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**A NECESSARY AND SUFFICIENT CONDITION
FOR σ -HURWITZ STABILITY
OF THE CONVEX COMBINATION
OF THE POLYNOMIALS**

Abstract. In the paper are given a necessary and sufficient condition for σ -Hurwitz stability of the convex combination of the polynomials.

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1. INTRODUCTION

We will consider the set of real polynomials

$$F(x, Q) = \{a_n(q)x^n + a_{n-1}(q)x^{n-1} + \cdots + a_1(q)x + a_0(q)\},$$

where $q = (q_1, q_2, \dots, q_k) \in Q \subset R^k$, Q is a compact set, $a_i(q): Q \rightarrow R$ ($i = 0, 1, \dots, n$), $a_n(q) \neq 0$ for each $q \in Q$.

Let $\sigma \in R$ and $\sigma > 0$.

Definition 1. We shall say that the real polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n (x - x_1)(x - x_2) \cdots (x - x_n) \quad (1)$$

where $a_n \neq 0$, is Hurwitz stable if $Re(x_i) < 0$ ($i = 1, 2, \dots, n$). The polynomial (1) is called σ -Hurwitz stable if $Re(x_i) < -\sigma$ ($i = 1, 2, \dots, n$).

Definition 2. The set of the polynomials $F(x, Q)$ is called σ -Hurwitz stable if each polynomial $g(x) \in F(x, Q)$ is σ -Hurwitz stable.

Consider the interval polynomial

$$G(x) = [\underline{a}_n, \bar{a}_n]x^n + [\underline{a}_{n-1}, \bar{a}_{n-1}]x^{n-1} + \cdots + [\underline{a}_1, \bar{a}_1]x + [\underline{a}_0, \bar{a}_0],$$

and the set of the polynomials

$$W(x) = \{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 : a_i \in \{\underline{a}_i, \bar{a}_i\} (i = 0, 1, \dots, n)\}.$$

The following theorem is true

Theorem 1 (Bhattacharyya, Chapellat, Keel [2]). *The interval real polynomial*

$$G(x) = [\underline{a}_n, \bar{a}_n]x^n + [\underline{a}_{n-1}, \bar{a}_{n-1}]x^{n-1} + \cdots + [\underline{a}_1, \bar{a}_1]x + [\underline{a}_0, \bar{a}_0],$$

where $0 \notin [\underline{a}_n, \bar{a}_n]$, is σ -Hurwitz stable if and only if the set of the polynomials $W(x)$ is σ -Hurwitz stable.

Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n (x - x_1)(x - x_2) \cdots (x - x_n),$$

where $a_n \neq 0$.

Denote by $H(f)$ the Hurwitz matrix for the polynomial $f(x)$, i.e.

$$H(f) = \begin{bmatrix} a_{n-1} & a_n & 0 & 0 & 0 & \cdots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & a_0 \end{bmatrix}.$$

It is easy to see that $H(f) \in R^{n \times n}$.

Consider the real polynomials

$$f_j(x) = a_n^{(j)} x^n + a_{n-1}^{(j)} x^{n-1} + \cdots + a_1^{(j)} x + a_0^{(j)} \quad (2)$$

for $j = 1, 2, \dots, m$, where $a_n^{(j)} \neq 0$ ($j = 1, 2, \dots, m$), and the convex combinations of these polynomials

$$C(f_1, f_2, \dots, f_m) = \{\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x) : \alpha_j \geq 0 \quad (j = 1, 2, \dots, m), \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1\}.$$

In this paper we give the necessary and sufficient condition for σ -Hurwitz stability of the convex combination $C(f_1, f_2, \dots, f_m)$.

We assume that the polynomials (2) are Hurwitz stable. Hence, follows that there exists the inverse matrix $H^{-1}(f_j)$ ($j = 1, 2, \dots, m$).

Let

$$\lambda_k (H^{-1}(f_j)H(f_i)) \quad (k = 1, 2, \dots, n; i, j = 1, 2, \dots, m; j < i)$$

denote the eigenvalues of the matrix $H^{-1}(f_j)H(f_i)$.

The following theorems are true:

Theorem 2 (Białaś [3]). *If the real polynomials*

$$\begin{aligned} f_1(x) &= a_n^{(1)}x^n + a_{n-1}^{(1)}x^{n-1} + \cdots + a_1^{(1)}x + a_0^{(1)}, \\ f_2(x) &= a_n^{(2)}x^n + a_{n-1}^{(2)}x^{n-1} + \cdots + a_1^{(2)}x + a_0^{(2)}, \end{aligned}$$

where $a_n^{(1)} \neq 0$, $a_n^{(2)} \neq 0$, are Hurwitz stable, then the convex combination

$$C(f_1, f_2) = \{\alpha_1 f_1(x) + \alpha_2 f_2(x) : \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1\}$$

is Hurwitz stable if and only if

$$\lambda_k(H^{-1}(f_1)H(f_2)) \notin (-\infty, 0) \quad (k = 1, 2, \dots, n).$$

Theorem 3 (Bartlett, Hollot, Huang [1]). *If the polynomials (2) are Hurwitz stable, then the convex combination*

$$\begin{aligned} C(f_1, f_2, \dots, f_m) &= \{\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x) : \\ &\alpha_j \geq 0 \quad (j = 1, 2, \dots, m), \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1\} \end{aligned} \quad (3)$$

is Hurwitz stable if and only if the convex combinations $C(f_i, f_j)$ are Hurwitz stable for each $i, j = 1, 2, \dots, m; i < j$.

2. MAIN RESULT

Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$.

It is easy to note that for $\alpha \in R$ we have

$$g(s) = f(s + \alpha) = b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0,$$

where

$$\begin{aligned} b_0 &= f(\alpha), \\ b_i &= \frac{1}{i!} \frac{d^i f(x)}{dx^i} \Big|_{x=\alpha} \quad (i = 1, 2, \dots, n). \end{aligned}$$

As it is easy to see, we have the following result.

Lemma 1. *The real polynomial*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$, is σ -Hurwitz stable if and only if the polynomial

$$g(s) = f(s - \sigma)$$

is Hurwitz stable.

Now, we will prove

Theorem 4. *If the polynomials (2) are σ -Hurwitz stable, then the convex combination*

$$C(f_1, f_2, \dots, f_m) = \{\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x) : \\ \alpha_j \geq 0 \quad (j = 1, 2, \dots, m), \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1\}$$

is σ -Hurwitz stable if and only if

$$\lambda_k(H^{-1}(g_i)H(g_j)) \notin (-\infty, 0) \quad (k = 1, 2, \dots, n) \quad (4)$$

for $i, j = 1, 2, \dots, m$; $i < j$, where $g_i(s) = f_i(s - \delta)$, $g_j(s) = f_j(s - \delta)$.

Proof. From Lemma 1, it follows that the convex combination $C(f_i, f_j)$ is σ -Hurwitz stable if and only if the convex combination $C(g_i, g_j)$ is Hurwitz stable.

However, from Theorem 2 and 3 follows that the set $C(g_i, g_j)$ is Hurwitz stable if and only if the conditions (4) holds. This completes the proof of Theorem 4. \square

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