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Bogdan Choczewski, Anna Wach-Michalik

EULER'S BETA FUNCTION DIAGONALIZED AND A RELATED FUNCTIONAL EQUATION

Abstract. Euler's Gamma function is the unique logarithmically convex solution of the functional equation (1), cf. the Proposition. In this paper we deal with the function $\beta\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\beta(x) := B(x,x)$, where $B(x,y)$ is the Euler Beta function. We prove that, whenever a function h is asymptotically comparable at the origin with the function $a\log + b$, $a > 0$, if $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies equation (5) and the function $h \circ \varphi$ is continuous and ultimately convex, then $\varphi = \beta$.

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1. PRELIMINARIES

The Euler Gamma function (cf., e.g., [8]), usually defined by the formula:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad x \in \mathbb{R}_+$$

(where $\mathbb{R}_+$ stands for $(0, +\infty)$), satisfies the functional equation

$$\varphi(x+1) = x\,\varphi(x), \quad x \in \mathbb{R}_+; \quad \varphi(1) = 1. \tag{1}$$

It is also the limit

$$\Gamma(x) = \lim_{n\to\infty} \Gamma_n(x), \quad x \in \mathbb{R}_+. \tag{2}$$

of the functional sequence

$$\Gamma_n(x) := \frac{n^x n!}{x^{[n]}}, \quad x \in \mathbb{R}_+, \tag{3}$$

where $x^{[n]}$ is the forward product polynomial

$$x^{[n]} = x(x+1)\dots(x+n), \quad n \in \mathbb{N} \cup \{0\}, \quad x \in \mathbb{R}_+.$$

The general solution of equation (1) is given by:

$$\varphi(x) = p(x)\Gamma(x), \quad x \in \mathbb{R}_+,$$

where $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a periodic function with period 1. One may ask under what additional conditions the function Γ is the unique solution of (1). The following famous result of H. Bohr and J. Mollerup [2] (cf. [1]) gives an answer to the question.

Proposition 1. *If a function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies equation* (1) *and* $\log \circ \varphi$ *is a convex function on an interval* $(c, +\infty) \subset \mathbb{R}_+$, *then* $\varphi = \Gamma$.

The proof of the Proposition may also be found in [5], or in [7], Theorem 10.4.1, p. 407). The role of convexity in the theory of the Gamma function is thoroughly discussed in [4].

In the Ph.D. thesis [9] of the second author (cf. also [10]) there is showed that in the Proposition the logarithmic function may be replaced by any function asymptotically comparable at infinity with the logarithm.

2. BETA FUNCTION DIAGONALIZED

The Euler Beta function (cf., e. g., [8]), usually defined by the formula:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x, y \in \mathbb{R}_+$$

is connected with Γ:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y \in \mathbb{R}_+.$$

We define the function $\beta\colon \mathbb{R}_+ \to \mathbb{R}_+$ by the formula

$$\beta(x) := B(x,x), \quad x \in \mathbb{R}_+.$$

Therefore

$$\beta(x) = \frac{\Gamma(x)^2}{\Gamma(2x)}, \quad x \in \mathbb{R}_+, \tag{4}$$

and we get from (1)

$$\beta(x+1) = \frac{\Gamma(x+1))^2}{\Gamma(2x+2)} = \frac{x^2(\Gamma(x))^2}{(2x+1)2x\Gamma(2x)} = \frac{x}{2(2x+1)}\beta(x),$$

Thus the function $\varphi = \beta$ is a particular solution of the functional equation

$$\varphi(x+1) = \frac{x}{2(2x+1)}\,\varphi(x), \quad x \in \mathbb{R}_+, \tag{5}$$

such that $\varphi(1) = 1$.

Given a function $h\colon \mathbb{R}_+ \to \mathbb{R}$ let $\mathbf{\Phi}_h$ be the following class of functions:

$$\mathbf{\Phi}_h = \{\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+ : \varphi \text{ satisfies (5)}, \ \varphi(1) = 1, \ h \circ \varphi \text{ is continuous and convex}\}$$

We aim at proving that $\mathbf{\Phi}_h = \{\beta\}$ whenever h is a function asymptotically comparable at the origin with log. The paper is mostly based on results taken from [9]. Some facts may be found in our paper [3] in which the problem of characterization of q-Gamma functions has been dealt with.

3. LINEAR DIFFERENCE EQUATION

Equation (5) is a particular case of the linear difference equation of the first order:

$$\varphi(x+1) = g(x)\ \varphi(x), \quad x \in \mathbb{R}_+, \tag{6}$$

which, in turn, is a special case of the iterative functional equation

$$\varphi(f(x)) = g(x)\varphi(x),$$

with the given functions f and g. The properties of solutions of equation (6), collected in the subsequent lemma, can be deduced from formula (3.1.4) and Theorem 3.1.1. on p. 97 of [7], valid for the latter equation. The sketch of a direct proof of the lemma is supplied for the sake of completeness.

Lemma 1.

1) *If a function* $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies equation* (6) *then*

$$\varphi(x+n) = \varphi(x) \prod_{k=0}^{n} g(x+k), \quad x \in \mathbb{R}_+, \quad n \in \mathbb{N}, \tag{7}$$

and

$$\varphi(n+1) = \prod_{k=1}^{n} g(k), \quad n \in \mathbb{N}. \tag{8}$$

2) *If the functions* $\varphi_1, \varphi_2\colon \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy equation* (6) *and* $\varphi_1|_{(0,1]} = \varphi_2|_{(0,1]}$, *then* $\varphi_1 = \varphi_2$.

Proof. Ad 1) Formulae (7) are checked by induction. Substituting $x = n+1$ in (6), and $x = 1$ in (7) we see that $g(n+1)\varphi(n+1) = g(n+1)\prod_{k=0}^{n-1} g(1+k)$, and (8) is also true.

Ad 2) This follows from the form $\varphi = p\varphi_0$ of the general solution to (6), where p is a 1-periodic function, and φ_0 is a particular solution to (6). $\square$

The other lemma contains a formula derived from (2), (3), (4), (5) and 1) of Lemma 1. We introduce the notion:

$$\beta_n(x) := \frac{\Gamma_n(x)^2}{\Gamma_{2n+1}(2x)}\,, \quad x \in \mathbb{R}_+, \quad n \in \mathbb{N}, \tag{9}$$

where Γ_n is defined by (3). Of course, because of (2) we have

$$\beta(x) = \lim_{n\to\infty} \beta_n(x) \quad x \in \mathbb{R}_+. \tag{10}$$

Lemma 2. *If a function* $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies equation* (5) *then*

$$\frac{\varphi(x+n+1)}{\varphi(n+1)} = \left(\frac{n}{2n+1}\right)^x \cdot \frac{\varphi(x)}{\beta_n(x)}\,, \quad x \in \mathbb{R}_+, \quad n \in \mathbb{N}. \tag{11}$$

Proof. . From (5) with x replaced by $x+n+1$ one gets

$$\varphi(x+n+1) = \frac{x^{[n]}}{2^{n+1}(2x+1)\cdots(2x+2n+1)}\varphi(x) = \frac{\left(x^{[n]}\right)^2}{(2x)^{[2n+1]}}\varphi(x), \tag{12}$$

Thanks to (3) we obtain

$$x^{[n]} = \frac{n^x n!}{\Gamma_n(x)}; \qquad (2x)^{[2n+1]} = \frac{(2n+1)^{2x}(2n+1)!}{\Gamma_{2n+1}(2x)}.$$

Coming back to (12) we find that

$$\varphi(x+n+1) = \left(\frac{n}{2n+1}\right)^{2x} \cdot \frac{\Gamma_{2n+1}(2x)}{\Gamma_n(x)^2} \cdot \frac{(n!)^2}{(2n+1)!} \cdot \varphi(x), \quad x \in \mathbb{R}_+, \quad n \in \mathbb{N}. \tag{13}$$

But (10) applied to $g(x) = x/(2(2x+1))$ yields

$$\varphi(n+1) = \frac{n!}{2^n(2n+1)!!} = \frac{(n!)^2}{(2n+1)!}.$$

This, (9) and (13) show that (11) actually holds true. □

4. MAIN RESULT

Now we are in position to formulate and prove the already announced result.

Theorem 1. *Let* $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, *and let* $h\colon \mathbb{R}_+ \to \mathbb{R}$ *be a continuous function which is convex on an interval* $(c, +\infty) \subset \mathbb{R}_+$ *and satisfies the condition*

$$h(x) = a\log x + b + o(1), \qquad x \to 0^+. \tag{\textbf{A}}$$

Then $\mathbf{\Phi}_h = \{\beta\}$.

Proof. Fix an $x \in (0,1]$, take a solution $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(1) = 1$, of equation (5) and assume that the function

$$f := h \circ \varphi$$

is continuous and convex on $(c, +\infty) \subset \mathbb{R}_+$. Thus the difference quotient of f is an increasing function of one argument with the other fixed, cf. [6], Theorem 7.3.2, p. 153. Taking the quotients in the intervals $[n, n+1]$, $[n+1, x+n+1]$, $[n+1, n+2]$, consecutively, we obtain the inequalities

$$f(n+1) - f(n) < \frac{1}{x}[f(x+n+1) - f(n+1)] \leq f(n+2) - f(n+1), \quad n \in \mathbb{N} \cap (c, +\infty). \tag{14}$$

Introducing, for short, the notion:

$$H_n(x) := f(x+n+1) - (x+1)f(n+1) + xf(n), \quad n \in \mathbb{N} \cup (c, \infty), \tag{15}$$

we may rewrite (14) in the form

$$0 \leq H_n(x) \leq x\ H_n(1). \tag{16}$$

Using equation (5) with x replaced by $x+n$ we get

$$\lim_{n\to\infty} \frac{\varphi(x+n+1)}{\varphi(x+n)} = \lim_{n\to\infty} \frac{(x+n)}{2(2x+2n+1)} = \frac{1}{4},$$

whence

$$\lim_{n\to\infty} \varphi(x+n) = 0, \quad n \in \mathbb{N}.$$

(This equality remains true when $x = 0$ but it requires a separate argument, however, the same as above. The departure point is then equation (5) with $x = n$ which yields $\varphi(n+1)/\varphi(n) = n/(2(2n+1))$.)

Now we may use the relation (**A**) with that x replaced by $\varphi(x+n)$ to arrive at

$$f(x+n) = h[\varphi(x+n)] = a(\log[\varphi(x+n)] + b + o(1), \quad n \to \infty. \tag{17}$$

(*In the rest of the proof all symbols* $o(1)$ *refer to* $x \to 0^+$.)

With g standing for $\log \circ \varphi$ we obtain from (17) the following relations:

$$\begin{aligned} f(x+n) &= ag(x+n) + b + o(1); \\ f(n+1) &= ag(n+1) + b + o(1); \\ f(n) &= ag(n) + b + o(1). \end{aligned}$$

Finally, on introducing $G_n(x)$ defined by (15) with that f replaced by g, we get

$$H_n(x) = aG_n(x) + o(1). \tag{18}$$

Now we want to find the limit of $G_n(1)$ as $n \to \infty$. According to (15) we may write

$$G_n(1) = g(n+2) - 2g(n+1) + g(n) = \log\left[\frac{\varphi(n+2)}{\varphi(n+1)} \cdot \frac{\varphi(n)}{\varphi(n+1)}\right] = \log\frac{(n+1)(2n+1)}{n(2n+3)}$$

Therefore $G_n(x) = o(1)$. Relation (18) for $x = 1$ implies $H_n(1) = o(1)$. Since inequalities (16) hold, $H_n(x) = o(1)$ as well. Finally, formula (18) yields

$$G_n(x) = o(1). \tag{19}$$

What is left is to calculate $G_n(x)$. First of all, from (15) (with g in place of f), we have for $n \in \mathbb{N} \cup (c, \infty)$

$$G_n(x) := g(x+n+1) - (x+1)g(n+1) + xg(n) = \log\left[\frac{\varphi(x+n+1)}{\varphi(n+1)} \cdot \left(\frac{\varphi(n)}{\varphi(n+1)}\right)^x\right].$$

Now we apply (11) and equation (5) for $x = n$ and observe that

$$G_n(x) = \log\left[\left(\frac{n}{2n+1}\right)^{2x} \cdot \frac{\varphi(x)}{\beta_n(x)} \cdot \left(\frac{2(2n+1)}{n}\right)^x\right] = \log\left[\left(\frac{2n}{2n+1}\right)^x \cdot \frac{\varphi(x)}{\beta_n(x)}\right].$$

This means that (cf. (10))

$$\lim_{n\to\infty} G_n(x) = \log\frac{\varphi(x)}{\beta(x)}.$$

On the other hand, (19) says that the limit is zero. Hence we have proved that

$$\varphi(x) = \beta(x), \quad x \in (0, 1].$$

Part 2) of Lemma 2 implies that such solutions of (5) coincide in their whole domain of definition, i.e., if $\varphi \in \mathbf{\Phi}_h$ then $\varphi = \beta$, as claimed. □

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Bogdan Choczewski
smchocze@cyf-kr.edu.pl

Anna Wach-Michalik
wach@agh.uci.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Cracow, Poland

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