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THE GEOMETRIC PROPERTIES OF REDUCED CANONICALLY SYMPLECTIC SPACES WITH SYMMETRY, THEIR RELATIONSHIP WITH STRUCTURES ON ASSOCIATED PRINCIPAL FIBER BUNDLES AND SOME APPLICATIONS. PART 1

Abstract. The canonical reduction method on canonically symplectic manifolds is analized in detail, the relationships with the geometric properties of associated principal fiber bundles endowed with connection structures are stated. Some results devoted to studying geometrical properties of nonabelian Yang–Mills type gauge field quations are presented.

Keywords: Hamiltonian reduction, symplectic structures, connections, principal fiber bundles, Yang–Mills type gauge fields.

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1. THE CANONICAL REDUCTION METHOD ON CANONICALLY SYMPLECTIC SPACES AND RELATED GEOMETRIC STRUCTURES ON PRINCIPAL FIBER BUNDLES: INTRODUCTORY BACKGROUNDS

The canonical reduction method in application to many geometric objects on symplectic manifolds with symmetry appears to be very effective tool for their studying, in particular for finding the effective phase space variables [1, 2, 3, 4, 5, 9] on integral submanifolds of Hamiltonian dynamical systems in which they are integrable [1, 2, 14] via the Liouville-Arnold theorem, for investigating related stability problems [1, 6, 12] of Hamiltonian dynamical systems under small perturbations and so on.

Let G denote a given Lie group with the unity element $e \in G$ and the corresponding Lie algebra $\mathcal{G} \simeq T_e(G)$. Consider a principal fiber bundle $M(N;G)$ with the projection $p\colon M \to N$, the structure group G and base manifold N, on which the Lie group G acts [1, 2, 13] by means of a smooth mapping $\varphi\colon M \times G \to M$. Namely, for each $g \in G$ there is a group of diffeomorphisms $\varphi_g\colon M \to M$, generating for any fixed $u \in M$ the following induced mapping: $\hat{u}\colon G \to M$, where

$$\hat{u}(g) = \varphi_g(u). \tag{1.1}$$

On the principal fiber bundle $p\colon (M, \varphi) \to N$ there is assigned [3, 7, 13] a connection $\Gamma(\mathcal{A})$ by means of such a morphism $\mathcal{A}\colon (T(M), \varphi_{g,*}) \to (\mathcal{G}, Ad_{g^{-1}})$, that for each $u \in M$ the mapping $\mathcal{A}(u)\colon T_u(M) \to \mathcal{G}$ is a left inverse one to the mapping $\hat{u}_*(e)\colon \mathcal{G} \to T_u(M)$ and the mapping $\mathcal{A}^*(u)\colon \mathcal{G}^* \to T_u^*(M)$ is a right inverse one to the mapping $\hat{u}^*(e)\colon T_u^*(M) \to \mathcal{G}^*$, that is

$$\mathcal{A}(u)\hat{u}_*(e) = 1, \quad \hat{u}^*(e)\mathcal{A}^*(u) = 1. \tag{1.2}$$

As usually, denote by $\varphi_g^*\colon T^*(M) \to T^*(M)$ the corresponding lift of the mapping $\varphi_g\colon M \to M$ at any $g \in G$. If $\alpha^{(1)} \in \Lambda^1(M)$ is the canonical G − invariant 1-form on M, the canonical symplectic structure $\omega^{(2)} \in \Lambda^2(T^*(M))$ given by

$$\omega^{(2)} := d\ pr_M^* \alpha^{(1)} \tag{1.3}$$

generates the corresponding momentum mapping $l\colon T^*(M) \to \mathcal{G}^*$, where

$$l(\alpha^{(1)})(u) = \hat{u}^*(e)\alpha^{(1)}(u) \tag{1.4}$$

for all $u \in M$. Remark here that the principal fiber bundle structure $p\colon M \to N$ means in part the exactness of the following two adjoint sequences of mappings:

$$\begin{aligned} 0 \leftarrow & \mathcal{G} \overset{\hat{u}^*(e)}{\leftarrow} T_u^*(M) \overset{p^*(u)}{\leftarrow} T_{p(u)}^*(N) \leftarrow 0, \\ 0 \rightarrow & \mathcal{G} \overset{\hat{u}_*(e)}{\rightarrow} T_u(M) \overset{p_*(u)}{\rightarrow} T_{p(u)}(N) \rightarrow 0, \end{aligned} \tag{1.5}$$

that is

$$p_*(u)\hat{u}_*(e) = 0, \quad \hat{u}^*(e)p^*(u) = 0 \tag{1.6}$$

for all $u \in M$. Combining (1.6) with (1.2) and (1.4), one obtains such an embedding:

$$[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) \in \text{range } p^*(u) \tag{1.7}$$

for the canonical 1-form $\alpha^{(1)} \in \Lambda^1(M)$ at $u \in M$. The expression (1.7) means, of course, that

$$\hat{u}^*(e)[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) = 0 \tag{1.8}$$

for all $u \in M$. Taking now into account that the mapping $p^*(u)\colon T_{p(u)}^*(N) \to T_u^*(M)$ is for each $u \in M$ injective, it has the unique inverse mapping $(p^*(u))^{-1}$ upon its image $p^*(u)T_{p(u)}^*(N) \subset T_u^*(M)$. Thereby for each $u \in M$ one can define a morphism $p_{\mathcal{A}}\colon (T^*(M), \varphi_g^*) \to (T^*(N), id)$ as

$$p_{\mathcal{A}}(u)\colon \alpha^{(1)}(u) \to (p^*(u))^{-1}[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u). \tag{1.9}$$

Based on the definition (1.9) one can easily check that the diagram

$$\begin{array}{ccc} T^*(M) & \xrightarrow{p_{\mathcal{A}}} & T^*(N) \\ {\scriptstyle pr_M}\downarrow & & \downarrow{\scriptstyle pr_N} \\ M & \xrightarrow{p} & N \end{array} \tag{1.10}$$

is commutative.

Let now an element $\xi \in \mathcal{G}^*$ be G-invariant, that is $Ad^*_{g^{-1}}\xi = \xi$ for all $g \in G$. Denote also by $p^{\xi}_{\mathcal{A}}$ the restriction of the mapping (1.9) upon the subset $\mathcal{M}_\xi := l^{-1}(\xi) \in T^*(M)$, that is the mapping $p^{\xi}_{\mathcal{A}} \colon \mathcal{M}_\xi \to T^*(N)$, where for all $u \in M$

$$p^{\xi}_{\mathcal{A}}(u) \colon \mathcal{M}_\xi \to (p^*(u))^{-1}\big[1 - \mathcal{A}^*(u)\hat{u}^*(e)\big]\mathcal{M}_\xi. \tag{1.11}$$

Now one can characterize the structure of the reduced phase space $\bar{\mathcal{M}}_\xi := \mathcal{M}_\xi/G$ by means of the following simple lemma.

Lemma 1.1. *The mapping $p^{\xi}_{\mathcal{A}}(u) \colon \mathcal{M}_\xi \to T^*(N)$ is a principal fiber G-bundle with the reduced space $\bar{\mathcal{M}}_\xi = \mathcal{M}_\xi)/G$ being diffeomorphic to $T^*(N)$.*

Denote by $\langle \cdot,\cdot \rangle_{\mathcal{G}}$ the standard *Ad*-invariant nondegenerate scalar product on $\mathcal{G}^* \times \mathcal{G}$. Based on Lemma 1.1 one derives the following characteristic theorem.

Theorem 1.2. *Given a principal fiber G-bundle with a connection $\Gamma(\mathcal{A})$ and a G-invariant element $\xi \in \mathcal{G}^*$, then every such connection $\Gamma(\mathcal{A})$ defines a symplectomorphism $\nu_\xi \colon \bar{\mathcal{M}}_\xi \to T^*(N)$ between the reduced phase space $\bar{\mathcal{M}}_\xi$ and cotangent bundle $T^*(N)$. Moreover, the following equality*

$$(p^{\xi}_{\mathcal{A}})\left(d\, pr^*_N \beta^{(1)} + pr^*_N \Omega^{(2)}_\xi\right) = d\, pr^*_{M_\xi}\alpha^{(1)} \tag{1.12}$$

holds for the canonical 1-form $\beta^{(1)} \in \Lambda^{(1)}(N)$ and $\alpha^{(1)} \in \Lambda^{(1)}(M)$, where $M_\xi := pr_M \mathcal{M}_\xi \subset M$, 2-form $\Omega^{(2)}_\xi \in \Lambda^{(2)}(N)$ is the ξ-component of the corresponding curvature 2-form $\Omega^{(2)} \in \Lambda^{(2)}(M) \otimes \mathcal{G}$.

Proof. . One has that on $\mathcal{M}_\xi \subset T^*(M)$ the following expression due to (1.9) holds:

$$p^*(u)p^{\xi}_{\mathcal{A}}(\alpha^{(1)}(u)) := p^*(u)\beta^{(1)}(pr_N(u)) = \alpha^{(1)}(u) - \mathcal{A}^*(u)\hat{u}^*(e)\alpha^{(1)}(u)$$

for any $\beta^{(1)} \in T^*(N), \alpha^{(1)} \in \mathcal{M}_\xi$ and $u \in M_\xi$. Thus we get easily that for such $\alpha^{(1)} \in \mathcal{M}_\xi$ there holds

$$\alpha^{(1)}(u) = (p^{\xi}_{\mathcal{A}})^{-1}\beta^{(1)}(p_N(u)) = p^*(u)\beta^{(1)})(pr_N(u)) + \langle \mathcal{A}(u), \xi \rangle_{\mathcal{G}}$$

for all $u \in M_\xi$. Recall now that in virtue of (1.10) one gets on M_ξ and $\mathcal{M}_\xi$ the following relationships:

$$p \cdot\ pr_{M_\xi} = pr_N \cdot\ p^{\xi}_{\mathcal{A}}, pr^*_{M_\xi} \cdot p^* = (p^{\xi}_{\mathcal{A}})^* \cdot pr^*_N.$$

Therefore we can write down now that for any $u \in M$

$$
\begin{aligned}
pr^*_{M_\xi}\alpha^{(1)}(u) &= pr^*_{M_\xi} p^*(u)\beta^{(1)}(p(u)) + pr^*_{M_\xi}\langle \mathcal{A}(u), \xi\rangle \\
&= (p^\xi_{\mathcal{A}})^* pr^*_N \beta^{(1)}(u) + pr^*_{M_\xi}\langle \mathcal{A}(u), \xi\rangle ,
\end{aligned}
$$

whence taking the external differential, one arrives at the following equality:

$$
\begin{aligned}
d\, pr^*_{M_\xi}\alpha^{(1)}(u) &= (p^\xi_{\mathcal{A}})^* d(pr^*_N\beta^{(1)})(u) + pr^*_{M_\xi}\langle d\mathcal{A}(u), \xi\rangle \\
&= (p^\xi_{\mathcal{A}})^* d(pr^*_N\beta^{(1)})(u) + pr^*_{M_\xi}\left\langle \Omega^{(2)}(u)), \xi\right\rangle \\
&= (p^\xi_{\mathcal{A}})^* d(pr^*_N\beta^{(1)})(u) + pr^*_{M_\xi} p^*\left\langle \Omega^{(2)}, \xi\right\rangle(u) \\
&= (p^\xi_{\mathcal{A}})^* d(pr^*_N\beta^{(1)})(u) + (p^\xi_{\mathcal{A}})^* pr^*_N\left\langle \Omega^{(2)}, \xi\right\rangle(u) \\
&= (p^\xi_{\mathcal{A}})^* \left[d(pr^*_N\beta^{(1)})(u) + pr^*_N\Omega^{(2)}_\xi(u)\right].
\end{aligned}
$$

When deriving the above expression we made use of the following property satisfied by the curvature 2-form $\Omega^{(2)} := d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} \in \Lambda^2(M)\otimes\mathcal{G}$:

$$
\begin{aligned}
\langle d\mathcal{A}(u), \xi\rangle_{\mathcal{G}} &= \langle d\mathcal{A}(u) + \mathcal{A}(u)\wedge\mathcal{A}(u), \xi\rangle_{\mathcal{G}} - \langle \mathcal{A}(u)\wedge\mathcal{A}(u), \xi\rangle_{\mathcal{G}} = \left\langle \Omega^{(2)}(u), \xi\right\rangle_{\mathcal{G}} = \\
&= \left\langle \Omega^{(2)}(u), Ad^*_g\xi\right\rangle_{\mathcal{G}} = \left\langle Ad_g\Omega^{(2)}(u), \xi\right\rangle_{\mathcal{G}} = \left\langle \Omega^{(2)}, \xi\right\rangle(p(u))_{\mathcal{G}} := p^*\Omega^{(2)}_\xi(u)
\end{aligned}
$$

at any $u \in M$, since for any $A, B \in \mathcal{G}$ there holds $\langle [A,B], \xi\rangle_{\mathcal{G}} = \langle B, ad^*_A\xi\rangle_{\mathcal{G}} = 0$ in virtue of of the invariance condition $Ad^*_g\xi = \xi$ for any $g \in G$. Thereby the proof is finished. □

Remark 1.3. *As the canonical 2-form $d\,pr^*_M\alpha^{(1)} \in \Lambda^{(2)}(T^*(M))$ is G-invariant on $T^*(M)$ due to the construction, it is evident that its restriction upon the G-invariant submanifold $\mathcal{M}_\xi \subset T^*(M)$ will be effectively defined only on the reduced space $\bar{\mathcal{M}}_\xi$, that ensures the validity of the equality sign in* (1.12).

As a simple but useful consequence of Theorem 1.2 one can formulate the following useful enough for applications results.

Theorem 1.4. *Let a momentum mapping value $l\left(\alpha^{(1)}\right)(u) = \hat{u}^*(e)\alpha^{(1)}(u) = \xi \in \mathcal{G}^*$ has the isotropy group G_ξ acting naturally on the subset $\mathcal{M}_\xi \subset T^*(M)$ invariantly, freely and properly, so that the reduced phase space $\left(\bar{\mathcal{M}}_\xi\, \bar{\omega}^{(2)}_\xi\right)$ is symplectic, where by definition* [1, 5], *for the natural embedding mapping $\pi_\xi \colon \mathcal{M}_\xi \to T^*(M)$ and the reduction mapping $r_\xi \colon \mathcal{M}_\xi \to \bar{\mathcal{M}}_\xi$ the defining equality*

$$
r^*_\xi\bar{\omega}^{(2)}_\xi := \pi^*_\xi(d\,pr^*_M\alpha^{(1)}) \tag{1.13}
$$

holds on $\mathcal{M}_\xi$. If an associated principal fiber bundle $p\colon M \to N$ has a structure group coinciding with G_ξ, then the reduced symplectic space $\left(\bar{\mathcal{M}}_\xi, \bar{\omega}^{(2)}_\xi\right)$ is symplectomorphic *to the cotangent symplectic space* $(T^*(N), \sigma_\xi^{(2)})$, *where*

$$\sigma_\xi^{(2)} = d\, pr_N^* \beta^{(1)} + pr_N^* \Omega_\xi^{(2)}, \tag{1.14}$$

and the corresponding symplectomorphism is given by the relation like (1.12).

Concerning some applications the following criterion can be useful when constructing associated fibre bundles with connections related with the symplectic structure reduced on the space $\bar{\mathcal{M}}_\xi$.

Theorem 1.5. *In order that two symplectic spaces* $(\bar{\mathcal{M}}_\xi, \bar{\omega}_\xi^{(2)})$ *and* $(T^*(N), d\, pr_N^* \beta^{(1)})$ *were symplectomorphic, it is necessary and sufficient that the element* $\xi \in \ker\, h$, *where for* G*-invariant element* $\xi \in \mathcal{G}^*$ *the mapping* $h\colon \xi \to [\Omega_\xi^{(2)}] \in H^2(N;\mathbb{Z})$ *with* $H^2(N;\mathbb{Z})$ *being the cohomology group of 2-forms on the manifold* N.

2. THE STRUCTURE OF REDUCED SYMPLECTIC STRUCTURES ON COTANGENT SPACES TO LIE GROUPS MANIFOLDS AND ASSOCIATED CANONICAL CONNECTIONS

In case when there is given a Lie group G, the tangent space $T(G)$ can be interpreted as a Lie group $\tilde{G}$ isomorphic to the semidirect product $G \otimes_{Ad} \bar{\mathcal{G}} \simeq \tilde{G}$ of the Lie group G and its Lie algebra $\bar{\mathcal{G}}$ under the adjoint Ad-action of G on $\bar{\mathcal{G}}$. The Lie algebra $\tilde{\mathcal{G}}$ of $\tilde{G}$ is, respectively, the semidirect product of $\mathcal{G}$ with itself, regarded as a trivial abelian Lie algebra, under the adjoint ad-action and has thus the induced bracket defined as $[(a_1, m_1), (a_2, m_2)] := ([a_1, a_2], [a_1, m_2] + [a_2, m_1])$ for all $(a_j, m_j) \in \mathcal{G} \otimes_{ad} \bar{\mathcal{G}}$, $j = \overline{1,2}$. Take now any element $\xi \in \mathcal{G}^*$ and compute its isotropy group G_ξ under the co-adjoint action Ad^* of G on $\mathcal{G}^*$, and denote by $\mathcal{G}_\xi$ its Lie algebra. The cotangent bundle $T^*(G)$ is obviously diffeomorphic to $M := G \times \mathcal{G}^*$ on which the Lie group G_ξ acts freely and properly (due to construction) by left translation on the first factor and Ad^*-action on the second one. The corresponding momentum mapping $l\colon G \times \mathcal{G}^* \to \mathcal{G}_\xi^*$ can be obtained as

$$l(h, \alpha) = Ad_{h^{-1}}^* \alpha|_{\mathcal{G}_\xi^*} \tag{2.1}$$

with no critical point. Let now $\eta \in \mathcal{G}^*$ and $\eta(\xi) := \eta|_{\mathcal{G}_\xi^*}$. Therefore the reduced space $\left(l^{-1}(\eta(\xi))/G_\xi^{\eta(\xi)}, \bar{\omega}_\xi^{(2)}\right)$ has to be symplectic due to the well known Marsden–Weinstein reduction theorem [1, 2, 5], where $G_\xi^{\eta(\xi)}$ is the isotropy subgroup of the G_ξ -coadjoint action on $\eta(\xi) \in \mathcal{G}_\xi^*$ and the condition $r_\xi^* \bar{\omega}_\xi^{(2)} := \pi_\xi^* d\ pr^* \alpha^{(1)}$ naturally induces the symplectic form $\bar{\omega}_\xi^{(2)}$ on the reduced space $\mathcal{M}_\xi := l^{-1}(\eta(\xi))/G_\xi^{\eta(\xi)}$ from the canonical symplectic structure on $T^*(G)$. Define now for $\eta(\xi) \in \mathcal{G}_\xi^*$ the one-form $\alpha_{\eta(\xi)}^{(1)} \in \Lambda^1(G)$ as

$$\alpha_{\eta(\xi)}^{(1)}(h) := R_h^* \eta(\xi), \tag{2.2}$$

where $R_h\colon G \to G$ is right translation by an element $h \in G$.

It is easy to check that the element (2.2) is right G-invariant and left $G_\xi^{\eta(\xi)}$-invariant, thus inducing a one-form on the quotient $N_\xi := G/G_\xi^{\eta(\xi)}$. Denote by $pr_{N_\xi}^*\alpha_{\eta(\xi)}^{(1)}$ its pull-back to $T^*(N_\xi)$ and form the symplectic manifold $(T^*(N_\xi), \sigma_\xi^{(2)})$, where $\sigma_\xi^{(2)} := d\,pr_{N_\xi}^*\beta^{(1)} + d\,pr_{N_\xi}^*\alpha_{\eta(\xi)}^{(1)})$ and $d\,pr_{N_\xi}^*\beta^{(1)} \in \Lambda^{(2)}(T^*(N_\xi))$ is the canonical symplectic form on $T^*(N_\xi)$. The construction above now can be summarized as the next theorem.

Theorem 2.1. *Let $\xi, \eta \in \mathcal{G}^*$ and $\eta(\xi) := \eta|_{\mathcal{G}_\xi^*}$ be fixed. Then the reduced symplectic manifold $(\bar{\mathcal{M}}_\xi, \bar{\omega}_\xi^{(2)})$ is a symplectic covering of the co-adjoint orbit $Or(\xi, \eta(\xi); \tilde{G})$ which symplectically embeds onto the subbundle over $N_\xi := G/G_\xi^{\eta(\xi)}$ of $(T^*(N_\xi), \sigma_\xi^{(2)})$, with $\omega_\xi^{(2)} := d\,pr_{N_\xi}^*\beta^{(1)} + d\,pr_{N_\xi}^*\alpha_{\eta(\xi)}^{(1)} \in \Lambda^2(T^*(N_\xi)$.*

The statement above fits into the conditions of Theorem 1.4 if one to define a connection 1-form $\mathcal{A}(g)\colon T_g(G) \to \mathcal{G}_\xi$ as follows:

$$\langle \mathcal{A}(g), \xi \rangle_{\mathcal{G}} := R_g^*\eta(\xi) \tag{2.3}$$

for any $g \in G$. The expression (2.3) generates a completely horizontal 2-form $d\,\langle \mathcal{A}(g), \xi \rangle_{\mathcal{G}}$ on the Lie group G, which gives rise immediately to the symplectic structure $\sigma_\xi^{(2)}$ on the phase space $T^*(N_\xi)$, where $N_\xi := G/G_\xi^{\eta(\xi)}$.

3. THE GEOMETRIC STRUCTURE OF ABELIAN YANG–MILLS TYPE GAUGE FIELD EQUATIONS WITHIN THE REDUCTION METHOD

If one studies a motion of a charged particle under an abelian Yang–Mills type gauge field, it is convenient [4, 11, 12] to introduce a special fiber bundle structure $p\colon M \to N$, namely such one that $M = N \times G$, $N := D \subset \mathbb{R}^n$ and $G := \mathbb{R}/\{0\}$ being the corresponding (abelian) structure Lie group. An analysis similar to the above gives rise to a reduced upon the symplectomorphic space $\mathcal{M}_\xi := l^{-1}(\xi)/G \simeq T^*(N)$, $\xi \in \mathcal{G}$, with the symplectic structure $\sigma_\xi^{(2)}(q,p) = \langle dp, \wedge dq \rangle + d\,\langle \mathcal{A}(q,g), \xi \rangle_{\mathcal{G}}$, where $\mathcal{A}(q,g) := \langle A(q), dq \rangle + g^{-1}dg$ is a usual connection 1-form on M, with $(q,p) \in T^*(N)$ and $g \in G$. The corresponding canonical Poisson brackets on $T^*(N)$ are easily found to be

$$\{q^i, q^j\}_\xi = 0,\ \{p_j, q^i\}_\xi = \delta_j^i, \{p_i, p_j\}_\xi = F_{ji}(q) \tag{3.1}$$

for all $(q,p) \in T^*(N)$ and the curvature tensor $F_{ij}(q) := \partial A_j/\partial q^i - \partial A_i/\partial q^j$, $i,j = \overline{1,n}$. If one introduces a new momentum variable $\tilde{p} := p + A(q)$ on $T^*(N) \ni (q,p)$, it is easy to verify that $\sigma_\xi^{(2)} \to \tilde{\sigma}_\xi^{(2)} := \langle d\tilde{p}, \wedge dq \rangle \in \Lambda^{(2)}(N)$ giving rise to the following Poisson brackets [8] on $T^*(N)$:

$$\{q^i, q^j\}_\xi = 0, \{\tilde{p}_j, q^i\}_\xi = \delta_j^i,\ \{\tilde{p}_i, \tilde{p}_j\}_\xi = 0, \tag{3.2}$$

iff for all $i, j, k = \overline{1, n}$ the standard abelian Yang–Mills equations

$$\partial F_{ij}/\partial q_k + \partial F_{jk}/\partial q_i + \partial F_{ki}/\partial q_j = 0 \tag{3.3}$$

hold on N. Such a construction permits a natural generalization to the case of nonabelian structure Lie group yielding a description of nonabelian Yang–Mills type field equations within the reduction approach describe above.

4. THE GEOMETRIC STRUCTURE OF NONABELIAN YANG–MILLS TYPE GAUGE FIELD EQUATIONS WITHIN THE REDUCTION METHOD AND ITS PROPERTIES

As before, we start with defining a phase space M of a particle under a nonabelian Yang–Mills type gauge field given on a region $D \subset \mathbb{R}^3$ as $M := D \times G$, where G is a (not in general semisimple) Lie group, acting on M from the right. Over the space M one can define quite naturally a connection $\Gamma(\mathcal{A})$ if to consider the following trivial principal fiber bundle $p \colon M \to N$, where $N := D$, with the structure group G. Namely, if $g \in G$, $q \in N$, then a connection 1-form on $M \ni (q, g)$ can be written down [1, 3, 7] as

$$\mathcal{A}(q; g) := g^{-1}(d + \sum_{i=1}^{n} a_i A^{(i)}(q))g, \tag{4.1}$$

where $\{a_i \in \mathcal{G} : i = \overline{1, m}\}$ is a basis of the Lie algebra $\mathcal{G}$ of the Lie group G, and $A_i \colon D \to \Lambda^1(D)$, $i = \overline{1, n}$, are the Yang–Mills fields on the physical n-dimensional open space $D \subset \mathbb{R}^n$.One can define the natural left invariant Liouville type form on M as

$$\alpha^{(1)}(q; g) := \langle p, dq \rangle + \left\langle y, g^{-1} dg \right\rangle_{\mathcal{G}}, \tag{4.2}$$

where $y \in T^*(G)$ and $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ denotes as before the usual Ad-invariant nondegenerate bilinear form on $\mathcal{G}^* \times \mathcal{G}$, as evidently $g^{-1}dg \in \Lambda^1(G) \otimes \mathcal{G}$. The main assumption we need to accept for further is that the connection 1-form is in accordance with the Lie group G action on M. The latter means that the condition

$$R_h^* \mathcal{A}(q; g) = Ad_{h^{-1}} \mathcal{A}(q; g) \tag{4.3}$$

is satisfied for all $(q, g) \in M$ and $h \in G$, where $R_h \colon G \to G$ means the right translation by an element $h \in G$ on the Lie group G.

Having stated all preliminary conditions needed for the reduction Theorem 1.4 to be applied to our model, suppose that the Lie group G canonical action on M is naturally lifted to that on the cotangent space $T^*(M)$ endowed due to (4.2) with the following G-invariant canonical symplectic structure

$$\begin{aligned} \omega^{(2)}(q, p; g, y) &:= d\, pr^* \alpha^{(1)}(q, p; g, y) = \\ &= \langle dp, \wedge dq \rangle + \left\langle dy, \wedge g^{-1} dg \right\rangle_{\mathcal{G}} + \left\langle y dg^{-1}, \wedge dg \right\rangle_{\mathcal{G}} \end{aligned} \tag{4.4}$$

for all $(q,p;g,y) \in T^*(M)$. Take now an element $\xi \in \mathcal{G}^*$ and assume that its isotropy subgroup $G_\xi = G$, that is $Ad_h^* \xi = \xi$ for all $h \in G$. In the general case such an element $\xi \in \mathcal{G}^*$ can not exist but trivial $\xi = 0$, as it happens to the Lie group $G = SL_2(\mathbb{R})$. Then owing to (1.12) one can construct the reduced phase space $\bar{\mathcal{M}}_\xi := \mathcal{M}_\xi / G$ symplectomorphic to $(T^*(N), \sigma_\xi^{(2)})$, $\mathcal{M}_\xi := l^{-1}(\xi) \in T^*(M)$, where for any $(q,p) \in T^*(N)$

$$\begin{aligned} \sigma_\xi^{(2)}(q,p) &= \langle dp, \wedge dq \rangle + \Omega_\xi^{(2)}(q) = \\ &= \langle dp, \wedge dq \rangle + \sum_{s=1}^{m} \sum_{i,j=1}^{n} e_s F_{ij}^{(s)}(q) dq^i \wedge dq^j . \end{aligned} \tag{4.5}$$

In the above we have expanded the element $\mathcal{G}^* \ni \xi = \sum_{i=1}^n e_i a^i$ with respect to the bi-orthogonal basis $\{a^i \in \mathcal{G}^* \colon \langle a^i, a_j \rangle_{\mathcal{G}} = \delta_j^i,\ i,j = \overline{1,m}\}$ with $e_i \in \mathbb{R}$, $i = \overline{1,n}$, being some constants, and as well we denoted by $F_{ij}^{(s)}(q)$, $i,j = \overline{1,n}$, $s = \overline{1,m}$, the respectively reduced on N components of the curvature 2-form $\Omega^{(2)} \in \Lambda^2(N) \otimes \mathcal{G}$, that is

$$\Omega^{(2)}(q) := \sum_{s=1}^{m} \sum_{i,j=1}^{n} a_s F_{ij}^{(s)}(q) dq^i \wedge dq^j \tag{4.6}$$

at any point $q \in N$. Summarizing calculations accomplished similarly to these above, we can formulate instantly the following result.

Theorem 4.1. *Suppose that an nonabelian Yang–Mills type field* (4.1) *on the fiber bundle $p \colon M \to N$ with $M = D \times G$ is invariant with respect to the Lie group G action $G \times M \to M$. Suppose also that an element $\xi \in G^*$ is chosen so that $Ad_G^* \xi = \xi$. Then for the naturally constructed momentum mapping $l \colon T^*(M) \to G^*$ the reduced phase space $\bar{\mathcal{M}}_\xi := l^{-1}(\xi)/G$, being symlectomorhic to the space $(T^*(N), \sigma^{(2)})$, is endowed with the symplectic structure* (4.5) *on $T^*(N)$, having the following component-wise Poisson brackets:*

$$\{p_i, q^j\}_\xi = \delta_i^j, \{q^i, q^j\}_\xi = 0, \{p_i, p_j\}_\xi = \sum_{s=1}^{n} e_s F_{ji}^{(s)}(q) \tag{4.7}$$

for all $i,j = \overline{1,n}$ and $(q,p) \in T^*(N)$.

The respectively extended Poisson bracket on the whole cotangent space $T^*(M)$ owing to (4.4) amounts to the following set of Poisson brackets relationships:

$$\begin{aligned} &\{y_s, y_k\}_\xi = \sum_{r=1}^{n} c_{sk}^r y_r, \{p_i, q^j\}_\xi = \delta_i^j, \\ &\{y_s, p_j\}_\xi = 0 = \{q^i, q^j\}, \{p_i, p_j\}_\xi = \sum_{s=1}^{n} y_s F_{ji}^{(s)}(q), \end{aligned} \tag{4.8}$$

where $i,j=\overline{1,n}$, $c^r_{sk}\in\mathbb{R}$, $s,k,r=\overline{1,m}$, are the structure constants of the Lie algebra $\mathcal{G}$, and we made use of the expansion $A^{(s)}(q)=\sum_{j=1}^n A^{(s)}_j(q)dq^j$ as well we introduced alternative fixed values $e_i:=y_i$, $i=\overline{1,n}$. The result (4.8) can be seen easily if one to make a shift within the expression (4.4) as $\sigma^{(2)}\to\sigma^{(2)}_{ext}$, where $\sigma^{(2)}_{ext}:=\sigma^{(2)}\big|_{\mathcal{A}_0\to\mathcal{A}}$, $\mathcal{A}_0(g):=g^{-1}dg$, $g\in G$. Thereby one can obtain in virtue of the invariance properties of the connection $\Gamma(\mathcal{A})$ that

$$\begin{aligned}\sigma^{(2)}_{ext}(q,p;u,y)&=\langle dp,\wedge dq\rangle+d\left\langle y(g),Ad_{g^{-1}}\mathcal{A}(q;e)\right\rangle_{\mathcal{G}}=\\&=\langle dp,\wedge dq\rangle+\left\langle d\,Ad^*_{g^{-1}}y(g),\wedge\mathcal{A}(q;e)\right\rangle_{\mathcal{G}}=\\&=\langle dp,\wedge dq\rangle+\sum_{s=1}^m dy_s\wedge du^s+\sum_{j=1}^n\sum_{s=1}^m A^{(s)}_j(q)dy_s\wedge dq-\left\langle Ad^*_{g^{-1}}y(g),\mathcal{A}(q,e)\wedge\mathcal{A}(q,e)\right\rangle_{\mathcal{G}}\\&+\sum_{k\geq s=1}^m\sum_{l=1}^m y_l c^l_{sk}du^k\wedge du^s+\sum_{s=1}^n\sum_{i\geq j=1}^3 y_sF^{(s)}_{ij}(q)dq^i\wedge dq^j,\end{aligned}\tag{4.9}$$

where coordinate points $(q,p;u,y)\in T^*(M)$ are defined as follows: $\mathcal{A}_0(e):=\sum_{s=1}^m du^i\ a_i$, $Ad^*_{g^{-1}}y(g)=y(e):=\sum_{s=1}^m y_s\ a^s$ for any element $g\in G$. Whence one gets right away the Poisson brackets (2.8) plus additional brackets connected with conjugated sets of variables $\{u^s\in\mathbb{R}\colon s=\overline{1,m}\}\in\mathcal{G}^*$ and $\{y_s\in\mathbb{R}\colon s=\overline{1,m}\}\in\mathcal{G}$:

$$\{y_s,u^k\}_\xi=\delta^k_s,\quad \{u^k,q^j\}_\xi=0,\quad \{p_j,u^s\}_\xi=A^{(s)}_j(q),\quad \{u^s,u^k\}_\xi=0,\tag{4.10}$$

where $j=\overline{1,n}$, $k,s=\overline{1,m}$, and $q\in N$. Note here that the suggested above transition from the symplectic structure $\sigma^{(2)}$ on $T^*(N)$ to its extension $\sigma^{(2)}_{ext}$ on $T^*(M)$ just consists formally in adding to the symplectic structure $\sigma^{(2)}$ an exact part, which transforms it into equivalent one. Looking now at the expressions (4.9), one can infer immediately that an element $\xi:=\sum_{s=1}^m e_sa^s\in\mathcal{G}^*$ will be invariant with respect to the Ad^*-action of the Lie group G iff

$$\{y_s,y_k\}_\xi|_{y_s=e_s}=\sum_{r=1}^m c^r_{sk}e_r\equiv 0\tag{4.11}$$

identically for all $s,k=\overline{1,m}$, $j=\overline{1,n}$ and $q\in N$. In this and only this case the reduction scheme elaborated above will go through.

Returning attention to the expression (4.10), one can easily write down the following exact expression:

$$\omega^{(2)}_{ext}(q,p;u,y)=\omega^{(2)}\left(q,p+\sum_{s=1}^n y_s\,A^{(s)}(q);u,y\right),\tag{4.12}$$

on the phase space $T^*(M)\ni(q,p;u,y)$, where we abbreviated for brevity $\left\langle A^{(s)}(q),dq\right\rangle$ as $\sum_{j=1}^n A^{(s)}_j(q)\ dq^j$. The transformation like (4.12) was discussed within somewhat different context in article [8] containing also a good background for the infinite dimensional generalization of symplectic structure techniques. Having observed from (4.12) that the simple change of variable

$$\tilde{p} := p + \sum_{s=1}^{m} y_s \; A^{(s)}(q) \tag{4.13}$$

of the cotangent space $T^*(N)$ recasts our symplectic structure (4.9) into the old canonical form (4.4), one obtains that the following new set of Poisson brackets on $T^*(M) \ni (q, \tilde{p}; u, y)$:

$$\begin{gathered} \{y_s, y_k\}_\xi = \sum_{r=1}^{n} c_{sk}^r y_r, \quad \{\tilde{p}_i, \tilde{p}_j\}_\xi = 0, \quad \{\tilde{p}_i, q^j\} = \delta_i^j, \\ \{y_s, q^j\}_\xi = 0 = \{q^i, q^j\}_\xi, \quad \{u^s, u^k\}_\xi = 0, \quad \{y_s, \tilde{p}_j\}_\xi = 0, \\ \{u^s, q^i\}_\xi = 0, \quad \{y_s, u^k\}_\xi = \delta_s^k, \quad \{u^s, \tilde{p}_j\}_\xi = 0, \end{gathered} \tag{4.14}$$

where $k, s = \overline{1, m}$ and $i, j = \overline{1, n}$, holds iff the nonabelian Yang–Mills type field equations

$$\begin{aligned} &\partial F_{ij}^{(s)}/\partial q^l + \partial F_{jl}^{(s)}/\partial q^i + \partial F_{li}^{(s)}/\partial q^j + \\ &\qquad + \sum_{k,r=1}^{m} c_{kr}^s (F_{ij}^{(k)} A_l^{(r)} + F_{jl}^{(k)} A_i^{(r)} + F_{li}^{(k)} A_j^{(r)}) = 0 \end{aligned} \tag{4.15}$$

are fulfilled for all $s = \overline{1, m}$ and $i, j, l = \overline{1, n}$ on the base manifold N. This effect of complete reduction of gauge Yang–Mills type variables from the symplectic structure (4.9) is known in literature [8, 11] as the principle of minimal interaction and appeared to be useful enough for studying different interacting systems as in [9, 10]. In part 2 of this work we plan to continue a study the geometric properties of reduced symplectic structures connected with infinite dimensional coupled dynamical systems like Yang–Mills–Vlasov, Yang–Mills–Bogolubov and Yang–Mills–Josephson ones [9, 10] as well as their relationships with associated principal fiber bundles endowed with canonical connection structures.

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