Opuscula Mathematica • Vol. 25 • No. 2 • 2005

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# A NOTE ON GEODESIC AND ALMOST GEODESIC MAPPINGS OF HOMOGENEOUS RIEMANNIAN MANIFOLDS

**Abstract.** Let $M$ be a differentiable manifold and denote by $\nabla$ and $\widetilde{\nabla}$ two linear connections on $M$. $\nabla$ and $\widetilde{\nabla}$ are said to be geodesically equivalent if and only if they have the same geodesics. A Riemannian manifold $(M, g)$ is a naturally reductive homogeneous manifold if and only if $\nabla$ and $\widetilde{\nabla} = \nabla - T$ are geodesically equivalent, where $T$ is a homogeneous structure on $(M, g)$ ([7]). In the present paper we prove that if it is possible to map geodesically a homogeneous Riemannian manifold $(M, g)$ onto $(M, \widetilde{\nabla})$, then the map is affine. If a naturally reductive manifold $(M, g)$ admits a nontrivial geodesic mapping onto a Riemannian manifold $(\overline{M}, \overline{g})$ then both manifolds are of constant cutvature. We also give some results for almost geodesic mappings $(M, g) \to (M, \widetilde{\nabla})$.

**Keywords:** homogeneous Riemannian manifold, geodesic, almost geodesic, geodesic mapping, almost geodesic mapping..

**Mathematics Subject Classification:** 53B30, 53C25.

## 1. INTRODUCTION

Let $(M, g)$ be an $n$-dimensional Riemannian manifold of class $C^\infty$. Let $\mathfrak{F}(M)$ be the ring of differentiable functions and $\mathfrak{X}(M)$ the $\mathfrak{F}(M)-$module of differentiable vector fields on $M$. A complete and simply connected manifold $(M, g)$ is homogeneous if there exists a transitive and effective group $G$ of isometries of $M$. Ambrose and Singer proved (see [7]) that a complete and simply connected Riemannian manifold $(M, g)$ is homogeneous if and only if there exists a tensor field $T$ of type $(1, 2)$ such that:

$$\begin{aligned} &\text{(i)} \quad g(T_X Y, Z) + g(Y, T_X Z) = 0, \\ &\text{(ii)} \quad (\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{T_X Y Z} - R_{Y T_X Z}, \qquad (1.1) \\ &\text{(iii)} \quad (\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}, \end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(M)$. Here $\nabla$ and $R$ denote the Levi–Civita connection and the Riemannian tensor field, respectively. A tensor field $T$ satisfying the conditions (1.1) on $M$ is called a homogeneous structure on $(M, g)$. It is easy to see that the conditions (1.1) are equivalent to

$$\left(\widetilde{\nabla}_X g\right)(Y,Z) = 0, \quad \left(\widetilde{\nabla}_X R\right)(Y,Z) = 0, \quad \left(\widetilde{\nabla}_X T\right)(Y,Z) = 0 \tag{1.2}$$

where $\widetilde{\nabla}$ is the connection determined by

$$\widetilde{\nabla}_X Y = \nabla_X Y - T\,(X,Y)\,. \tag{1.3}$$

where $T(X,Y) = T_X Y,\ X, Y, Z \in \mathfrak{X}(M)$.

In [7] F. Tricerri and L. Vanhecke studied the decomposition of the space of all the tensors $T$ satisfying the conditions (1.1) into the irreducible components under the action of orthogonal group. As is well-known, a geodesic in a Riemannian manifold $M$ is a curve of $c : I \to M$ whose tangent vector field $\dot{c}$ is parallel along $c$ ($I$ is an open interval in the real line $R^1$). A curve $c$ is almost geodesic in a Riemannian manifold $M$ if there exists a 2− dimensional distribution $E^2$ complanar along $c$, to which the tangent vector $\dot{c}$ of this curve belongs at every point. Let $(\overline{M}, \overline{\nabla})$ be a differentiable manifold with a linear symmetric connection $\overline{\nabla}$. A mapping $f\colon (M,g) \to (\overline{M}, \overline{\nabla})$ is called geodesic or projective if $f$ carries geodesics in $M$ to geodesics in $\overline{M}$. The mapping $f$ is an almost geodesic mapping if, as a result of $f$, every geodesic in the manifold $M$ passes into an almost geodesic curve in the manifold $\overline{M}$. If $\overline{M}$ coincides with $M$ and $f$ is a diffeomorphism, $f$ is called a geodesic or an almost geodesic transformation of $M$.

It is well known, that the identity transformation is geodesic if and only the connection deformation tensor $P(X,Y) = \overline{\nabla}_X Y - \nabla_X Y$ has the form ([1, 2, 3, 5])

$$P(X,Y) = \psi(X)Y + \psi(Y)X, \tag{1.4}$$

where $\psi$ is a certain 1-form and $\nabla$ denotes the Levi–Civita connection of $(M, g)$, $X, Y \in \mathfrak{X}(M)$.

In this case, $\overline{\nabla}$ and $\nabla$ are said to be geodesically (or projectively) equivalent or geodesically (projectively) related. Two such connections define the same system of geodesics. Obviously $\sim$ is an equivalence relation and an equivalence class $[\nabla]$ containing $\nabla$ is called a projective structure on $M$.

Sinyukow [5] defined three kinds of almost geodesic mappings, namely $\pi_1$, $\pi_2$ and $\pi_3$ which are characterized, respectively, by the conditions:

$$\pi_1\colon \underset{X,Y,Z}{\mathfrak{S}}\left[(\nabla_X P)(Y,Z) + P\left(P(X,Y),Z\right) - a(X,Y)Z - P(X,Y)b(Z)\right] = 0 \tag{1.5}$$

($\mathfrak{S}$ is cyclic sum),

$$\pi_2\colon\; P(X,Y)=\psi(X)Y+\psi(Y)X+F(X)\varphi(Y)+F(Y)\varphi(X),$$
$$(\nabla_X F)(Y)+(\nabla_Y F)(X)+F(F(X))\varphi(Y)+F(F(Y))\varphi(X)= \tag{1.6}$$
$$=\mu(X)F(Y)+\mu(Y)F(X)+\rho(X)Y+\rho(Y)X;$$

$$\begin{aligned}\pi_3\colon\; P(X,Y)&=\psi(X)Y+\psi(Y)X+a(X,Y)\nu,\\ \nabla_X\nu&=\theta(X)\nu+\lambda X,\qquad \lambda\in\mathfrak{F}(M);\; X,Y,Z\in\mathfrak{X}(M),\end{aligned} \tag{1.7}$$

where $P(X,Y)=\overline{\nabla}_X Y-\nabla_X Y$ is the connection deformation tensor and $\varphi$, $\psi$, $b$, $\theta$, $\rho$, $\nu$, $a$, $F$ are tensors of the corresponding types.

In the present paper we shall study a geodesic and an almost geodesic related connections $\nabla$ and $\widetilde{\nabla}=\nabla-T$, where $T$ is a homogeneous structure on $(M,g)$.

## 2. GEODESIC MAPPINGS OF HOMOGENEOUS RIEMANNIAN MANIFOLDS

By [7], Theorem 6.8, a complete and simply connected Riemannian manifold $(M,g)$ is naturally reductive homogeneous manifold if and only if there exists a tensor field $T$ of type $(1,2)$ satisfying the conditions (1.1) and such that $\widetilde{\nabla}$ and $\nabla$ are geodesically equivalent.

Now we shall prove

**Lemma 2.1.** *If it is possible to map geodesically a homogeneous Riemannian manifold $(M,g)$ onto a manifold $(M,\widetilde{\nabla})$, then the map is affine.*

*Proof.* The connections $\nabla$ and $\widetilde{\nabla}$ are geodesically equivalent if and only if the connection deformation $D$ have the form

$$D(X,Y)=-T(X,Y)=\psi(X)Y+\psi(Y)X+S(X,Y) \tag{2.1}$$

where $\psi$ is a 1-form and the tensor field S satisfies

$$S(X,Y)+S(Y,X)=0.$$

We put

$$\overset{0}{P}(X,Y)=\frac{1}{2}\left(T(X,Y)+T(Y,X)\right)$$

and

$$\overset{0}{P}(X,Y,Z)=g(\overset{0}{P}(X,Y),Z).$$

From (1.1 (i)) we obtain

$$\underset{X,Y,Z}{\mathfrak{S}}\overset{0}{P}(X,Y,Z)=0.$$

Hence and from (2.1) we have

$$\psi(X)g(Y,Z)+\psi(Y)g(X,Z)+\psi(Z)g(X,Y)=0.$$

Therefore $\psi(X)=0$ for all $X\in\mathfrak{X}(M)$. This completes the proof. □

If $T = 0$, then (1.1) implies that $(M, g)$ is a symmetric manifold. In view of the Sinyukov theorem we obtain: if it is possible to map geodesically a complete and simply connected Riemannian manifold with the homogeneous structure $T = 0$ into a manifold $(\overline{M}, \overline{g})$ then both manifolds are of constant sectional curvature.

Let $(M, g)$ be a connected Riemannian manifold and suppose $M$ admits a non-trivial homogeneous structure $T$ by

$$T(X, Y, Z) + T(Y, X, Z) = 0, \tag{2.2}$$

where $X, Y, Z \in \mathfrak{X}(M)$.

From (1.1) and (2.2) we get easily.

**Lemma 2.2.** *Let $(M, g)$ be a connected Riemannian manifold with the homogeneous structure of type* (2.2). *Then Ricci tensor on $M$ satisfies*

$$\underset{X,Y,Z}{\mathfrak{S}}(\nabla_X Ric)(Y, Z) = 0. \tag{2.3}$$

Now we shall prove

**Theorem 2.1.** *If it is possible to map geodesically on $(M, g)$ satysfying* (2.3) *onto a manifold $(\overline{M}, \overline{g})$, then both manifolds are of constant curvature.*

*Proof.* As is well-known a manifold $(M, g)$ admits a geodesic mapping if and only if there exists a function $\varphi \in \mathfrak{F}(M)$ and a symmetric non-singular bilinear form a on $M$ satisfying

$$(\nabla_X a)(Y, Z) = (Y\varphi)g(X, Z) + (Z\varphi)g(X, Y) \tag{2.4}$$

for all $X, Y, Z \in \mathfrak{X}(M)$ ([5]).

Let $p \in M$ be such that $d\varphi \neq 0$ and (2.4) hold at $p$. Choose a local coordinate system $(U, x)$ so that $p \in U$. By $R^l_{ijk}$, $R_{ik}$, $g_{ik}$, $a_{ik}$, $\varphi_{ik}$ we denote the components of the tensors $R$, $Ric$, $g$, $a$ and the Hessian $H\varphi$ of $\varphi$ in this coordinate system. Differentiating covariantly (2.4) and applying the Ricci identity we get

$$a_{it}R^t_{jkl} + a_{tj}R^t_{ikl} = \varphi_{li}g_{jk} + \varphi_{lj}g_{ik} - \varphi_{ki}g_{jl} - \varphi_{kj}g_{il}. \tag{2.5}$$

Differentiating covariantly (2.5) with respect to $x^m$, contracting with $g^{lm}$ and applying the Ricci identity, by (2.4) and (2.3), we obtain

$$4\varphi_t R^t_{jki} = 3R^t_k\varphi_t g_{ji} - 4\varphi_k R_{ji} + 4\psi_i R_{jk} - 3g_{jk}R^t_i\varphi_t + a_j g_{ik} - a_k g_{ji}, \tag{2.6}$$

where $a_i = \nabla_s\varphi_{it}g^{ts}$. Transvecting (2.6) with $g^{jk}$ we get $R^t_i\varphi_t = \rho\varphi_i$, $\rho \in \mathfrak{F}(U)$. Following considerations made in [6] we get

$$a^t_i\varphi_t = \tau\varphi_i, \qquad \varphi^t_i\varphi_t = \lambda\varphi_i, \qquad \tau, \lambda \in \mathfrak{F}(U),$$

and finally we obtain

$$H_\varphi(X, Y) = \Phi(\varphi)g(X, Y) \tag{2.7}$$

where $H_\varphi$ is the Hessian of $\varphi$ and $\Phi \in \mathfrak{F}(M)$.

By [7] if a complete and simply connected manifold with homogeneous structure $T$ admits condition (2.7), then the manifold $(M, g)$ is of constant curvature. This completes the proof. □

From Lemmas 2.1 i 2.2 and Theorem 2.1 we obtain

**Theorem 2.2.** *On a homogeneous manifold the geodesic of* $\nabla$ *and* $\widetilde{\nabla} = \nabla - T$ *are the same if and only if* $M$ *is naturally reductive. The geodesic mapping* $(M, g) \to (M, \widetilde{\nabla})$ *is affine. If a naturally reductive manifold* $(M, g)$ *admits a non-trivial geodesic mapping onto a Riemannian manifold* $(\overline{M}, \overline{g})$*, then both manifolds are of constant curvature.*

## 3. ALMOST GEODESIC MAPPINGS OF HOMOGENEOUS MANIFOLDS

On the basis [7] the most general form of the structure tensor $T$ is following

$$T(X, Y) = g(X, Y)\Phi - g(\Phi, Y)X + \overset{2}{T}(X, Y) \tag{3.1}$$

where $\Phi$ is a given vector field on $(M, g)$ and $\overset{2}{T}$ is a tensor field such that

$$\begin{gathered} g(\overset{2}{T}(X, Y), Z) + g(Y, \overset{2}{T}(X, Z)) = 0, \\ \widetilde{\nabla}\overset{2}{T} = 0, \\ C_{12}(\overset{2}{T}) = \sum_{i=1}^{n} \overset{2}{T}(X_i, X_i) = 0, \end{gathered} \tag{3.2}$$

where $X_i$ is the base vector of the natural frame.

We put

$$\begin{aligned} \overset{1}{P}(X, Y) &= \frac{1}{2}\left(\psi(X)Y + \psi(Y)X\right) - g(X, Y), \\ \overset{1}{S}(X, Y) &= \frac{1}{2}\left(\psi(X)Y - \psi(Y)X\right), \\ \overset{2}{P}(X, Y) &= -\frac{1}{2}\left(\overset{2}{T}(X, Y) + \overset{2}{T}(Y, X)\right), \\ \overset{2}{S}(X, Y) &= -\frac{1}{2}\left(\overset{2}{T}(X, Y) - \overset{2}{T}(Y, X)\right), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} P(X, Y) &= \overset{1}{P}(X, Y) + \overset{2}{P}(X, Y) \\ S(X, Y) &= \overset{1}{S}(X, Y) + \overset{2}{S}(X, Y) \end{aligned}$$

where $\psi(X) = g(X, \Phi)$.

Here $P$ denotes the symmetric part of the tensor field $T$ and $S$ – the skew-symmetric one.

Then we have

$$\widetilde{\nabla}_X Y = \nabla_X Y + P\,(X,Y) + S(X,Y). \tag{3.4}$$

and the connection deformation tensor $D$ have the form

$$D(X,Y) = P(X,Y) + S(X,Y) \tag{3.5}$$

for all $X, Y \in \mathfrak{X}(M)$.

We shall prove

**Theorem 3.1.** *On the homogeneous Riemannian manifold the connections* $\nabla$ *and* $\widetilde{\nabla}$ *defined by* (3.3) *and* (3.4) *are almost geodesically related if and only if the tensor fields* $\overset{2}{P}$ *and* $\overset{2}{S}$ *satisfy the relations*

$$\underset{X,Y,Z}{\mathfrak{S}}\Bigg[(\nabla_X \overset{2}{P})(Y,Z) + \overset{2}{P}(\overset{2}{P}(X,Y),Z) - \overset{2}{P}(X,Y)b(Z) - \overset{2}{P}(X,\psi)g(Y,Z) + \\ - \overset{2}{P}(\overset{2}{S}(X,Y),Z) - \overset{2}{S}(X,\psi)g(Y,Z) + \\ - h(X,Y)\nabla_Z\psi + k(X,Y,Z)\psi + q(X,Y)Z\Bigg] = 0, \tag{3.6}$$

*where: b, d, h, k, q are tensors of the corresponding types.*

*Proof.* By [5] the mapping $\nabla \to \widetilde{\nabla}$ is almost geodesic if and only if the connection deformation tensor $D$ satisfies the relations

$$\left(\nabla_\gamma D^h_{\alpha\beta} + D^h_{\delta\alpha} D^\delta_{\beta\gamma}\right)\lambda^\alpha\lambda^\beta\lambda^\gamma = bD^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + a\lambda^h \tag{3.7}$$

where $\lambda^i = \frac{dc^i}{dt}$ denotes the vector tangent to the geodesic $c(t) = \left(c^i(t)\right)$. We conclude from (3.3), (3.4), (3.5), (3.7) that (3.6) holds. This proves the theorem. □

**Corollary 3.1.** *If* $\overset{2}{P} = 0$ *and* $\overset{2}{S} = 0$ *then the almost geodesic mapping is of the kind* (1.7).

**Corollary 3.2.** *If* $\overset{2}{P} = 0$ *and* $\overset{2}{S} = 0$ *then a homogeneous Riemannian manifold is a manifold of constant curvature (see [7]).*

**Corollary 3.3.** *If* $g(X,Y)\nabla_Z\psi + g(X,Y)\overset{2}{P}(Z,\psi) + g(X,Y)\overset{2}{S}(Z,\psi) + k(X,Y,Z)\psi = 0$ *then the almost geodesic mapping* (3.6) *is of the kind* (1.5).

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*Received: September 27, 2004.*