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THE MINIMUM EXPONENT OF THE PRIMITIVE DIGRAPHS ON THE GIVEN NUMBER OF ARCS

Abstract. Primitive digraphs on n vertices, k arcs and girth s are considered. By $a(n,k,s)$ we mean the minimum exponent taken over all such digraphs. We estimate the number $a(n,k,s)$ using the Frobenius number for special values of k and s.

Keywords: primitive directed graph, exponent, Frobenius number.

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1. INTRODUCTION

Let D be a directed graph (digraph) on n vertices, $n \geq 2$, with vertex set $V(D)$ and arc set $A(D)$. Loops are permitted. A sequence $v_0, e_1, v_1, ..., e_m, v_m$ such that $e_i = (v_{i-1}, v_i) \in A(D)$, for $i = 1, 2, ..., m$, is called a walk from v_0 to v_m in D. The length of a walk is the number of arcs in it. Note that vertices and arcs may repeat in this sequence. The length of the shortest walk from v_0 to v_m is denoted by $d(v_0, v_m)$. If $v_0 = v_m$ and $m > 0$, then a walk is called a cycle. A simple cycle is a cycle in which $v_i \neq v_j$ for $i \neq j$, except $i = 0$ and $j = m$, $m > 0$. We will use the symbol γ_k to denote a simple cycle of length k. A set of the lengths of all simple cycles in the digraph D is denoted by $l(D)$. A girth of D is denoted by $g(D)$ and it is defined by $g(D) = \min\{k \in \mathbb{N} : k \in l(D)\}$.

A digraph D is called primitive if there exists positive integer t such that there exists a walk from v_i to v_j of length t, for every ordered pair of vertices v_i,v_j (not necessarily distinct). The smallest such t is denoted by $\exp(D)$ and it is called the exponent of D. It is easy to see that, for every $m \geq \exp(D)$, there exists a walk from v_i to v_j of length m, for every ordered pair of vertices v_i,$v_j \in V(D)$.

Let $l(D) = \{p_1, p_2, ..., p_t\}$. Strong connectedness is a necessary but not sufficient condition for a digraph to be primitive.

Theorem 1. ([3]) *A digraph D is primitive if and only if D is strongly connected and* $\gcd(p_1, p_2, ..., p_t) = 1$.

If there are some loops in a strongly connected digraph D, then D is primitive. It is not difficult to determine the exponent in this case. To estimate the exponent of a primitive digraph without loops we use the Frobenius number.

The Frobenius number $F(p_1, p_2, ..., p_t)$ is defined if $\gcd(p_1, p_2, ..., p_t) = 1$. It is the smallest integer k, such that every $m \geq k$ can be expressed by the combination $m = a_1p_1 + a_2p_2 + ... + a_tp_t$, for some nonnegative integers a_1, a_2, ..., a_t.There are known the following properties of the Frobenius number:

$$F(p_1, p_2) = (p_1 - 1)(p_2 - 1), \tag{1}$$

$$F(p, p+1, p+2, ..., n) = p \left\lfloor \frac{n-2}{n-p} \right\rfloor, \quad \text{see [2]}. \tag{2}$$

Using (2) we obtain

$$F(p, p+1, ..., n) = p, \quad \text{for } n \geq 2p - 1. \tag{3}$$

Given $V(D) = \{v_1, ..., v_n\}$, define the number $r_{i,j}$ as follows: if $i = j$ and for every $k \in l(D)$ there exists a simple cycle γ_k containing vertex v_i, then $r_{i,j} = 0$; otherwise $r_{i,j}$ is the length of the shortest walk α from v_i to v_j such that, for every $k \in l(D)$, there exists a simple cycle γ_k which has at least one vertex in common with walk α. Dulmage and Mendelsohn gave the upper bound of the exponent of a primitive digraph expressed by the lengths of the simple cycles in it.

Theorem 2. ([2]) *Let D be a primitive digraph on n vertices and $l(D) = \{p_1, p_2, ..., p_t\}$. Then* $\exp(D) \leq F(p_1, ..., p_t) + \max_{1 \leq i,j \leq n} r_{i,j}$.

In this paper we estimate the number

$$a(n, k, s) = \min\{\exp(D)\colon\ |V(D)| = n, |A(D)| = k, g(D) = s\}.$$

The case $s = 1$ was discussed in [4]. It is interest to find the upper bound for $s > 1$.

Note that Theorem 1 implies $k \geq n + 1$.

Proposition 1. *$a(n, k+1, s) \leq a(n, k, s)$ for $k \geq n+1$ and $s \geq 1$.*

Proof. Let D_0 be primitive digraph on n vertices, k arcs and girth s, satisfying $\exp(D_0) = a(n, k, s)$. Let D be a primitive digraph on n vertices and girth s obtained from digraph D_0 by adding one arc, i.e. $V(D) = V(D_0)$ and $A(D) = A(D_0) \cup \{e\}$, for some $e \in A(\overline{D_0})$. Since every walk existing in the digraph D_0 is a walk in the digraph D, then we have $\exp(D) \leq \exp(D_0)$. Thus $a(n, k+1, s) \leq \exp(D) \leq \exp(D_0) = a(n, k, s)$. □

In other words, the number $a(n, n+1, s)$ is the upper bound of the number $a(n, k, s)$, for every $k > n + 1$.

It is interesting to study not so much the largest possible value of k as the minimum possible value of the number $a(n,k,s)$. Next lemma gives the upper bound of the minimum value of $a(n,k,s)$:

Lemma 1. ([2]) *If D is a primitive digraph with n vertices and girth s, then*

$$\min_D\{\exp(D)\} \leq s + s\left\lfloor \frac{n-2}{n-s} \right\rfloor.$$

MAIN RESULTS

If $n \geq 2s-1$ and $s \geq 2$ then by (3) and Lemma 1, we obtain $\min_D\{\exp(D)\} \leq 2s$. Moreover, the smallest possible exponent of a digraph on n vertices and girth s is the smallest possible value of $a(n,k,s)$. It turns out that this bound can be improved.

Theorem 3. *If $s \geq 3$, $n \geq 2s-1$ and $k \geq 3n-2s+1$, then $a(n,k,s) \leq 2s-2$.*

Proof. Let $s \geq 3$ and D be a digraph with the set of vertices $V(D) = \{v_1, ..., v_n\}$ and $A(D) = \bigcup_{i=1}^{s-2}\{(v_i, v_{i+1})\} \cup \bigcup_{i=s}^{n}\{(v_{s-1}, v_i), (v_i, v_1)\} \cup \bigcup_{i=s}^{n-1}\{(v_i, v_{i+1})\} \cup \{(v_n, v_s)\}$ be the set of arcs of the digraph D (see Fig. 1). It is not difficult to observe that $l(D) = \{s, s+1, ..., n\}$, so D is primitive.

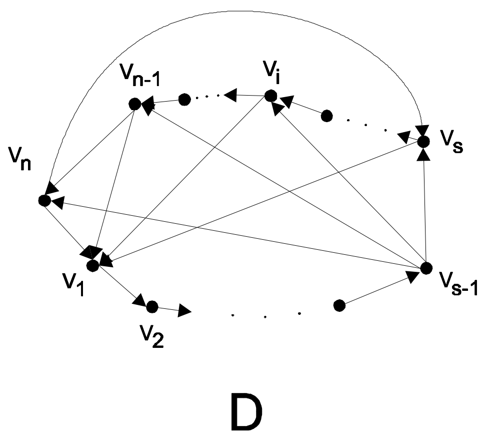

Fig. 1

Because $|A(D)| = 3n-2s+1$ and $g(D) = s$, then we have $a(n, 3n-2s+1, s) \leq$ $\leq \exp(D)$. Note that the inequality $n \geq 2s-1$ is equivalent to $n-s+1 \geq s$, so there exists a simple cycle γ_{n-s+1} through vertices: v_s, v_{s+1}, ..., v_n (otherwise, if $n \leq 2s-2$, then the length of this cycle is less than s). Moreover, every vertex of $V(D)$ belongs to some simple cycle γ_m, for every $m \in l(D)$. Hence, we have $r_{ii} = 0$

for $i = 1, 2, ..., n$ and $r_{i,j} = d(v_i, v_j)$ for $i \neq j$. For every ordered pair of vertices v_i, v_j, we have $d(v_i, v_j) \leq s$, so $\max r_{i,j} \leq s$. Thus by (3) and Theorem 2 we obtain $\exp(D) \leq 2s$.

Now we prove that $\exp(D) = 2s - 2$. Let $W_{i,j}$ be the set of the lengths of all walks from v_i to v_j in D and let $W = \bigcap_{1 \leq i,j \leq n} W_{i,j}$. Since D is primitive, there exists the natural number t such that, for every $m \geq t$, $m \in W$. The smallest integer $m \in W$ determines $\exp(D)$. We observe that $W_{1,s-1} = \{s-2, 2s-2, 2s-1, 2s, ...\}$, so $\exp(D) \geq 2s - 2$. Because $\{2s-2, 2s-1, 2s, ...\} \subset W_{i,j}$ while $i \neq 1$ or $j \neq s-1$, we have $W = \{2s-2, 2s-1, 2s, ...\}$. Thus $a(n, 3n-2s+1, s) \leq \exp(D) = 2s-2$. By Lemma 1, the result follows. □

The lower bound of the number $a(n, k, s)$ is equal to s. For $s = 1$, it is reached. A complete digraph with loops is the only one, which satisfies $\exp(D) = 1$. Next theorem shows that this lower bound is also reached for $s = 2$.

Theorem 4. *$a(n, k, 2) = 2$, for $n \geq 3$ and $3n - 3 \leq k \leq n^2 - n$.*

Proof. We first note that the assumption $g(D) = 2$ implies $k \leq n^2 - n$. Let D be a digraph on n vertices and $A(D) = \bigcup_{i=2}^{n} \{(v_1, v_i), (v_i, v_1)\} \cup \bigcup_{i=2}^{n-1} \{(v_i, v_{i+1})\} \cup \{(v_n, v_2)\}$. We have $g(D) = 2$ and $l(D) = \{2, 3, ..., n\}$. So D is primitive. Since $|A(D)| = 3n-3$, we have $a(n, 3s-3, 2) \leq \exp(D)$. It is easy to see that any two vertices are joined by a walk of length 2, so $\exp(D) = 2$. This is the smallest possible value of the exponent in this case, so $a(n, 3n-3, 2) = 2$ and, by Lemma 1, $a(n, k, 2) = 2$, for $k \geq 3n-3$. □

Now we consider digraphs with $n + 1$ arcs to obtain the lower bound for the number $a(n, n+1, s)$. If there are $n+1$ arcs in a primitive digraph, then there exist exactly two simple cycles in it.

Theorem 5. *If $s \geq 1$ and $n \geq 2s$, then $a(n, n+1, s) \geq (s+1)(n-s)$. Moreover, if $\gcd(s, n+1) = 1$, then the upper bound is attained.*

Proof. Let D be a primitive digraph on n vertices, $n+1$ arcs and girth s. Let $l(D) = \{s, m\}$. Since D is primitive and $g(D) = s$, we have $m \geq s+1$ and $\gcd(s, m) = 1$. Let p be the number of common vertices in both cycles. Of course $1 \leq p \leq s$. We have $n = m + s - p$, so $m = n + p - s$. Note that $p = 1$ is possible while $n \geq 2s$. The length of the longest walk which does not contain common vertices, is equal to $m - p - 1$, so $\max r_{i,j} = 2m - p - 1$. It is easy to calculate that $\exp(D) = F(m, s) + 2m - p - 1$. By (1) we obtain $\exp(D) = (m-1)(s-1) + 2m - 1 - p = (s+1)(m-1) - p + 1 = (n-s)(s+1) + s(p-1)$. Since $p \geq 1$, we have $\exp(D) \geq (n-s)(s+1)$. Hence, $a(n, n+1, s) \geq (n-s)(s+1)$. If $p = 1$ then $m = n+1-s$ and $\exp(D) = (n-s)(s+1)$, while $\gcd(n+1, s) = 1$. □

Theorem 6. *If $|A(D)| = n+1$, $s \geq 2$ and $n \leq 2s-1$, then $a(n, k, s) = n + s(s-1)$.*

Proof. Let D be a primitive digraph on n vertices, girth s and $|A(D)| = n+1$. Let $l(D) = \{s, m\}$. Since D is primitive, we have $\gcd(s, m) = 1$ and $m \geq s+1$. Note, that the assumption $2s-1 \geq n$ implies $s+m-n \geq 2$. There exist $n-s$ vertices in a simple cycle γ_m, which are not in common with cycle γ_s. Let α be the longest walk in D which does not contain common vertices. The length of α is equal to $n-s-1$. Let x and y be the first and the last vertex of a walk α, respectively. We have $d(x,y) = n-s-1$. If a walk α' is any other walk from x to y, then the length of α' is equal to $n-s-1+m+am+bs$ for some non-negative integers a, b. There is no walk from x to y of the length $n-s-1+m+F(s,m)-1$, so $\exp(D) \geq n-s-1+m+F(s,m) = n-s-1+m+(s-1)(m-1) = n+s(m-2)$. Since $m \geq s+1$, we obtain $exp(D) \geq n+s(m-2) \geq n+s(s-1)$.

Note that the inequality $n \leq 2s-1$ is equivalent to $2s+1 \geq n+2$. This assumption implies the existence of a digraph D such that $m = s+1$. Indeed: if there are $2s+1-n$ common vertices in cycles γ_m and γ_s, then $m = s+1$. Thus, for every pair of the numbers n, s satisfying $n \leq 2s-1$, there exists a digraph D with $l(D) = \{s, s+1\}$ and hence, $a(n, n+1, s) = n+s(s-1)$. □

In [4] it was proved that, for $n \geq 2$, $s \geq 1$, $a(n, n+1, s) \geq 2n-2$. We can prove this inequality by using Theorem 5 and Theorem 6.

Theorem 7. *For $n \geq 2$ and $s \geq 1$, $a(n, n+1, s) \geq 2n-2$.*

Proof. If $s = 1$, then by Theorem 5, we have $a(n, n+1, 1) \geq 2n-2$. Let $s \geq 2$. Suppose that $a(n, n+1, s) < 2n-2$. Two cases are considered:

(i) $n \geq 2s$
By Theorem 5 we have following inequality: $(n-s)(s+1) < 2n-2$, hence, we obtain equivalently $(s-1)(n-s-2) < 0$. The assumption $s \geq 2$ implies $n-s-2 < 0$. It means that $n-s \leq 1$. This is a contradiction because $n-s \geq s \geq 2$.

(ii) $n \leq 2s-1$
The Theorem 6 gives $a(n, k, s) = n+s(s-1)$. The inequality $n+s(s-1) < 2n-2$ implies $n > s^2-s+2$. Since $s \geq 2$, we have $s^2-s+2 \geq 2s$ and hence $n \geq 2s$. It contradicts the assumption $n \leq 2s-1$.

Finally, $a(n, n+1, s) \geq 2n-2$, for every $n \geq 2$ and $s \geq 1$. □

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