Opuscula Mathematica • Vol. 25 • No. 2 • 2005

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ON THE STRUCTURE OF CHARACTERISTIC SURFACES RELATED WITH PARTIAL DIFFERENTIAL EQUATIONS OF FIRST AND HIGHER ORDERS. PART 1.

Abstract. The geometric structure of characteristic surfaces related with partial differential equations of first and higher orders is studied making use the vector field technique on hypersurfaces. It is shown, that corresponding characteristics are defined uniquely up to some smooth tensor fields, thereby supplying additonal information about the suitable set of their solutions. In particular, it may be very useful for studying asymptotic properties of solutions to our partial differential equations under some boundary conditions.

Keywords: characteristic surface, vector fields, tangency, Monge cone, tensor fields.

Mathematics Subject Classification: Primary 34A30, 34B05, 35B12, 35A15; Secondary 35J50, 35J65, 46T15, 34B15.

1. INTRODUCTION AND GEOMETRIC SETTING

Assume we are given a general partial differential equation of the first order in the form

$$H(x;\, u, u_x) = 0, \tag{1.1}$$

where $H\colon \mathbb{R}^n \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}$ is a twice-differentiable function. We will call a mapping $u \in C^2(\mathbb{R}^n;\mathbb{R})$ by the solution to (1.1) if it transforms the expression (1.1) into identity. The equation (1.1) allows a geometric interpretation [1, 2, 6] based still on Monge's classical considerations. Namely, the equation (1.1) defines naturally the so called Monge cone of normals $K(x;u) \subset \mathbb{R}^n$ at each point $(x;u) \in \mathbb{R}^{n+1}$. Consider now hypersurface $\bar{S}_H \subset \mathbb{R}^{n+1}$ such that

$$\bar{S}_H := \{(x;u) \in \mathbb{R}^n \times \mathbb{R} : \quad u = \psi(x)\} \tag{1.2}$$

for some $\psi \in C^2(\mathbb{R}^n;\mathbb{R})$. Let $n(x) = (-1, \nabla u(x))|_{u=\psi(x)} \in \mathbb{R}^{n+1}$ be the set of normal vectors to $\bar{S}_H$. Then the surface $\bar{S}_H$ like (1.2) will be a solution to (1.1) if and only if the normal $n(x) \in K(x;u)$ at each $(x;u) \in \bar{S}_H$. One can also define a family of hyperplanes orthogonal to the cone $K(x;u)$ at $(x;u) \in \bar{S}_H$ and take the enveloping them set $K^*(x;u) \subset \mathbb{R}^{n+1}$ which is called a dual Monge cone. Then the surface will be a solution to (1.1) if it is tangent to the dual Monge cone $K^*(x;u)$ at each point $(x;u) \in \mathbb{R}^{n+1}$. It is seen that directions, along which our surface S_H is tangent to the Monge cone K^*, generate some vector fields on $\bar{S}_H$ whose orbits fulfill completely the surface $\bar{S}_H$. For constructing these vector fields on $\bar{S}_H$ let us consider our surface $\bar{S}_H$ as a surface imbedded into some surface $S_H \subset \mathbb{R}^{2n} \times \mathbb{R}^1 \ni (x;u,p)$, where $p := u_x \in \mathbb{R}^n$ for all $(x;u) \in \mathbb{R}^{n+1}$, that is

$$S_H := \{(x;u,p) \in \mathbb{R}^{2n} \times \mathbb{R} : \quad H(x;u,p) = 0\} \tag{1.3}$$

under the condition that $\|\bigtriangledown_p H\| = 0$. There exist many of vector fields on S_H in the general form

$$dx/d\tau = a_H(x;u,p), \quad dp/d\tau = b_H(x;u,p), \quad du/d\tau = c_H(x;u,p), \tag{1.4}$$

where $\tau \in \mathbb{R}$ is an evolution parameter, $a_H, b_H \in C^1(\mathbb{R}^{2n+1};\mathbb{R}^n)$ and $c_H \in C^1(\mathbb{R}^{2n+1};\mathbb{R})$ are some expressions depending on the function $H \in C^2(\mathbb{R}^{2n+1};\mathbb{R})$, defining the surface S_H (1.3). In particular, from the tangency condition of vector field (1.4) to the surface S_H one gets easily that

$$\left\langle \frac{\partial H}{\partial x}, a_H \right\rangle + \left\langle \frac{\partial H}{\partial p}, b_H \right\rangle + \frac{\partial H}{\partial u} c_H = 0 \tag{1.5}$$

for all $(x;u,p) \in S_H$. Assume that at some while of time $\tau \in \mathbb{R}$ the vector field (1.4), if reduced upon the sub-surface $\bar{S}_H \subset S_H$, satisfies the tangency condition concerning the corresponding dual Monge cone K^*. The latter can be realized if to take into account that upon the sub-surface $\bar{S}_H \subset S_H$ the following constraint

$$du - \langle p, dx \rangle = 0 \tag{1.6}$$

holds for all $(x;u) \in \bar{S}_H \subset S_H$. As a result of (1.6) one gets easily that along the vector field (1.4) the expression

$$c_H - \langle p, a_H \rangle = 0 \tag{1.7}$$

holds for all $(x;u,p) \in S_H$. Thus the condition (1.5), owing (1.7), will take the form

$$\left\langle \frac{\partial H}{\partial x} + p\frac{\partial H}{\partial u}, a_H \right\rangle + \left\langle \frac{\partial H}{\partial p}, b_H \right\rangle = 0, \tag{1.8}$$

being true for all $(x;u,p) \in S_H$. The simplest way to satisfy the identity condition (1.8) is to take a smooth tensor field $s^{(1|1)} \in C^1(\mathbb{R}^{n+1} \times \mathbb{R}^n; \mathbb{R}^n \otimes \mathbb{R}^n)$ and put, by definition,

$$b_H := -\left\langle s^{(1|1),*}, \frac{\partial H}{\partial x} + p\frac{\partial H}{\partial u} \right\rangle, \quad a_H := \left\langle s^{(1|1)}, \frac{\partial H}{\partial p} \right\rangle, \tag{1.9}$$

where $s^{(1|1),*}$ is the adjoint to $s^{(1|1)}$ tensor, being the generalized expressions modeling the well known [4, 5, 6] classical Hamilton type equations, generated by the Hamilton function (1.1).

Definition 1.1. *The vector fields* (1.4) *defined by means of expressions* (1.9) *are called characteristics of the equation* (1.1).

Denote now by $Or(x;u,p)$ the orbit of the vector field (1.4), (1.9) through a point $(x;u,p) \in \mathbb{R}^{2n+1}$.

Definition 1.2. *The set* $\Sigma_H := \cup_{(x;u,p)\in\not\Sigma} Or(x;u,p)$*, where a set* $\Sigma \subset S_H$*, is called a characteristic strip of the equation* (1.1).

Remark 1.1. *It is to be noted here that the choice* (1.9) *is not unique which satisfies the identity condition* (1.8).

Before proceeding further we need to formulate the following lemma.

Lemma 1.1. *If a function* $u \in C^2(\mathbb{R}^{n+1};\mathbb{R})$ *with the graph* $\Gamma_H := \{(x;u,u_x) \in \mathbb{R}^{2n+1}\}$ *solves the equation* (1.1) *and its characteristic strip* Σ_H *has a common point with the graph* $\Gamma_H \subset S_H$*, then the whole strip* $\Sigma_H \subset \Gamma_H$.

Proof. Take a point $(x^{(0)};u^{(0)},p^{(0)}) \in \Sigma_H \subset \Gamma_H$ and put $\tau_0 \in \mathbb{R}$ being the value of parameter $\tau \in \mathbb{R}$, corresponding to this point. Denote by $\bar{l}_H$ a curve in $\mathbb{R}^n$, such that

$$dx/d\tau = \left\langle s^{(1|1)}, \partial H/\partial p \right\rangle, \ x|_{\tau=\tau_0} = x^{(0)}. \tag{1.10}$$

Let now $l_H \subset \Gamma_H$ be a curve lying over the curve $\bar{l}_H$, that is its equations have the form:

$$\begin{gathered} dx/d\tau = \left\langle s^{(1|1)}, \partial H/\partial p \right\rangle, \quad dp/d\tau = \left\langle \partial(\bigtriangledown u)/\partial x, \left\langle s^{(1|1)}, \partial H/\partial p \right\rangle \right\rangle, \\ du/d\tau = \left\langle \partial u/\partial x, \left\langle s^{(1|1)}, \partial H/\partial p \right\rangle \right\rangle, \end{gathered} \tag{1.11}$$

being considered at $(x;u(x),p(x)) \in \Gamma_H$. For the theorem to be proved it is enough to state that the vector-function (1.11) satisfies the characteristic set of equations (1.4) at conditions (1.7) and (1.9) . Since $(x^{(0)};u^{(0)},p^{(0)}) \in l_H \subset \Sigma_H$, owing to the existence and unicity theorem for the characteristic equations (1.4) one can assert that $\Sigma_H \subset \Gamma_H$. But it is easy to see that

$$\left\langle \partial u/\partial x, \left\langle s^{(1|1)}, \partial H/\partial p \right\rangle \right\rangle = \left\langle p, \left\langle s^{(1|1)}, \partial H/\partial p \right\rangle \right\rangle = c_H, \tag{1.12}$$

and also, owing to (1.1), upon Γ_H

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial u} p + \left\langle \frac{\partial H}{\partial p}, \frac{\partial p}{\partial x} \right\rangle = 0. \tag{1.13}$$

Taking now into account (1.13) and (1.11), one finds easily that at $p = \partial u(x)/\partial x$, $x \in \mathbb{R}^n$, $\langle \partial(\bigtriangledown u)/\partial x, \langle s^{(1|1)}, \partial H/\partial p \rangle \rangle = -\langle s^{(1|1)}, *, \frac{\partial H}{\partial x} + p\frac{\partial H}{\partial u} \rangle = b_H$, thereby finishing the proof. □

As a consequence of this Lemma, one can formulate the following theorem.

Theorem 1.1. *A function $u \in C^2(\mathbb{R}^n;\mathbb{R})$ solves the equation* (1.1) *if and only if its graph Γ_H is woven of characteristic strips $\Sigma_H \subset S_H$, generated by vector field* (1.4), (1.7) *and* (1.9).

Proof. Let a function $u \in C^2(\mathbb{R}^n;\mathbb{R})$ solves the equation (1.1). Through points if its graph Γ_H one can draw a characteristic strip Σ_H, belonging to Γ_H owing to the Lemma 1.1. Conversely, let $u \in C^2(\mathbb{R}^n;\mathbb{R})$ and graph Γ_H is made of characteristic bounds Σ_H. Since at the same time there holds the condition $H(x;u,p) = 0$ along characteristics Σ_H the relationship $p = u_x$ holds too at each point $(x;u,p) \in \Sigma_H \subset \Gamma_H$. Thereby, the function $u \in C^2(\mathbb{R}^n;\mathbb{R})$ solves the equation (1.1). □

2. CHARACTERISTIC SURFACES RELATED WITH PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Consider the following differential expression

$$H(x;u,u_x,u_{xx},\ldots,u_{mx}) = 0, \tag{2.1}$$

where a function $u \in C^{m+1}(\mathbb{R}^n;\mathbb{R})$ turns (2.1) into the identity. Such a function is called the solution to (2.1), where the function $H \in C^{m+1}(\mathbb{R}^{n+1}\times\mathbb{R}^n\times\mathbb{R}^{\frac{n(n+1)}{2}},\ldots;\mathbb{R})$ satisfies the natural condition $|\bigtriangledown_{u_{mx}}H| = 0$. Similarly to the reasoning in Section 1 consider a surface $S_H \subset \mathbb{R}^{n+1} \times \mathbb{R}^{N(m,n)}$, where $N_{(m,n)} \in \mathbb{Z}_+$ counts the whole number of independent partial derivatives u_{kx}, where $k = \overline{1,m}$:

$$S_H := \left\{ (x;u,p^{(1)},\ldots,p^{(m)}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{N(m,n)} \colon H(x;u,p^{(1)},\ldots,p^{(m)}) = 0 \right\} \tag{2.2}$$

and $\left|\nabla_{p^{(m)}}H\right| \neq 0$ on S_H. Now we proceed to constructing a characteristic strip $\Sigma_H \subset S_H$ by means of orbits of vector fields:

$$\begin{aligned} dx/d\tau &= a_x(x;u,p^{(1)},\ldots,p^{(m)}), \\ du/d\tau &= c_H(x;u,p^{(1)},\ldots,p^{(m)}), \\ dp^{(j)}/d\tau &= b_H^{(j)}(x;u,p^{(1)},\ldots,p^{(m)}), \end{aligned} \tag{2.3}$$

where a multi-index $j \in \mathbb{Z}_+^n$ and $|j| = \overline{1,m}$, and $(x;u,p^{(1)},\ldots,p^{(m)}) \in \mathbb{R}^{n+1}\times\mathbb{R}^{N_{(m,n)}}$. The right handsides of (2.3) should be now chosen in such a way that for all admissible $\tau \in \mathbb{R}$ points $(x(\tau);u(\tau),p^{(1)}(\tau),\ldots,p^{(m)}) \in S_H$ and, simultaneously, the set of conditions:

$$\begin{aligned} \left\langle p^{(1)}, dx \right\rangle - du &= 0, \\ \left\langle p^{(1)}, dx \right\rangle - dp^{(1)} &= 0, \\ \left\langle p^{(m)}, dx \right\rangle - dp^{(m-1)} &= 0 \end{aligned} \tag{2.4}$$

together with

$$\left\langle \frac{\partial H}{\partial x}, a_H \right\rangle + \sum_{|j|=0}^{m} \left\langle \frac{\partial H}{\partial p^{(j)}}, b_H^{(j)} \right\rangle + \frac{\partial H}{\partial u} c_H = 0 \tag{2.5}$$

to be satisfied. From (2.4) one easily finds that:

$$\begin{aligned} &\left\langle p^{(1)}, a_H \right\rangle - c_H = 0, \\ &\left\langle p^{(1)}, a_H \right\rangle - b_H^{(1)} = 0, \\ &\qquad \cdots \\ &\left\langle p^{(m)}, a_H \right\rangle - b_H^{(m-1)} = 0 \end{aligned} \tag{2.6}$$

for all points on S_H. For the condition (2.5) to be satisfied identically on S_H, one can put owing to (2.6) that:

$$\begin{aligned} a_H &:= \left\langle s^{(1|m)}, \frac{\partial H}{\partial p^{(m)}} \right\rangle, \\ b_H^{(m)} &:= \left\langle s^{(1|m),*}, \frac{\partial H}{\partial x} + p^{(1)} \frac{\partial H}{\partial u} + \left\langle p^{(2)}, \frac{\partial H}{\partial p^{(1)}} \right\rangle + \cdots + \left\langle p^{(m)}, \frac{\partial H}{\partial p^{(m-1)}} \right\rangle \right\rangle, \end{aligned} \tag{2.7}$$

where $s^{(1|m)} \in C^1(\mathbb{R}^{n+1} \times \mathbb{R}^{N_{(m,n)}}; \mathbb{R}^n \otimes \mathbb{R}^{N_{(m,n)}})$ is some arbitrarily chosen tensor field. Thereby the vector field (2.3) is now completely determined, generating the characteristic strips Σ_H in the unique way. Following the reasonings similar to those from Section 1, one can formulate the following theorem.

Theorem 2.1. *A function* $u \in C^2(\mathbb{R}^n; \mathbb{R})$ *solves the partial differential equation* (2.1) *iff its graph* $\Gamma_H = \{(x; u, u_x) \in \mathbb{R}^{2n+1}\}$ *is woven of characteristics strips* Σ_H*, generated by vector field* (2.3) *under conditions* (2.6) *and* (2.7)*.*

3. THE STRUCTURE OF CHARACTERISTIC SURFACES RELATED WITH SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

Consider the following system of two partial differential equations:

$$\begin{aligned} H^{(1)}(x; u^{(1)}, u^{(2)}, u_x^{(1)}, u_x^{(2)}) = 0, \\ H^{(2)}(x; u^{(1)}, u^{(2)}, u_x^{(1)}, u_x^{(2)}) = 0, \end{aligned} \tag{3.1}$$

where scalar functions $H^{(1)}, H^{(2)} \in C^2(\mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n}; \mathbb{R})$ one mapping $u_1, u_2 \in C^2(\mathbb{R}^n; \mathbb{R})$ are called solutions to (3.1), if they turn it into identity. Define, as above, the following surface

$$\begin{aligned} S_H := \Big\{ &(x; u^{(1)}, u^{(2)}, p_1, p_2) \in \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n} : \\ &0 = H^{(1)}(x; u^{(1)}, u^{(2)}, p_1, p_2),\ 0 = H^{(2)}(x; u^{(1)}, u^{(2)}, p_1, p_2) \Big\} \end{aligned} \tag{3.2}$$

under the condition $\left|\nabla_p H^{(j)}\right| \neq 0$, $j=\overline{1,2}$, and construct upon it some vector fields both tangent to S_H and compatible with the following set of natural Monge type constraints:

$$\begin{aligned} du^{(1)} - \langle p_1, dx\rangle &= 0,\\ du^{(2)} - \langle p_2, dx\rangle &= 0, \end{aligned} \tag{3.3}$$

valid for all points of some graph $\Gamma_H := \left\{(x; u^{(1)}(x), u^{(2)}(x), u_x^{(1)}(x), u_x^{(2)}(x))\right\} \subset S_H$, generated by two mappings $u^{(1)}, u^{(2)} \in C^2(\mathbb{R}^m;\mathbb{R})$. We will seek our tangent to S_H vector fields in the following general form:

$$\begin{gathered} dx/d\tau = a_H, \quad du^{(1)}/d\tau = c_H^{(1)}, \quad du^{(2)}/d\tau = c_H^{(2)},\\ dp_1/d\tau = b_H^{(1)}, \quad dp_2/d\tau = b_H^{(2)}, \end{gathered} \tag{3.4}$$

where $\tau \in \mathbb{R}$ is an evolution parameter, mappings a_H and $b_H^{(j)} \in C^2(\mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n};\mathbb{R}^n)$, $j=\overline{1,2}$, and mappings $c_H^{(j)} \in C^2(\mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n};\mathbb{R})$, $j=\overline{1,2}$, simultaneously satisfy upon S_H the following tangency conditions:

$$\begin{aligned} \left\langle \frac{\partial H^{(1)}}{\partial x}, a_H \right\rangle + \sum_{j=1}^{2}\left(\frac{\partial H^{(1)}}{\partial u^{(j)}} c_H^{(j)} + \left\langle \frac{\partial H^{(1)}}{\partial p_j}, b_H^{(j)} \right\rangle \right) &= 0,\\ \left\langle \frac{\partial H^{(2)}}{\partial x}, a_H \right\rangle + \sum_{j=1}^{2}\left(\frac{\partial H^{(2)}}{\partial u^{(j)}} c_H^{(j)} + \left\langle \frac{\partial H^{(1)}}{\partial p_j}, b_H^{(j)} \right\rangle \right) &= 0. \end{aligned} \tag{3.5}$$

From expressions (3.3) and (3.4) one easily gets that $c_H^{(1)} = \langle p^{(1)}, a_H\rangle$, $c_H^{(2)} = = \langle p^{(2)}, a_H\rangle$,

$$\begin{gathered} \sum_{j=1}^{2}\left\langle s_{1,j}^{(1|1)}, \frac{\partial H^{(1)}}{\partial p_j}\right\rangle = a_H = \sum_{j=0}^{2}\left\langle s_{2,j}^{(1|1)}, \frac{\partial H^{(2)}}{\partial p_j}\right\rangle,\\ \left\langle s_{1,1}^{(1|1),*}, \frac{\partial H^{(1)}}{\partial x} + \sum_{k=1}^{2} p_k \frac{\partial H^{(1)}}{\partial u^{(k)}}\right\rangle = -b_H^{(1)} = \left\langle s_{2,1}^{(1|1)}, \frac{\partial H^{(2)}}{\partial x} + \sum_{k=1}^{2} p_k \frac{\partial H^{(2)}}{\partial u^{(k)}}\right\rangle,\\ \left\langle s_{1,2}^{(1|1),*}, \frac{\partial H^{(1)}}{\partial x} + \sum_{k=1}^{2} p_k \frac{\partial H^{(1)}}{\partial u^{(k)}}\right\rangle = -b_H^{(2)} = \left\langle s_{2,2}^{(1|1,*)}, \frac{\partial H^{(2)}}{\partial x} + \sum_{k=1}^{2} p_k \frac{\partial H^{(2)}}{\partial u^{(k)}}\right\rangle, \end{gathered} \tag{3.6}$$

where tensor fields $s_{j,k}^{(1|1)} \in C^2(\mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n};\mathbb{R}^n \otimes \mathbb{R}^n)$, $j,k=\overline{1,2}$, must be compatible with conditions (3.6). If the surface S_H defined by (3.2) is not empty, one can easily show that the suitable tensor fields satisfying (3.6) do exist, thereby there exist the corresponding characteristic strips Σ_H, woven with orbits of the vector field (3.4).

Remark 3.1. *It is clear enough that in the case of only one partial differential equation of the form* (3.1) *conditions* (3.6) *determine the sought tangent vector field* (3.4) *uniquely, that is seen from the constraint* $H_1 = H_2$ *and natural conditions* $s_{i,j}^{(1|1)} := s_j^{(1|1)} \in C^2(\mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n};\mathbb{R}^n \otimes \mathbb{R}^n)$, $i,j=\overline{1,2}$, *being suitable smooth tensor fields. Similar analysis as above can be evidently applied also to systems of partial differential equations of higher order.*

4. CONCLUSION

We considered some aspects of existence characteristic surfaces related with partial differential equations of first and higher orders. In particular, the geometric structure of characteristic surfaces was analyzed by means of some specially generalized vector fields, constructed in accordance with a given partial differential equation of first order, and then generalized for that of higher order making use of specially defined tensor fields. Since the constructed vector fields can possess in the general case some important properties, it is a very interesting problem to make use of them for studying asymptotic behavior of solutions [3, 7, 8, 9] to our partial differential equations under suitable boundary conditions. These and related problems we plan to study in more detail in part 2 of this work.

Acknowledgements

The authors are thankful to Professors D. L. Blackmore (NJIT, Newark, USA), M. Malec and St. Brzychczy (WMS AGH, Krakow, Poland) for many valuable discussions and comments on the results obtained.

REFERENCES

[1] Malgrange B.: *Existence et approximation des solution des equations aux derivees partielles et des equations de convolution.* Ann. Inst. Fouier, (6) 1956, 271–351.

[2] Evans L. C.: *Partial differential equations.* AMB, USA, 1998.

[3] Benton S.: *The Hamilton-Jakobi equation: a global approach.* Academic Press USA, 1977.

[4] Garabedian P.: *Partial differential equations.* Wiley Publ., USA, 1964.

[5] John F.: *Partial differential equations.* Berlin, Springer, 1970.

[6] Arnold V. I.: *Lectures on partial differential equations.* Moscow, Fazis, 1999 (Russian).

[7] Prykarpatska N. K., Blackmore D. L., Prykarpatsky A. K., Pytel-Kudela M.: *On the inf-type extremality solutions to Hamilton-Jacobi equations and some generalizations.* Miskolc Mathematical Notes (4) 2003, 157–180.

[8] Prykarpatska N. K., Prytula M. M.: *On the inf-type extremality solution to a Hamilton-Jacobi equation on the sphere* $\mathbb{S}^N$. Nonlinear Oscillations, (3) 2000 1, 74–77.

[9] Mykytiuk Ya. V., Prykarpatsky A. K., Blackmore D.: *The Lax solution to a Hamilton-Jacobi equation and its generalization.* Nonlinear Analysis, (2003) 55, 629–640.

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Received: February 18, 2005.