max_{x\in\bar{G}} |v(x)|$. In this space, $v \le \tilde{v}$ means that $v(x) \le \tilde{v}(x)$ for every $x \in \bar{G}$. By $C^{l+\alpha}(\bar{G})$, where $l = 0, 1, 2, \ldots$ and $\alpha \in (0,1)$, we denote the space of all functions continuous in $\bar{G}$ with derivatives of order less or equal l being Hölder continuous with exponent α in G (see [6, pp. 52–53]) and by $C^{l+\alpha}(\bar{D})$, where $l = 0, 1, 2, \ldots$ and $\alpha \in (0,1)$, we denote the space of all functions continuous in $\bar{D}$ with all derivatives $\frac{\partial^{r+s}}{\partial t^r \partial x^s}$ being Hölder continuous with exponent α in D if $0 \le 2r + s \le l$ (see [5, pp. 37-38]). By $H^{l,p}(G)$ we denote the Sobolev space of all functions whose weak derivatives of order l are in $L^p(G)$ (see [1, pp. 44-46]). A notation $g \in C^{l+\alpha}(\partial G)$ (resp. $g \in H^{l,p}(\partial G)$) means that there exists a function $\mathbf{g} \in C^{l+\alpha}(\bar{G})$ (resp. $\mathbf{g} \in H^{l,p}(G) \cap C(\bar{G})$) such that $\mathbf{g}(x) = g(x)$ for every $x \in \partial G$. In these spaces, norms are defined as $\|g\|_{C^{l+\alpha}(\partial G)} := \inf_{\mathbf{g}\in C^{l+\alpha}(\bar{G}):\forall x\in\partial G:\mathbf{g}(x)=g(x)} \|\mathbf{g}\|_{C^{l+\alpha}(\bar{G})}$ and $\|g\|_{H^{2,p}(\partial G)} := \inf_{\mathbf{g}\in H^{2,p}(G)\cap C(\bar{G}):\forall x\in\partial G:\mathbf{g}(x)=g(x)} \|\mathbf{g}\|_{H^{2,p}(G)}$, respectively.

We denote $z = (z^i)_{i\in S} \in C_S(\bar{G})$ if $z : \bar{G} \to \mathcal{B}(S)$ and $z^i : \bar{G} \to \mathbb{R}$ $(i \in S)$ is a continuous function with $\sup_{i\in S} \|z^i(x)\|_{C(\bar{G})} < \infty$. The space $C_S(\bar{G})$ is a Banach space with the norm $\|z\|_{C_S(\bar{G})} := \sup_{i\in S} \|z^i(x)\|_{C(\bar{G})}$ and the partial order $z \le \tilde{z}$ defined as $z^i(x) \le \tilde{z}^i(x)$ for every $x \in \bar{G}$, $i \in S$. The space $C^{l+\alpha}_S(\bar{G})$ is the space of all functions $(z^i)_{i\in S}$ such that $z^i \in C^{l+\alpha}(\bar{G})$ for every $i \in S$ and $\sup_{i\in S} \|z^i(x)\|_{C^{l+\alpha}(\bar{G})} < \infty$. In this space, the norm is defined as $\|z(x)\|_{C^{l+\alpha}_S(\bar{G})} = \sup_{i\in S} \|z^i(x)\|_{C^{l+\alpha}(\bar{G})}$. In similar way, $C^{l+\alpha}_S(\bar{D})$ is the space of all functions $(z^i)_{i\in S}$ such that $z^i \in C^{l+\alpha}(\bar{D})$ for every $i \in S$ and $\sup_{i\in S} \|z^i(x)\|_{C^{l+\alpha}(\bar{D})} < \infty$ with the norm $\|z(x)\|_{C^{l+\alpha}_S(\bar{D})} = \sup_{i\in S} \|z^i(x)\|_{C^{l+\alpha}(\bar{D})}$. We will write that $z = (z^i)_{i\in S} \in L^p_S(G)$ if $z^i \in L^p(G)$ for every $i \in S$ and

$\sup_{i\in S}\|z^i(x)\|_{L^p(G)} < \infty$. A notation $z = (z^i)_{i\in S} \in H^{l,p}_S(G)$ means that $z^i \in H^{l,p}(G)$ for every $i \in S$ and $\sup_{i\in S}\|z^i(x)\|_{H^{l,p}(G)} < \infty$. In these spaces, norms are defined as $\|z(x)\|_{L^p_S(G)} = \sup_{i\in S}\|z^i(x)\|_{L^p(G)}$ and $\|z(x)\|_{H^{l,p}_S(G)} = \sup_{i\in S}\|z^i(x)\|_{H^{l,p}(G)}$, respectively.

A function $\tilde{u}$ is said to be *regular in* $\bar{D}$ if $\tilde{u} \in C_S(\bar{D})$ and $\tilde{u}$ has continuous derivatives $\frac{\partial\tilde{u}}{\partial t}$, $\frac{\partial\tilde{u}}{\partial x_j}$, $\frac{\partial^2\tilde{u}}{\partial x_j\partial x_k}$ in D for $j,k = 1..m$. A function $\tilde{u}$ is said to be a *classical (regular) solution* of problem (1), (2), (3) in $\bar{D}$ if $\tilde{u}$ is regular in $\bar{D}$ and fulfils the system of equations (1) in D with conditions (2) and (3). A function $\tilde{u}$ is said to be a *weak solution* of problem (1), (2), (3) in $\bar{D}$ if $\tilde{u}(t,\cdot) \in L^2_S(G)$, $\frac{\partial\tilde{u}^i(t,\cdot)}{\partial t} \in L^2(G)$, $\mathcal{L}^i[\tilde{u}^i](t,\cdot) \in L^2(G)$ and

$$\left\langle \frac{\partial\tilde{u}^i(t,x)}{\partial t}, \xi(x) \right\rangle - \left\langle \mathcal{L}^i[\tilde{u}^i](t,x), \xi(x) \right\rangle = \left\langle f^i(x, \tilde{u}(t,x), \tilde{u}(t,\cdot)), \xi(x) \right\rangle$$

for every $t > 0$, $i \in S$ and for any test function $\xi \in C_0^\infty(\bar{G})$, where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(G)$, i.e., $\langle f, g\rangle = \int_G fg\,dx$ and $\tilde{u}$ fulfils conditions (2), (3) in trace sense.

A function $\hat{u}$ is said to be *regular in* $\bar{G}$ if $\hat{u} \in C_S(\bar{G}) \cap C^2_S(G)$. A function $\hat{u}$ is said to be a *classical (regular) solution* of problem (4), (5) in $\bar{G}$ if $\hat{u}$ is regular in $\bar{G}$ and fulfils the system of equations (4) in G with condition (5). A function $\hat{u}$ is said to be a *weak solution* of problem (4), (5) in $\bar{G}$ if $\hat{u} \in L^2_S(G)$, $\mathcal{L}^i[\hat{u}^i] \in L^2(G)$ and

$$-\left\langle \mathcal{L}^i[\hat{u}^i](x), \xi(x) \right\rangle = \left\langle f^i(x, \hat{u}(x), \hat{u}(\cdot)), \xi(x) \right\rangle$$

for every $i \in S$ and for any test function $\xi \in C_0^\infty(\bar{G})$ and $\hat{u}$ fulfils condition (5) in trace sense.

A solution $\hat{u}(x)$ of elliptic problem (4), (5) is said to be a *stable solution of parabolic problem (1), (2), (3)* if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\hat{u}(\cdot) - h(\cdot)\|_{C_S(\bar{G})} < \delta$ implies $\|\hat{u}(\cdot) - \tilde{u}(t,\cdot)\|_{C_S(\bar{G})} < \epsilon$ for each $t > 0$, where $\tilde{u}(t,x)$ is a solution of parabolic problem (1), (2), (3). A solution $\hat{u}(x)$ is called an *asymptotically stable solution of parabolic problem (1), (2), (3)* if it is a stable solution of parabolic problem (1), (2), (3) and $\lim_{t\to\infty}\|\hat{u}(\cdot) - \tilde{u}(t,\cdot)\|_{C_S(\bar{G})} = 0$.

Functions $\tilde{u}_0 = \tilde{u}_0(t,x)$ and $\tilde{v}_0 = \tilde{v}_0(t,x)$ regular in $\bar{D}$, satisfying the infinite systems of inequalities

$$\begin{cases} \frac{\partial\tilde{u}_0^i(t,x)}{\partial t} - \mathcal{L}^i[\tilde{u}_0^i](t,x) \le f^i(x, \tilde{u}_0(t,x), \tilde{u}_0(t,\cdot)) & \text{for } t > 0,\ x \in G,\ i \in S, \\ \tilde{u}_0^i(t,x) \le h^i(x) & \text{for } t > 0,\ x \in \partial G,\ i \in S, \\ \tilde{u}_0^i(0,x) \le h^i(x) & \text{for } x \in \bar{G},\ i \in S \end{cases} \tag{6}$$

$$\begin{cases} \frac{\partial\tilde{v}_0^i(t,x)}{\partial t} - \mathcal{L}^i[\tilde{v}_0^i](t,x) \ge f^i(x, \tilde{v}_0(t,x), \tilde{v}_0(t,\cdot)) & \text{for } t > 0,\ x \in G,\ i \in S, \\ \tilde{v}_0^i(t,x) \ge h^i(x) & \text{for } t > 0,\ x \in \partial G,\ i \in S, \\ \tilde{v}_0^i(0,x) \ge h^i(x) & \text{for } x \in \bar{G},\ i \in S \end{cases} \tag{7}$$

are called a *lower* and an *upper function*, respectively, for parabolic problem (1), (2), (3) in $\bar{D}$. In a similar way, functions $\hat{u}_0 = \hat{u}_0(x)$ and $\hat{v}_0 = \hat{v}_0(x)$ regular in $\bar{G}$, satisfying the infinite systems of inequalities

$$\begin{cases} -\mathcal{L}^i[\hat{u}_0^i](x) \le f^i(x, \hat{u}_0(x), \hat{u}_0(\cdot)) & \text{for } \ x \in G, \ i \in S, \\ \hat{u}_0^i(x) \le h^i(x) & \text{for } \ x \in \partial G, \ i \in S, \end{cases} \tag{8}$$

$$\begin{cases} -\mathcal{L}^i[\hat{v}_0^i](x) \ge f^i(x, \hat{v}_0(x), \hat{v}_0(\cdot)) & \text{for } \ x \in G, \ i \in S, \\ \hat{v}_0^i(x) \ge h^i(x) & \text{for } \ x \in \partial G, \ i \in S \end{cases} \tag{9}$$

are called a *lower* and an *upper function*, respectively, for elliptic problem (4), (5) in $\bar{G}$.

We define

$$\mathcal{K} := \{(x, y, z) \colon \ x \in \bar{G}, \ y \in [m_0, M_0], \ z \in \langle u_0, v_0 \rangle\},$$

where $m_0 := (m_0^i)_{i \in S}$, $M_0 := (M_0^i)_{i \in S}$, $m_0^i := \min_{x \in \bar{G}} u_0^i(x)$, $M_0^i := \max_{x \in \bar{G}} v_0^i(x)$ and $\langle u_0, v_0 \rangle := \{\zeta \in L_S^p(G) \colon \ u_0(x) \le \zeta(x) \le v_0(x) \text{ for } x \in G\}$, if $u_0 \le v_0$.

Assumptions. We make the following assumptions:

(a) $\mathcal{L}$ is uniformly elliptic operator in $\bar{G}$, i.e., there exists a constant $\mu > 0$ such that
$$\sum_{j,k=1}^{m} a_{jk}^i(x) \xi_j \xi_k \ge \mu \sum_{j=1}^{m} \xi_j^2, \quad i \in S,$$
for all $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$, $x \in G$ and the adjoint operator $\mathcal{L}^\star$ to $\mathcal{L}$ exists.

(b) The functions $a_{jk} = (a_{jk}^i)_{i \in S}$, $b_j = (b_j^i)_{i \in S}$ for $j, k = 1, \ldots, m$ are of class $C_S^{0+\alpha}(\bar{G})$ and fulfil the Lipschitz condition on ∂G; also $a_{jk}^i(x) = a_{kj}^i(x)$ for every $i \in S$, $j, k = 1, \ldots, m$ and $x \in \bar{G}$.

(c) $h \in C_S^{2+\alpha}(\bar{G})$.

(d) There exists at least one ordered pair $u_0, v_0 \in C_S^{2+\alpha}(\bar{G})$ of a lower and an upper function for problem (4), (5) in $\bar{G}$, i.e.,
$$u_0(x) \le v_0(x) \qquad \text{for } x \in \bar{G}.$$

(e) $f(\cdot, y, z) \in C_S^{0+\alpha}(\bar{G})$ for $y \in [m_0, M_0]$, $z \in \langle u_0, v_0 \rangle$.

(f) For every $i \in S$, $x \in \bar{G}$, $y, \tilde{y} \in \mathcal{B}(S)$ and $z, \tilde{z} \in C_S(\bar{G})$
$$|f^i(x, y, z) - f^i(x, \tilde{y}, \tilde{z})| \le L_f(\|y - \tilde{y}\|_{\mathcal{B}(S)} + \|z - \tilde{z}\|_{C_S(\bar{G})}),$$
where $L_f > 0$ is a constant independent of $i \in S$.

(g) f^i is an increasing function with respect to the second and third variables for every $i \in S$.

(h) $u_0(x) \le h(x) \le v_0(x)$ for every $x \in \bar{G}$.

Remark. *Let us see that if assumptions (d) and (h) are fulfiled, then the functions* $u_0(x)$ *and* $v_0(x)$ *are a lower and an upper function for parabolic problem (1), (2), (3) in* $\bar{D}$.

By applying the monotone iterative method, we may proove the following theorem.

Theorem 1. *If assumption (a)–(h) hold then problem (1), (2), (3) has the unique solution* $\tilde{u} \in C_S^{2+\alpha}(\bar{D})$ $(0 < \alpha < 1)$ *within the sector* $\langle u_0, v_0 \rangle$.

We will outline a proof of the theorem (cf. [5, pp. 49–56, 61–62]). We start from the lower function u_0 and the upper function v_0 and we define by induction two monotone sequences $\{u_n\}$ and $\{v_n\}$ as regular solutions of problem (1), (2), (3) with u_{n-1} and v_{n-1} substituted for u in the right-hand sides of the system.

The essential part of the proof is showing that if the functions f^i fulfil assumptions (e) and (f) and the substituted function $\beta \in C_S^{0+\alpha}(\bar{D})$ (where $\beta = u_{n-1}$ and $\beta = v_{n-1}$, respectively), then the function $f^i(x, \beta(t,x), \beta(t,\cdot)) \in C_S^{0+\alpha}(\bar{D})$.

The function $\beta \in C_S^{0+\alpha}(\bar{D})$, so

$$\|\beta(t,x) - \beta(t',x')\|_{\mathcal{B}(S)} \le H_\beta(|t-t'|^{\frac{\alpha}{2}} + |x-x'|^\alpha),$$

where $H_\beta > 0$ is some constant. Therefore,

$$\begin{aligned}
|f^i(x, \beta(t,x), \beta(t,\cdot)) - f^i(x', \beta(t',x'), \beta(t',\cdot))| \le \\
\le |f^i(x, \beta(t,x), \beta(t,\cdot)) - f^i(x', \beta(t,x), \beta(t,\cdot))| + \\
+ |f^i(x', \beta(t,x), \beta(t,\cdot)) - f^i(x', \beta(t',x'), \beta(t',\cdot))| \le \\
\le H_f|x-x'|^\alpha + L_f(\|\beta(t,x) - \beta|(t',x')\|_{\mathcal{B}(S)} + \|\beta(t,\cdot) - \beta(t',\cdot)\|_{C_S(\bar{G})}) \le \\
\le H_f|x-x'|^\alpha + L_f H_\beta(|t-t'|^{\frac{\alpha}{2}} + |x-x'|^\alpha) + L_f H_\beta |t-t'|^{\frac{\alpha}{2}} \le \\
\le H(|t-t'|^{\frac{\alpha}{2}} + |x-x'|^\alpha),
\end{aligned}$$

where $H = H_f + 2L_f H_\beta$.

We obtain a solution of problem (1), (2), (3) as the limit of the sequences $\{u_n\}$ and $\{v_n\}$.

For elliptic problem (4), (5), the following existence theorem is known (cf. [9]):

Theorem 2. *If assumptions (a)–(g) hold, then problem (4), (5) has a solution* $\hat{u} \in C_S(\bar{G}) \cap C_S^2(G)$.

Let assumptions (a)–(h) hold. We study the behavior of solutions of the parabolic problem with conditions independent of t.

Theorem 3. *Let* $v_0(x)$ *be an upper function of elliptic problem (4), (5) in* $\bar{G}$ *and*

$$\begin{cases}
\frac{\partial u^i(t,x)}{\partial t} - \mathcal{L}^i[u^i](t,x) = f^i(x, u(t,x), u(t,\cdot)) & \text{for } t > 0, x \in G, i \in S, \\
u^i(t,x) = v_0^i(x) & \text{for } t > 0,\ x \in \partial G,\ i \in S, \\
u^i(0,x) = v_0^i(x) & \text{for } x \in G,\ i \in S.
\end{cases} \quad (10)$$

Then problem (10) has a solution $\tilde{v}(t,x) \in C_S^{2+\alpha}(\bar{D})$, which is nonincreasing with respect to t and $\tilde{v}(t,x) \leq v_0(x)$ in $\bar{D}$.

Proof. By virtue of Theorem 1, parabolic problem (10) has the unique solution $\tilde{v}(t,x) \in C_S^{2+\alpha}(\bar{D})$.

The function $v_0(x)$ is an upper function for elliptic problem (4), (5) and is independent of t, so $v_0(x)$ is a solution of the following problem:

$$\begin{cases} \frac{\partial v_0^i(x)}{\partial t} - \mathcal{L}^i[v_0^i](x) \geq f^i(x, v_0(x), v_0(\cdot)) & \text{for } t > 0, \ x \in G, \ i \in S, \\ v_0^i(x) = v_0^i(x) & \text{for } t > 0, \ x \in \partial G, \ i \in S, \\ v_0^i(x) = v_0^i(x) & \text{for } x \in G, \ i \in S. \end{cases} \tag{11}$$

Applying the Szarski theorem on weak partial differential-functional inequalities [8] to problems (10) and (11) we obtain:

$$\tilde{v}(t,x) \leq v_0(x) \text{ in } \bar{D}.$$

Now let

$$\tilde{v}_\tau(t,x) := \tilde{v}(t+\tau, x) \text{ for } \tau > 0.$$

The function $\tilde{v}_\tau(t,x)$ satisfies the following problem:

$$\begin{cases} \frac{\partial \tilde{v}_\tau^i(t,x)}{\partial t} - \mathcal{L}^i[\tilde{v}_\tau^i](t,x) = f^i(x, \tilde{v}_\tau(t,x), \tilde{v}_\tau(t,\cdot)) & \text{for } t > 0, \ x \in G, \ i \in S, \\ \tilde{v}_\tau^i(t,x) = v_0^i(x) & \text{for } t > 0, \ x \in \partial G, \ i \in S, \\ \tilde{v}_\tau^i(0,x) = \tilde{v}^i(\tau, x) \leq v_0^i(x) & \text{for } x \in G, \ i \in S. \end{cases} \tag{12}$$

Applying again the Szarski theorem on weak partial differential-functional inequalities [8] to problems (10) and (12) we obtain:

$$\tilde{v}_\tau(t,x) \leq \tilde{v}(t,x) \text{ in } \bar{D}.$$

Let $t_1, t_2 > 0$ and $t_1 \leq t_2$; for $\tau = t_2 - t_1$ there is

$$\tilde{v}(t_1, x) \geq \tilde{v}_\tau(t_1, x) = \tilde{v}(t_1 + \tau, x) = \tilde{v}(t_2, x) \text{ in } \bar{D},$$

so $\tilde{v}(t,x)$ is nonincreasing with respect to t. □

Theorem 4. *Let $u_0(x)$ be a lower function for elliptic problem (4), (5) in $\bar{G}$ and*

$$\begin{cases} \frac{\partial u^i(t,x)}{\partial t} - \mathcal{L}^i[u^i](t,x) = f^i(x, u(t,x), u(t,\cdot)) & \textit{for } t > 0, \ x \in G, \ i \in S, \\ u^i(t,x) = u_0^i(x) & \textit{for } t > 0, \ x \in \partial G, \ i \in S, \\ u^i(0,x) = u_0^i(x) & \textit{for } x \in G, \ i \in S. \end{cases} \tag{13}$$

Then problem (13) has a solution $\tilde{u}(t,x) \in C_S^{2+\alpha}(\bar{D})$, which is nondecreasing with respect to t and $\tilde{u}(t,x) \geq u_0(x)$ in $\bar{D}$.

Now we show the main result of this paper. We prove that the uniform limit at $t \to \infty$ of a solution of the parabolic problem is a solution of the elliptic problem.

Theorem 5. *If $u(t,x)$ is a regular uniformly bounded solution of the parabolic boundary initial value problem*

$$\begin{cases} \frac{\partial u^i(t,x)}{\partial t} - \mathcal{L}^i[u^i](t,x) = f^i(x,u(t,x),u(t,\cdot)) & \text{for } t>0,\ x\in G,\ i\in S, \\ u^i(t,x) = v_0^i(x) & \text{for } t>0,\ x\in \partial G,\ i\in S, \\ u^i(0,x) = v_0^i(x) & \text{for } x\in G,\ i\in S, \end{cases} \tag{14}$$

and there exists a $\hat{u}$ such that $\lim_{t\to\infty}\|u(t,\cdot)-\hat{u}(\cdot)\|_{C_S(\bar{G})}=0$, then the function $\hat{u}$ is a regular solution of elliptic boundary value problem (4), (5).

Proof. First we will prove that $\hat{u}$ is a weak solution of elliptic problem (4), (5).

Parabolic problem (14) has the unique regular solution by Theorem 1. Multiplying the equations in (14) by a test function $\xi \in C_0^\infty(G)$ and integrating, we get

$$\left\langle \frac{\partial u^i(t,x)}{\partial t}, \xi(x) \right\rangle - \left\langle \mathcal{L}^i[u^i](t,x), \xi(x) \right\rangle = \left\langle f^i(x,u(t,x),u(t,\cdot)), \xi(x) \right\rangle$$

for every $\xi \in C_0^\infty(G)$, $t>0$, $i\in S$, and using the adjoint operator $\mathcal{L}^{\star i}$ to $\mathcal{L}^i$

$$\mathcal{L}^{\star i}[g^i](x) := \sum_{j,k=1}^{m} \frac{\partial^2}{\partial x_j \partial x_k}(a^i_{jk}(x) g^i(x)) - \sum_{j=1}^{m} \frac{\partial}{\partial x_j}(b^i_j(x) g^i(x))$$

we obtain

$$\left\langle \frac{\partial u^i(t,x)}{\partial t}, \xi(x) \right\rangle - \left\langle u^i(t,x), \mathcal{L}^{\star i}[\xi](x) \right\rangle = \left\langle f^i(x,u(t,x),u(t,\cdot)), \xi(x) \right\rangle$$

for every $\xi \in C_0^\infty(G)$, $t>0$, $i\in S$. Next, we choose any $T>0$, and integrating with respect to t on the interval $[0,T]$ and multiplying by $\frac{1}{T}$ we get

$$\frac{1}{T}\int_0^T \left\langle \frac{\partial u^i(t,x)}{\partial t}, \xi(x) \right\rangle dt - \frac{1}{T}\int_0^T \left\langle u^i(t,x), \mathcal{L}^{\star i}[\xi](x) \right\rangle dt = \\ = \frac{1}{T}\int_0^T \left\langle f^i(x,u(t,x),u(t,\cdot)), \xi(x) \right\rangle dt \tag{15}$$

for every $\xi \in C_0^\infty(G)$, $i\in S$. Now, we pass to the limits in (15) as $T\to\infty$. For every $i\in S$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle \frac{\partial u^i(t,x)}{\partial t}, \xi(x) \rangle dt = \\ = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\partial}{\partial t}\left\langle u^i(t,x), \xi(x) \right\rangle dt = \lim_{T\to\infty} \frac{1}{T}\left\langle u^i(T,x), \xi(x) \right\rangle - \left\langle u^i(0,x), \xi(x) \right\rangle = \\ = \lim_{T\to\infty} \left\langle \frac{u^i(T,x)-u^i(0,x)}{T}, \xi(x) \right\rangle = 0,$$

because $\left|\frac{u^i(T,x)-u^i(0,x)}{T}\xi(x)\right| \leq \frac{1}{T}2C\max_{x\in\bar{G}}|\xi| \to 0$ as $T\to\infty$ and G is a bounded domain.

Next, for every $i \in S$

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T \left\langle u^i(t,x), \mathcal{L}^{\star i}[\xi](x)\right\rangle dt = \lim_{T\to\infty}\left\langle \frac{\int_0^T u^i(t,x)dt}{T}, \mathcal{L}^{\star i}[\xi](x)\right\rangle =$$
$$= \left\langle \lim_{T\to\infty}\frac{\int_0^T u^i(t,x)dt}{T}, \mathcal{L}^{\star i}[\xi](x)\right\rangle = \left\langle \lim_{T\to\infty}\frac{\frac{\partial}{\partial t}\int_0^T u^i(t,x)dt}{1}, \mathcal{L}^{\star i}[\xi](x)\right\rangle =$$
$$= \left\langle \lim_{T\to\infty} u^i(T,x), \mathcal{L}^{\star i}[\xi](x)\right\rangle = \left\langle \hat{u}^i(x), \mathcal{L}^{\star i}[\xi](x)\right\rangle.$$

And for every $i \in S$

$$\lim_{T\to\infty}\left|\frac{1}{T}\int\limits_0^T \langle f^i(x,u(t,x),u(t,\cdot)),\xi(x)\rangle dt - \left\langle f^i(x,\hat{u}(x),\hat{u}(\cdot)),\xi(x)\right\rangle\right| =$$
$$= \left|\left\langle \lim_{T\to\infty}\frac{\int_0^T f^i(x,u(t,x),u(t,\cdot)) - f^i(x,\hat{u}(x),\hat{u}(\cdot))dt}{T}, \xi(x)\right\rangle\right| =$$
$$= \left|\left\langle \lim_{T\to\infty}\frac{\frac{\partial}{\partial t}\int_0^T f^i(x,u(t,x),u(t,\cdot)) - f^i(x,\hat{u}(x),\hat{u}(\cdot))dt}{1}, \xi(x)\right\rangle\right| =$$
$$= \left|\left\langle \lim_{T\to\infty}(f^i(x,u(T,x),u(T,\cdot)) - f^i(x,\hat{u}(x),\hat{u}(\cdot))), \xi(x)\right\rangle\right| \leq$$
$$\leq \left\langle \lim_{T\to\infty}|f^i(x,u(T,x),u(T,\cdot)) - f^i(x,\hat{u}(x),\hat{u}(\cdot))|, |\xi(x)|\right\rangle \leq$$
$$\leq \left\langle \lim_{T\to\infty}(L_1\|u(T,x)-\hat{u}(x)\|_{\mathcal{B}(S)} + L_2\|u(T,\cdot)-\hat{u}(\cdot)\|_{C_S(G)}), |\xi(x)|\right\rangle =$$
$$= \langle 0, |\xi(x)|\rangle = 0,$$

so

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T \left\langle f^i(x,u(t,x),u(t,\cdot)),\xi(x)\right\rangle dt = \left\langle f^i(x,\hat{u}(x),\hat{u}(\cdot)),\xi(x)\right\rangle.$$

Therefore,

$$-\left\langle \hat{u}^i(x), \mathcal{L}^{\star i}[\xi](x)\right\rangle = \left\langle f^i(x,\hat{u}(x),\hat{u}(\cdot)),\xi(x)\right\rangle$$

and

$$-\left\langle \mathcal{L}^i[\hat{u}^i](x), \xi(x)\right\rangle = \left\langle f^i(x,\hat{u}(x),\hat{u}(\cdot)),\xi(x)\right\rangle,$$

so $\hat{u}$ is a weak solution of elliptic problem (4), (5).

Since $\hat{u}$ is bounded in G, $\hat{u} \in L^p_S(G)$ for any $p \in [1,\infty]$. Let $p > m$. Thus $f(x,\hat{u}(x),\hat{u}(\cdot)) \in L^p_S(G)$.

Now we consider problem

$$\begin{cases} -\mathcal{L}^i[w^i](x) = f^i(x, \hat{u}(x), \hat{u}(\cdot)) & \text{for } x \in G,\ i \in S, \\ w^i(x) = h^i(x) & \text{for } x \in \partial G,\ i \in S, \end{cases} \tag{16}$$

System (16) is a system of Dirichlet problems with a single equation each. We apply the Agmon–Douglis–Nirenberg theorem to every problem separately and get $w = (w^i)_{i=1}^{\infty} \in H_S^{2,p}(G)$. By the Agmon–Douglis–Nirenberg theorem, each of the problems included in (16) has the unique solution, so $\hat{u} = w \in H_S^{2,p}(G)$.

Because the Sobolev space $H^{2,p}(\bar{G})$ is continuously imbeddable in $C^{0+\alpha}(\bar{G})$ for $p > m$ and then

$$\|u^i\|_{C^{0+\alpha}(\bar{G})} \le C\|u^i\|_{H^{2,p}(G)}, \quad i \in S,$$

where C is independed of i [1, p. 144], we get

$$\hat{u} \in C_S^{0+\alpha}(\bar{G}).$$

Applying the Schauder theorem to (16) for $\hat{u} \in C_S^{0+\alpha}(\bar{G})$ separately for every $i \in S$, we obtain

$$\hat{u} \in C_S^{2+\alpha}(\bar{G}).$$ □

Now using Theorem 5 we show the stability of solutions of the parabolic problem with the conditions independent of time.

Theorem 6. *Let assummptions (a)–(h) hold.*

(i) *If $\overline{u}$ is a maximal regular solution of problem (4), (5) such that h fulfils $\overline{u}(x) \le h(x)$ for $x \in \bar{G}$, then the function $\overline{u}$ is an asymptotically stable solution from above of parabolic problem (1), (2), (3).*

(ii) *If $\underline{u}$ is a minimal regular solution of problem (4), (5) such that h fulfils $h(x) \le \underline{u}(x)$ for $x \in \bar{G}$, then the function $\underline{u}$ is an asymptotically stable solution from below of parabolic problem (1), (2), (3).*

(iii) *If u is the unique (i.e., $\overline{u} = \underline{u} = u$) regular solution of problem (4), (5), then the function u is an asymptotically stable solution of parabolic problem (1), (2), (3).*

Proof. (i) From the theorem on weak partial differential-functional inequalities [8], Theorem 3 and Theorem 5, each solution $u(t,x)$ of problem (1), (2), (3) such that

$$u_0(x) \le h(x) \le v_0(x) \text{ in } \bar{G}$$

satisfies

$$u_0(x) \le \tilde{u}(t,x) \le u(t,x) \le \tilde{v}(t,x) \le v_0(x) \text{ in } \bar{D},$$

where $\tilde{u}$, $\tilde{v}$ are solutions of problems (13) and (10), respectively.

The function $\overline{u}$ is a maximal regular solution of problem (4), (5); thus the solution $\tilde{v}(t,x)$ satisfies

$$\overline{u}(x) \le \tilde{v}(t,x) \le v_0(x).$$

Hence $\tilde{v}(t,x)$ is bounded from below and, by Theorem 3, is a nondecreasing function with respect to t, so $\lim_{t\to\infty}\tilde{v}(t,x)$ exists. From Theorem 1 we know that

$$\|\tilde{u}\|_{C_S^{2+\alpha}(\bar{D})} \le B$$

for $0<\alpha<1$, where $B>0$ is a constant indepedent of i, t, x, so $\{\tilde{u}(t,\cdot)\}_{t\in[0,\infty)}$ are equicontinuous functions. $\bar{G}$ is a compact set. Thus $\tilde{v}(t,x)$ converges uniformly as $t\to\infty$. By Theorem 5, this limit is a solution of problem (4), (5). $\overline{u}(x)\le\lim_{t\to\infty}\tilde{v}(t,x)$ and $\overline{u}(x)$ is a maximal solution of problem (4), (5). Consequently,

$$\lim_{t\to\infty}\tilde{v}(t,x)=\overline{u}(x).$$

Hence $u(t,x)$ such that $\overline{u}(x)\le u(t,x)\le\tilde{v}(t,x)$ converges uniformly to $\overline{u}(x)$, so $\overline{u}(x)$ is asymptotically stable solution from above of problem (1), (2), (3).

The proofs of (ii) and (iii) run similarly. □

Corollary. *If the function $h=h(t,x)$ depends on t, but is bounded by functions $\check{h}(x),\hat{h}(x)\in C_S^{2+\alpha}(\bar{G})$ independent of t such that $u_0(x)\le\check{h}(x)\le h(t,x)\le\hat{h}(x)\le v_0(x)$ for $t>0$, $x\in\bar{G}$ and an asymptotically stable solution u of*

$$\begin{cases} \frac{\partial u^i(t,x)}{\partial t}-\mathcal{L}^i[u^i](t,x)=f^i(x,u(t,x),u(t,\cdot)) & \text{for } t>0,\ x\in G,\ i\in S,\\ u^i(t,x)=h^i(t,x) & \text{for } t>0,\ x\in\partial G,\ i\in S,\\ u^i(0,x)=h^i(0,x) & \text{for } x\in G,\ i\in S \end{cases}$$

exists, then

$$\check{u}(x)\le u(x)\le\hat{u}(x) \quad \text{for } x\in\bar{G},$$

where $\check{u}$ is the minimal solution of

$$\begin{cases} -\mathcal{L}^i[u^i](t,x)=f^i(x,u(t,x),u(t,\cdot)) & \text{for } x\in G,\ i\in S,\\ u^i(t,x)=\check{h}^i(x) & \text{for } x\in\partial G,\ i\in S, \end{cases}$$

and $\hat{u}$ is the maximal solution of

$$\begin{cases} -\mathcal{L}^i[u^i](t,x)=f^i(x,u(t,x),u(t,\cdot)) & \text{for } x\in G,\ i\in S,\\ u^i(t,x)=\hat{h}^i(x) & \text{for } x\in\partial G,\ i\in S. \end{cases}$$

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Tomasz S. Zabawa
zabawa@mat.agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Kraków, Poland

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