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THE EXACT VALUES OF NONSQUARE CONSTANTS FOR A CLASS OF ORLICZ SPACES

Abstract. We extend the M_{Δ} -condition from [10] and introduce the Φ_{Δ} -condition at zero. Next we discuss nonsquare constant in Orlicz spaces generated by an N -function $\Phi(u)$ which satisfy Φ_{Δ} -condition. We obtain exact value of nonsquare constant in this class of Orlicz spaces equipped with the Luxemburg norm.

Keywords: nonsquare constant Orlicz space, Φ_{Δ} -condition.

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1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a Banach space, $S(X) = \{x: \|x\| = 1, x \in X\}$ denote the unit sphere of X . Following James [1], X is called *uniformly nonsquare* if there is a constant $0 < c < 1$ such that for $x, y \in S(X)$, we have

$$\|x + y\| \leq 2 - 2c, \quad \text{or} \quad \|x - y\| \leq 2 - 2c.$$

To discuss the property of uniform nonsquareness, Gao and Lau [2] introduce the following concept.

Definition 1. For a Banach space X , the parameter $J(X)$ is termed a nonsquare constant (in the sense of James) where

$$J(X) = \sup\{\min(\|x - y\|, \|x + y\|): x, y \in S(X)\} \quad (1)$$

It is easy to deduce that (cf. Gao and Lau [2]) X is uniformly nonsquare iff $J(X) < 2$.

Let:

$$\Phi(u) = \int_0^{|u|} \phi(t) dt, \quad \Psi(v) = \int_0^{|v|} \psi(s) ds$$

be a pair of complementary N -functions, i.e., $\phi(t)$ is right continuous, $\phi(0) = 0$, and $\phi(t) \nearrow \infty$ as $t \nearrow \infty$ and the same properties has ψ . Let (Ω, Σ, μ) be a measure space. The Orlicz space is defined by

$$L^\Phi(\Omega) = \{x: \Omega \rightarrow \mathbf{R}, \text{ measurable, } \rho_\Phi(\lambda x) d\mu < \infty \text{ for some } \lambda > 0\}.$$

Luxemburg norm (gauge norm) and *Orlicz norm* in $L^\Phi(G)$ are defined, respectively, by

$$\|x\|_{(\Phi)} = \inf \left\{ c > 0: \rho_\Phi \left(\frac{x}{c} \right) \leq 1 \right\}$$

and

$$\|x\|_\Phi = \inf_{k>0} \frac{1}{k} [1 + \rho_\Phi(kx)].$$

For simplicity, we use the notations L^Φ and $L^{(\Phi)}$ for the Orlicz spaces equipped with the Orlicz norm $(L^\Phi(\Omega), \|\cdot\|_\Phi)$ and the Orlicz spaces equipped with the Luxemburg norm $(L^{(\Phi)}(\Omega), \|\cdot\|_{(\Phi)})$, respectively. i.e. we denote $L^{(\Phi)} = (L^\Phi, \|\cdot\|_{(\Phi)})$ and $L^\Phi = (L^\Phi, \|\cdot\|_\Phi)$.

An N -function $\Phi(u)$ is said to satisfy the Δ_2 -condition for small u (for all $u \geq 0$ or for large u), in symbol $\Phi \in \Delta_2(0)$ ($\Phi \in \Delta_2$ or $\Phi \in \Delta_2(\infty)$), if there exists $u_0 > 0$ and $c > 0$ such that $\Phi(2u) \leq c\Phi(u)$ for $0 \leq u \leq u_0$ (for all $u \geq 0$ or for $u \geq u_0$). An \mathcal{N} -function $\Phi(u)$ is said to satisfy the ∇_2 -condition for small u (for all $u \geq 0$ or for large u), in symbol $\Phi \in \nabla_2(0)$ ($\Phi \in \nabla$ or $\Phi \in \nabla_2(\infty)$), if its complementary \mathcal{N} -function $\Psi \in \Delta_2(0)$ ($\Psi \in \Delta_2$ or $\Psi \in \Delta_2(\infty)$). The basic facts on Orlicz spaces can be found in Krasnoselskii and Rutickii [11], Lindenstrauss and Tzafriri [12], Rao and Ren [4] and Chen [3].

For nonsquare constant for the Orlicz function and sequence spaces equipped with Luxemburg norm with Φ satisfying the Δ_2 -condition, Ji and Wang [5] and Ji and Zhan [6] gave some expressions. Letter on, Y. Q. Yan [7] gave the corresponding results for the Orlicz spaces equipped with Orlicz norm with Φ satisfying the Δ_2 -condition. Using this expression, it is not easy to compute nonsquare constant for specific Orlicz spaces. For computation, Rao and Ren [9] and Y. Q. Yan [7, 8] gave estimates of nonsquare constants by Semenov and Simonenko indices of Φ , and obtain its exact value in some class of Orlicz spaces.

In view of some their results for latter use, we review the Semenov indices of Φ here:

$$\begin{aligned} \alpha_\Phi &= \liminf_{u \rightarrow \infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, & \beta_\Phi &= \limsup_{u \rightarrow \infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}; \\ \alpha_\Phi^0 &= \liminf_{u \rightarrow 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, & \beta_\Phi^0 &= \limsup_{u \rightarrow 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}; \end{aligned}$$

$$\bar{\alpha}_\Phi = \inf_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \bar{\beta}_\Phi = \sup_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}.$$

Using the Semenov indices, Ren gave the following estimate of nonsquare constant.

Lemma 1 (Rao and Ren [9, p. 54]). *Let Φ and Ψ be a pair of complementary N -function. Then:*

$$\begin{aligned} \max\left(\frac{1}{\alpha_\Phi}, 2\beta_\Phi\right) &\leq J\left(L^{(\Phi)}[0, 1]\right), \\ \max\left(\frac{1}{\bar{\alpha}_\Phi}, 2\bar{\beta}_\Phi\right) &\leq J(L^{(\Phi)}[0, \infty)), \\ \max\left(\frac{1}{\alpha_\Phi^0}, 2\beta_\Phi^0\right) &\leq J(l^{(\Phi)}). \end{aligned}$$

Lemma 2 (Rao and Ren [9, p. 66]). *Let Φ be an N -function, and Φ_s be the inverse of:*

$$\Phi_s^{-1}(u) = [\Phi^{-1}(u)]^{1-s} [\Phi_0^{-1}(u)]^s, \quad u \geq 0, 0 < s \leq 1$$

with $\Phi_0(u) = u^2$. If (Ω, Σ, μ) is a σ -finite space, then:

(i) *for $L^{(\Phi_s)}(\Omega)$ on (Ω, Σ, μ) with Luxemburg norm,*

$$J(L^{(\Phi_s)}(\Omega)) \leq 2^{1-\frac{s}{2}},$$

(ii) *the same result holds also for $L^{\Phi_s}(\Omega)$ with Orlicz norm.*

Definition 2. *An N -function Φ is said to satisfy $\Phi_\Delta(0)$, written as $\Phi \in \Phi_\Delta(0)$, if $p = \lim_{u \rightarrow 0} \frac{\ln \Phi(u)}{\ln u} < \infty$. An N -function Φ is said to satisfy $\Phi_\Delta(\infty)$, written as $\Phi \in \Phi_\Delta(\infty)$, if $p = \lim_{u \rightarrow \infty} \frac{\ln \Phi(u)}{\ln u} < \infty$.*

By the definition of N -function, we easily see that $p \geq 1$. Using a similar method as [10], we have

Lemma 3.

(i) *If $\Phi \in \Phi_\Delta(0)$ and $\lim_{u \rightarrow 0} \frac{\ln \Phi(u)}{\ln u} = p > 1$, then*

$$\lim_{u \rightarrow 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = 2^{-\frac{1}{p}},$$

where $\Phi^{-1}(u)$ is an inverse function of Φ .

(ii) *If $\Phi \in \Phi_\Delta(\infty)$ and $\lim_{u \rightarrow \infty} \frac{\ln \Phi(u)}{\ln u} = p > 1$, then*

$$\lim_{u \rightarrow \infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = 2^{-\frac{1}{p}},$$

Proof. (ii) was proved in [10]. The authors use the definition of limit on $\lim_{u \rightarrow 0} \frac{\ln \Phi(u)}{\ln u} = p$, and then give the estimate of $\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ to get the result. (i) is similar to (ii). But we adjust the proof in [10] and prove (ii) here.

Noting $\lim_{u \rightarrow 0} \frac{\ln \Phi(u)}{\ln u} = p$ iff $\lim_{u \rightarrow 0} \frac{\ln \Phi^{-1}(u)}{\ln u} = \frac{1}{p}$. Let $\beta(u) = \ln \Phi^{-1}(u) - \frac{1}{p} \ln u$. Then $\lim_{u \rightarrow 0} \frac{\beta(u)}{\ln u} = 0$ and

$$\Phi^{-1}(u) = u^{\frac{1}{p}} e^{\beta(u)}.$$

So

$$\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = \frac{u^{\frac{1}{p}} e^{\beta(u)}}{(2u)^{\frac{1}{p}} e^{\beta(2u)}} = 2^{-\frac{1}{p}} \frac{\left(e^{\frac{\beta(u)}{\ln u}} \right)^{\ln u}}{\left(e^{\frac{\beta(2u)}{\ln 2u}} \right)^{\ln 2u}}.$$

Noting that $\ln u - \ln 2u = \ln \frac{1}{2}$ and $\lim_{u \rightarrow 0} e^{\frac{\beta(u)}{\ln u}} = \lim_{u \rightarrow 0} e^{\frac{\beta(2u)}{\ln 2u}} = 1$, we have

$$\lim_{u \rightarrow 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = 2^{-\frac{1}{p}}. \quad \square$$

The next lemma, which has been proved in [10], is useful for our goal.

Lemma 4. *Let $\Phi \in \Phi_{\Delta}(\infty)$ and Ψ be its complementary N -function, $\lim_{u \rightarrow \infty} \frac{\ln \Phi(u)}{\ln u} = p > 1$ and $\Phi_0(u) = u^{p_0}$ where $p_0 > 1$. Then:*

- (i) $\lim_{v \rightarrow \infty} \frac{\ln \Psi(v)}{\ln v} = q > 1$, where $\frac{1}{p} + \frac{1}{q} = 1$,
- (ii) $\lim_{u \rightarrow \infty} \frac{\ln \Phi^{-1}(u)}{\ln u} = \frac{1}{p}$,
- (iii) $\lim_{u \rightarrow \infty} \frac{\ln \Phi(u) \Phi_0(u)}{\ln u} = p + p_0$,
- (iv) $\lim_{u \rightarrow \infty} \frac{\Phi_0(\Phi(u))}{\ln u} = pp_0$.

Lemma 4 is also true for $\Phi \in \Phi_{\Delta}(0)$.

2. NONSQUARE CONSTANTS FOR ORLICZ SPACES WITH LUXEMBURG NORM

Now we give our main results.

Theorem 1. *Let $\Phi \in \Phi_{\Delta}(\infty)$ and $\lim_{u \rightarrow \infty} \frac{\ln \Phi(u)}{\ln u} = p > 1$. Then*

$$J\left(L^{(\Phi)}[0, 1]\right) = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right). \quad (2)$$

Proof. First, by Lemma 1, we have

$$J(L^{(\Phi)}[0, 1]) \geq \max\left(\frac{1}{\alpha_\Phi}, 2\beta_\Phi\right).$$

By Lemma 3, we have $\alpha_\Phi = \beta_\Phi = 2^{-\frac{1}{p}}$. Hence

$$J(L^{(\Phi)}[0, 1]) \geq \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right).$$

Next, we will show the inequality \leq in (2). Now let $1 < p \leq 2$. We choose l such that $1 < l < p \leq 2$ and take $s = \frac{2(p-l)}{p(2-l)}$. Obviously, $0 < s < 1$. Let M be the inverse function of $M^{-1}(u) = u^{-\frac{1}{2(1-s)}}[\Phi^{-1}(u)]^{\frac{1}{1-s}}$ and $\Phi_0(u) = u^2$. Then $\Phi^{-1}(u) = [M^{-1}(u)]^{1-s}[\Phi_0^{-1}(u)]^s$, i.e. $\Phi_s^{-1} = [M^{-1}]^{1-s}[\Phi_0^{-1}]^s = \Phi^{-1}$. Therefore, by Lemma 2, we have

$$J(L^{(\Phi)}[0, 1]) = J(L^{(\Phi_s)}[0, 1]) < 2^{1-\frac{s}{2}} = 2^{1-\frac{p-l}{p(2-l)}}.$$

Since $\lim_{l \rightarrow 1} \frac{p-l}{p(2-l)} = \frac{p-1}{p}$, we get

$$J(L^{(\Phi)}[0, 1]) \leq 2^{\frac{1}{p}} = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right).$$

Let $2 < p < \infty$, we choose l such that $2 < p < l < \infty$ and take $s = \frac{2(l-p)}{p(l-2)}$. Obviously, $0 < s < 1$. Let M be the inverse function of $M^{-1}(u) = u^{-\frac{1}{2(1-s)}}[\Phi^{-1}(u)]^{\frac{1}{1-s}}$ and $\Phi_0(u) = u^2$. Then $\Phi^{-1}(u) = [M^{-1}(u)]^{1-s}[\Phi_0^{-1}(u)]^s$, i.e. $\Phi_s^{-1} = [M^{-1}]^{1-s}[\Phi_0^{-1}]^s = \Phi^{-1}$. Therefore,

$$J(L^{(\Phi)}[0, 1]) = J(L^{(\Phi_s)}[0, 1]) < 2^{1-\frac{s}{2}} = 2^{1-\frac{l-p}{p(l-2)}}.$$

Since $\lim_{l \rightarrow \infty} \frac{l-p}{p(l-2)} = \frac{1}{p}$, we get

$$J(L^{(\Phi)}[0, 1]) \leq 2^{1-\frac{1}{p}} = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right). \quad \square$$

For Orlicz function spaces $L^{(\Phi)}[0, \infty)$ and Orlicz sequence spaces $l^{(\Phi)}$, we have similar results.

Theorem 2. Let $\Phi \in \Phi_\Delta(\infty)$ and $\lim_{u \rightarrow \infty} \frac{\ln \Phi(u)}{\ln u} = p > 1$. Then

$$J(L^{(\Phi)}[0, \infty)) = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right). \tag{3}$$

Proof. By Lemma 1, we get

$$J(L^{(\Phi)}[0, \infty)) \geq \max\left(\frac{1}{\alpha_\Phi}, 2\bar{\beta}_\Phi\right).$$

By Lemma 3, we have $\bar{\alpha}_\Phi \leq \alpha_\Phi = 2^{-\frac{1}{p}}, \bar{\beta}_\Phi \geq \beta_\Phi = 2^{-\frac{1}{p}}$.

So

$$J(L^{(\Phi)}[0, \infty)) \geq \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right).$$

The rest of the proof is similar to the proof of Theorem 1. □

Theorem 3. Let $\Phi \in \Phi_\Delta(0)$ and $\lim_{u \rightarrow 0} \frac{\ln \Phi(u)}{\ln u} = p > 1$.

Then

$$J(l^{(\Phi)}) = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right). \tag{4}$$

Proof. The proof is similar to the proof of Theorem 1. □

Example 1. Let $L^p \in \{L^p[0, 1], L^p[0, \infty), l^p\}$, $1 < p < \infty$.

Then

$$J(L^p) = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right).$$

In fact, if we take $\Phi(u) = |u|^p$, then the results is easy to be deduce from Theorems 1, 2 and 3.

Example 2. Let $\Phi(u) = |u|^{2p} + 2|u|^p$, $1 < p < \infty$. Then $\lim_{u \rightarrow \infty} \frac{\ln \Phi(u)}{\ln u} = 2p > 1$ and $\lim_{u \rightarrow 0} \frac{\ln \Phi(u)}{\ln u} = p > 1$. By Theorems 1, 2 and 3, we have $J(l^{(\Phi)}) = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right)$, $J(L^{(\Phi)}[0, 1]) = J(L^{(\Phi)}[0, \infty)) = \max\left(2^{\frac{1}{2p}}, 2^{1-\frac{1}{2p}}\right)$.

Remark 1. Since $\phi(u) = \Phi'(u) = 2pu^{2p-1} + 2pu^{p-1}$ is not convex or concave on $[0, \infty)$, so computation method of Y. Q. Yan in [6] and [8] is not suitable for Example 2.

Example 3. Let $\Phi_{p,r} = |u|^p \ln^r(1 + |u|)$. Then $\lim_{u \rightarrow \infty} \frac{\ln \Phi(u)}{\ln u} = p > 1$ and $\lim_{u \rightarrow 0} \frac{\ln \Phi(u)}{\ln u} = p + r > 1$. By Theorems 1, 2 and 3, we have $J(l^{(\Phi)}) = \max(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}})$, $J(L^{(\Phi)}[0, 1]) = J(L^{(\Phi)}[0, \infty)) = \max(2^{\frac{1}{p+r}}, 2^{1-\frac{1}{p+r}})$.

Example 4. Let Φ be a function defined as the inverse of

$$\Phi^{-1}(u) = [\ln(1 + u)]^{\frac{1}{2p}} u^{\frac{1}{4}}, \quad u \geq 0, \quad 1 < p < \infty.$$

Then:

$$\begin{aligned} \lim_{u \rightarrow \infty} \frac{\ln \Phi^{-1}(u)}{\ln u} &= \lim_{u \rightarrow \infty} \frac{\frac{1}{2p} \ln \ln(1 + u) + \frac{1}{4} \ln u}{\ln u} = \frac{1}{4}, \\ \lim_{u \rightarrow 0} \frac{\ln \Phi^{-1}(u)}{\ln u} &= \frac{1}{4} + \frac{1}{2p} \lim_{u \rightarrow 0} \frac{\frac{1}{\ln(1+u)} \cdot \frac{1}{1+u}}{\frac{1}{u}} = \frac{1}{4} + \frac{1}{2p}. \end{aligned}$$

By Lemma 4, we have $\lim_{u \rightarrow \infty} \frac{\ln \Phi(u)}{\ln u} = 4$, $\lim_{u \rightarrow 0} \frac{\ln \Phi(u)}{\ln u} = \frac{1}{\frac{1}{4} + \frac{1}{2p}}$.

So

$$J(L^{(\Phi)}[0, 1]) = J(L^{(\Phi)}[0, \infty)) = \max\left\{2^{\frac{1}{4}}, 2^{1-\frac{1}{4}}\right\} = 2^{\frac{3}{4}},$$
$$J(l^{(\Phi)}) = \max\left\{2^{\frac{1}{4}+\frac{1}{2p}}, 2^{\frac{3}{4}-\frac{1}{2p}}\right\}.$$

Remark 2. Example 4 improve the Example 8 in Chapter II of [9, p. 71], because in [9], the author didn't give the exact value of $J(l^{(\Phi)})$.

REFERENCES

- [1] James R. C.: *Uniformly non-square Banach spaces*. Ann. of Math. (1964) 80, 542–550.
- [2] Gao J., Lau K. S.: *On the geometry of spheres in normed linear spaces*. J. Austral. Math. Soc. Ser. A 48(1990), 101–112.
- [3] Chen S. T.: *Geometry of Orlicz Spaces*. Dissertationes Math. Warszawa, 1996, 356: 1–204.
- [4] Rao M. M., Ren Z. D.: *Theory of Orlicz spaces*. New York, Marcel Dekker 1991.
- [5] Ji D. H., Wang T. F.: *Nonsquare constants of normed spaces*. Acta. Sci. Math. (Szeged) 59 (1994), 719–723.
- [6] Ji D. H., Zhan D. P.: *Some equivalent representations of nonsquare constants and its applications*. Northeast. Math. J. (4) 15 (1999), 439–444.
- [7] Yan Y. Q.: *On the Nonsquare Constants of Orlicz Spaces with Orlicz Norm*. Canad. J. Math. Vol. 55 (1) (2003), 204–224.
- [8] Yan Y. Q.: *Computation of Nonsquare Constants of Orlicz Spaces*. J. Austral. Math. Soc. (to appear)
- [9] Rao M. M., Ren Z. D.: *Applications of Orlicz Spaces*. Marcel Dekker, New York, 2002.
- [10] Han J., Li X.: *On Exact Value of Packing for a Class of Orlicz Spaces*. (Chinese), Journal of Tongji Univ. 30 (2002) 7, 895–899.
- [11] Krasnosel'skii M. A., Rutickii Ya. B.: *Convex Functions and Orlicz Space*. Groningen, P. Noordhoff Ltd. 1961.
- [12] Lindenstrauss J., Tzafriri L.: *Classical Banach Spaces, I and II*. Berlin, Springer 1977 and 1979.

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