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DISCRETIZATION OF THE STATIONARY DISTRIBUTION OF HEAT IN THE NON-HOMOGENEOUS BODY

Abstract. We give a short survey on the theory of the mixed boundary-value problem for the stationary Fourier equation in a non-homogeneous medium defined on any Lipschitz domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$. The compatibility condition for the thermal flux has been established by the standard procedure of integration the divergence.

Keywords: elliptic partial differential equations, stationary distribution of heat, discretization method.

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1. INTRODUCTION

The present paper supplements and continues [2, 3, 4, 5, 8] and [1] and includes a fragment of the general theory of the mixed boundary-value problem for the diffusion equation in a non-homogeneous medium. The compatibility condition (4) for the thermal flux has been established by the standard procedure of integration the divergence (see e.g. [7] chap. I, (1, sec. 1.3)).

We look for a stationary distribution of a temperature T in a solid Ω (which arises by sticking of some homogeneous materials $\Omega_1, \ldots, \Omega_s$ with thermal conductivities $\theta_1, \ldots, \theta_s$. Heat sources f, a temperature b on a piece $\partial_I \Omega$ of the boundary of Ω (so-called Dirichlet boundary) and overall heat-transfer coefficients α, β on the remaining part of the boundary (that is, on $\partial_{III}\Omega := (\partial\Omega) \setminus (\partial_I \Omega)$, called the Fourier boundary) are treated as known. Let us say that in any homogeneous material Ω_j the heat propagates in agreement with the Fourier law (1). On the borderland of Ω_i and Ω_j, $i \neq j$, we postulate a consistency of normal components of the heat fluxes, i.e. the local heat balance (2). Statement of this condition is a result of our co-operation with M. Danielewski and W. Krzyżański [6]. Both of these laws have a local character (they are described by differential operators) and follow from the more fundamental integral law:

$$\frac{d}{dt}\int_B T(t,x)\,dx = \int_{\partial B} \mathbf{n}_B(\zeta) J(t,\zeta)\,d\sigma_B(\zeta) + \int_B f(t,x)\,dx \quad \text{for every } B \subset \Omega \text{ with a regular boundary} \tag{1}$$

where the vector-valued function J is interpreted as a heat flux, $\mathbf{n}_B$ means the inward normal with regard to B, and σ_B – the surface element on ∂B. In the present paper we suppose that J is a "plywood" of Fickxian fluxes, i.e.

$$J(t,x) = -\theta_i \cdot \left(\frac{\partial T}{\partial x_1}(t,x), \ldots, \frac{\partial T}{\partial x_n}(t,x)\right) \quad \text{for} \quad x \in \Omega_i \quad (i = 1, \ldots, s). \tag{2}$$

The law (1) is more general than appropriate differential equations, because it admits discontinuous heat fluxes; in particular the flux (2) is, in general, discontinuous.

2. STATIONARY DISTRIBUTION OF HEAT IN A NON-HOMOGENEOUS MEDIUM – INTRODUCTION

Let $\mathbb{N} \ni n \geq 2$. Consider a bounded Lipschitz domain Ω in $\mathbb{R}^n$ (i.e. Ω is bounded, open and connected, while the boundary $\partial\Omega$ of Ω satisfies the Lipshitz condition). Suppose that $1 \leq s \in \mathbb{N}$ and $\Omega_1, \ldots, \Omega_s \in \operatorname{top}\Omega$ (i.e. Ω_i is open in Ω, $\forall i$). Moreover $\Omega_1, \ldots, \Omega_s$ are pairwise disjoint and $m\left(\Omega \setminus \bigcup_{j=1}^{s} \Omega_j\right) = 0$, where m stands for the Lebesgue measure in $\mathbb{R}^n$. We assume that for every j the boundary of fulfills the Lipschitz condition. So, the field $\mathbf{n}^j : \partial\Omega_j \to \mathbb{R}^n$ of normal vectors, inward with respect to Ω_j, has a domain of full measure in the sense of the surface element $\sigma_j : \mathcal{B}(\partial\Omega_j) \to \mathbb{R}_+$ on $\partial\Omega_j$. The $\mathit{dom}\,\mathbf{n}^j$ may be only a subset of $\partial\Omega_j$ and the symbol $\mathcal{B}(\partial\Omega_j)$ stands for the σ-algebra of all Borel subsets of the topological space $\partial\Omega_j$. Lastly, let $\sigma : \mathcal{B}(\partial\Omega) \to \mathbb{R}_+$ denote the surface element of $\partial\Omega$. Distinguish an arbitrary Borel subset $\partial_I\Omega$ of $\partial\Omega$ and denote:

$$\partial_{III}\Omega := (\partial\Omega) \setminus (\partial_I\Omega), \qquad \partial_{III}\Omega_j := (\partial\Omega_j) \cap (\partial_{III}\Omega).$$

Now we shall give the classical formulation of the mixed boundary value-problem for the stationary Fourier equation with singularities.

At this stage we do not discuss the regularity of considered functions. Assuming the following
Data:

$$\theta_1, \ldots, \theta_s \in]0, \infty[;$$
$$f : \Omega \to \mathbb{R};$$

$$b\colon \partial_I\Omega \to \mathbb{R};$$
$$\alpha, \beta\colon \partial_{III}\Omega \to \mathbb{R},$$

we look for a function

$$T\colon \Omega \to \mathbb{R}$$

satisfying the following

Physical laws:

$$0 = \theta_j \cdot \Delta T + f \quad \text{in} \quad (j = 1, \dots, s) \tag{3}$$

and

$$\left(\theta_i \frac{\partial T}{\partial \mathbf{n}^i} + \theta_j \frac{\partial T}{\partial \mathbf{n}^j}\right)(x) = 0 \quad \text{for any} \quad i \neq j \\ \text{and } \sigma_i\text{-almost every } x \in (\partial\Omega_i) \cap (\partial\Omega_j) \tag{4}$$

and the

Boundary conditions:

$$T = b \qquad \text{on } \partial_I\Omega, \tag{5}$$

$$\frac{\partial T}{\partial \mathbf{n}} = \alpha T + \beta \quad \text{on } \partial_{III}\Omega, \tag{6}$$

where $\frac{\partial T}{\partial \mathbf{n}^i} := \lim_{\Omega_i \ni z \to x} \mathbf{n}^i(x) \nabla_z T$ (scalar product of the vectors $\mathbf{n}^i(x)$ and $\nabla_z T$), $\nabla := \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$, $\Delta := \mathrm{div}\nabla = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $\mathbf{n}\colon \partial\Omega \to \mathbb{R}^n$ is the inward normal with respect to Ω.

To put a definition of a solution of the problem (3)–(6) with full mathematical accuracy, we have to go across the following

Heuresis. Let $\varphi\colon \overline{\Omega} \to \mathbb{R}$ vanish on $\partial_I\Omega$. Using the Fourier equation in Ω_j, integrating by parts and applying (6) we calculate:

$$
\begin{aligned}
0 &= \int\limits_{\Omega_j} (\theta_j \Delta T + f)\, \varphi \, dm = \\
&= \int\limits_{\Omega_j} f\varphi \, dm - \theta_j \int\limits_{\partial\Omega_j} \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j - \theta_j \int\limits_{\Omega_j} (\nabla T)(\nabla\varphi)\, dm = \\
&= \int\limits_{\Omega_j} f\varphi \, dm - \int\limits_{\Omega\cap\partial\Omega_j} \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j - \theta_j \int\limits_{\partial_{III}\Omega_j} (\alpha T + \beta)\varphi \, d\sigma - \theta_j \int\limits_{\Omega_j} (\nabla T)(\nabla\varphi)\, dm.
\end{aligned}
$$

Adding the obtained equalities within the range of $1 \leq j \leq s$ we get:

$$\sum_{j=1}^s \theta_j \left(\int\limits_{\Omega_j} (\nabla T)(\nabla\varphi)\, dm + \int\limits_{\partial_{III}\Omega_j} (\alpha T + \beta)\varphi \, d\sigma \right) + \sum_{j=1}^s \int\limits_{\Omega\cap\partial\Omega_j} \theta j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j = \\ = \int\limits_{\Omega} f\varphi \, dm. \tag{7}$$

We denote

$$M_{ij} := \Omega \cap (\partial\Omega_i) \cap (\partial\Omega_j) \quad \text{for different} \quad i, j \in \{1, \dots, s\}$$

and remark that

$$\Omega \cap \partial\Omega_j = \bigcup_{i \neq j} M_{ij}, \quad \sigma_j \left(M_{i_1,j} \cap M_{i_2,j} \right) = 0 \quad \text{for} \quad i_1 \neq i_2.$$

Hence, using (4) and dividing the set $\mathcal{A} := \{(i,j) \in \{1,\dots,s\}^2 \colon i \neq j\}$ into packets, we calculate:

$$\sum_j \int\limits_{\Omega\cap\partial\Omega_j} \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j = \sum_{(i,j)\in\mathcal{A}} \int\limits_{M_{ij}} \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j = \sum_{(i,j)\in\mathcal{A}} \int\limits_{M_{ji}} \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_i =$$
$$= \sum_{(k,l)\in\mathcal{A}} \int\limits_{M_{kl}} \theta_k \frac{\partial T}{\partial \mathbf{n}^k} \varphi \, d\sigma_l = - \sum_{(k,l)\in\mathcal{A}} \int\limits_{M_{kj}} \theta_l \frac{\partial T}{\partial \mathbf{n}^l} \varphi \, d\sigma_l = - \sum_{(i,j)\in\mathcal{A}} \int\limits_{M_{ij}} \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j.$$

Thus $\sum_j \int_{\Omega\cap\partial\Omega_j} \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j = 0$, which reduces the equality (7) to the form:

$$a_{\theta,\alpha}(T, \varphi) = \xi(\varphi), \tag{8}$$

where

$$a_{\theta,\alpha}(T,\varphi) := \sum_{j=1}^{s} \theta_j \left(\int\limits_{\Omega_j} (\nabla T)(\nabla \varphi) \, dm + \int\limits_{\partial_{III}\Omega_j} \alpha T \varphi \, d\sigma \right), \tag{9}$$

$$\xi(\varphi) := \int\limits_{\Omega} f \varphi \, dm - \sum_{j=1}^{s} \theta_j \int\limits_{\partial_{III}\Omega_j} \beta \varphi \, d\sigma \tag{10}$$

and $\theta := (\theta_1, \dots, \theta_s)$ $(\in \mathbb{R}^s)$. □

Now we shall present an exact formulation of the boundary-value problem (3)–(6). In order to attain this we introduce the following special space of test functions:

$$\Phi := \{\varphi \in H^1 \colon \varphi_{|\partial_I \Omega} = 0\} \tag{11}$$

where as usually

$$H^1(\Omega) := \left\{ \varphi \in L^2(\Omega) \colon \frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_n} \in L^2(\Omega) \right\}$$

and $\frac{\partial \varphi}{\partial x_i}$ is the only element of $L^2(\Omega)$ for which

$$\left(\frac{\partial \varphi}{\partial x_i} \mid \psi \right)_{L^2(\Omega)} = - \left(\varphi \mid \frac{\partial \psi}{\partial x_i} \right)_{L^2(\Omega)} \tag{12}$$

Here

$$\mathfrak{D}(\Omega) := C^\infty(\Omega) \cap \{\text{compact support}\} = \left\{\psi\colon \Omega \to \mathbb{R} \text{ of class } C^\infty \;\mid\; \overline{\{\psi \neq 0\}} \subset \Omega\right\},$$

while $L^2(\Omega)$ is defined as the factor space:

$$L^2(\Omega) := \{\text{square summable real functions on } \Omega\} \,/ \\ /\, \{\text{almost everywhere vanishing functions on } \Omega\}\,.$$

The linear spaces $L^2(\Omega)$, $H^1(\Omega)$ are separable Hilbert spaces with the scalar products

$$\begin{aligned}(p_1 \mid p_2)_{L^2(\Omega)} &:= \int\limits_\Omega p_1 p_2 \\ (\varphi_1 \mid \varphi_2)_{H^1(\Omega)} &:= (\varphi_1 \mid \varphi_2)_{L^2(\Omega)} + (\nabla\varphi_1 \mid \nabla\varphi_2)_{L^2(\Omega,\mathbb{R}^n)} = \\ &= (\varphi_1 \mid \varphi_2)_{L^2(\Omega)} + \sum_{i=1}^n \left(\frac{\partial\varphi_1}{\partial x_i} \,\middle|\, \frac{\partial\varphi_2}{\partial x_i}\right)_{L^2(\Omega)}\end{aligned} \tag{13}$$

respectively.

The restriction $\varphi_{|\partial_I\Omega}$ on the right hand-side of (11) is meant in the sense of the trace theory (see [9, 10]), i.e. it is a limit of the sequence $\left((\varphi_\nu)_{|\partial_I\Omega}\right)_{\nu=1}^\infty$ in the Hilbert space $L^2(\partial_I\Omega)$ (provided with the scalar product $(b_1 \mid b_2)_{L^2(\partial_I\Omega)} := \int_{\partial_I\Omega} b_1 b_2\, d\sigma$) for any sequence $(\varphi_\nu)_{\nu=1}^\infty \in (C^\infty(\mathbb{R}^n))^\mathbb{N}$ such that $(\varphi_\nu)_{|\Omega} \overset{\nu\to\infty}{\longrightarrow} \varphi$ in $H^1(\Omega)$ (see also (26) below). So, Φ is a separable Hilbert space with the inner product (13), as a closed subspace of $H^1(\Omega)$.

Definition 1. *A function $T \in H^1(\Omega)$ is a weak solution of the problem* (3)–(6) *for given:*

a) $\theta_1, \ldots, \theta_s \in]0, \infty[$;

b) $0 \le \alpha \in \begin{cases} L^{n-1}(\partial_{III}\Omega), & \text{as } n > 2, \\ L^{1+\varepsilon}(\partial_{III}\Omega) \text{ for some } \varepsilon > 0, & \text{as } n = 2; \end{cases}$

c) $\xi \in \Phi'$;

d) $b = \overline{b}_{|\partial_I\Omega}$ *for some* $\overline{b} \in H^1(\Omega)$

if and only if $T_{|\partial_I\Omega} = b$ *and* (8) *holds for any* $\varphi \in \Phi$.

All the restrictions to $\partial_I\Omega$ or $\partial_{III}\Omega$ are here meant in the sense of the trace theory. In particular, the restriction $(\alpha T\varphi)_{|\partial_{III}\Omega}$ on the right hand-side of (9) is σ-summable, in virtue of (26). In Definition 1 we did not specify regularity of the data f, α, β considered in the classical statement of (3)–(6); we only assume that the linear functional (10) is well defined and continuous as a function on the Hilbert space Φ, or, in other words, ξ is a element of the dual space Φ'. The condition $T_{|\partial_I\Omega} = b$ and the variational equation (8) take in both the physical laws (3), (4) and the boundary conditions (5), (6).

Example 3. *Consider*

$$f \in \begin{cases} L^{\frac{2n}{n+2}}(\Omega) & \text{as } n > 2, \\ L^{1+\varepsilon}(\Omega) & \text{as } n = 2 \end{cases}$$

and

$$\beta \in \begin{cases} L^{2-\frac{2}{n}}(\partial_{III}\Omega) & \text{as } n > 2, \\ L^{1+\varepsilon}(\partial_{III}\Omega) & \text{as } n = 2 \end{cases} \tag{14}$$

for a certain $\varepsilon > 0$. Then the functional $\xi \colon \Phi \to \mathbb{R}$ given by the formula (1) *is well-defined and continuous. It follows from* (26)*.*

Physically, the functional ξ from Example 3 corresponds to *voluminal* heat sources f (that means that the second integral on the right hand-side of (1) is arbitrarily small for sufficiently small $m(B)$). Definition 1 admits also singular heat sources, e.g.

$$\xi : \Phi \ni \varphi \to \int_{\mathcal{M}} \rho\varphi \, d\sigma_{\mathcal{M}} - \sum_{j=1}^{s} \theta_j \int_{\partial_{III}\Omega_j} \beta\varphi \, d\sigma \in \mathbb{R} \tag{15}$$

where β satisfies (14), while $\mathcal{M} \subset \overline{\Omega}$ is a hypersurface in $\mathbb{R}^n$ (i.e. $\mathcal{M}$ is a submanifold of $\mathbb{R}^n$, $\operatorname{codim} \mathcal{M} = 1$), $\sigma_{\mathcal{M}}$ is the surface element on $\mathcal{M}$ and, finally,

$$\rho \in \begin{cases} L^{2-\frac{2}{n}}(\partial_{III}\Omega) & \text{as } n > 2, \\ L^{1+\varepsilon}(\partial_{III}\Omega) & \text{as } n = 2. \end{cases} \tag{16}$$

The functional (15) belongs to Φ' because the condition (16) has the same structure as (14). Physically, the function $\rho \colon \mathcal{M} \to \mathbb{R}$ is a density of the *surface* heat sources.

Proposition 1. *The following conditions are equivalent:*

a) $a_{\theta,\alpha}$ *is an admissible scalar product in Φ (i.e. it is induces in Φ the topology inherited from $H^1(\Omega)$);*

b) $\sigma\left(\left\{\alpha \neq 0\right\} \cup \partial_I\Omega\right) > 0$.

Proof. (a)⇒(b). Suppose, contrary to our claim, that

$$\sigma\left(\left\{\alpha \neq 0\right\} \cup \partial_I\Omega\right) = 0. \tag{17}$$

Then $\Phi = H^1(\Omega)$, while $\varphi \equiv 1$ is an element of Φ, so,

$$0 < a_{\theta,\alpha}(\varphi,\varphi) = \sum_{j=1}^{s} \theta_j \left(\int_{\Omega_j} (\nabla\varphi)(\nabla\varphi)\, dm + \underbrace{\int_{\partial_{III}\Omega_j} \alpha\varphi\varphi\, d\sigma}_{0} \right) = 0.$$

In this way the hypothesis (17) has led to a contradiction.

(b)⇒(a). The bilinear form $a_{\theta,\alpha}\colon \Phi \times \Phi \to \mathbb{R}$ is continuous, symmetric and non-negative, so it induces the semi-norm:

$$\Phi \ni \varphi \to |\varphi|_{\theta,\alpha} := \sqrt{a_{\theta,\alpha}(\varphi,\varphi)} \in \mathbb{R}_+. \tag{18}$$

It is sufficient to show that the semi-norm (18) is stronger than the original norm

$$\|\varphi\|_{H^1(\Omega)} := \sqrt{(\varphi \mid \varphi)_{H^1(\Omega)}} \tag{19}$$

induced by the inner product (13). Of course, for any $\varphi \in \Phi$

$$\begin{aligned} |\varphi|^2_{\theta,\alpha} &= \sum_{j=1}^{s} \theta_j \left(\int_{\Omega_j} |\nabla\varphi|^2 \, dm + \int_{\partial_{III}\Omega_j} \alpha\varphi^2 \, d\sigma \right) \geq \\ &\geq \left(\min_i \theta_i\right) \left(\|\nabla\varphi\|_{L^2} + \int_{\partial_{III}\Omega} \alpha\varphi^2 \, d\sigma \right). \end{aligned} \tag{20}$$

Consider an arbitrary Borel set $\Sigma \subset \partial\Omega$ for which $\sigma(\Sigma) > 0$. Then by Lemma 1 which will be proved below the functional

$$H^1(\Omega) \ni \varphi \to |\varphi|_\Sigma := \sqrt{\|\nabla\varphi\|^2_{L^2} + \frac{1}{\sigma(\Sigma)} \left(\int_\Sigma \varphi \, d\sigma \right)^2} \in \mathbb{R}_+ \tag{21}$$

is an admissible norm in $H^1(\Omega)$, so, $\exists\, c_\Sigma \in \mathbb{R}_+ \,\forall\, \varphi \in H^1(\Omega)\colon\ |\varphi|_\Sigma \geq c_\Sigma \|\varphi\|_{H^1(\Omega)}$.

In the case of $\sigma(\partial_I\Omega) > 0$ we put $\Sigma = \partial_I\Omega$ and, remembering (20), obtain:

$$\|\varphi\|_{\theta,\alpha} \geq \left(\min_i \theta_i\right) \|\nabla\varphi\|_{L^2} = \left(\min_i \theta_i\right) |\varphi|_\Sigma \geq \left(\min_i \theta_i\right) \cdot c_\Sigma \cdot \|\varphi\|_{H^1(\Omega)}$$

for any $\varphi \in H^1(\Omega)$. q.e.d.

It remains the case $\sigma(\partial_I\Omega) = 0$. Then

$$0 < \sigma(\{\alpha \neq 0\}) = \sigma\left(\bigcup_{\mathbb{Q}\ni\varepsilon>0} \{\alpha > \varepsilon\} \right) \leq \sum_{\mathbb{Q}\ni\varepsilon>0} \sigma(\{\alpha > \varepsilon\})$$

and so, there is an $\varepsilon_* \in]0,1[$ such that $\sigma(\{\alpha > \varepsilon_*\}) > 0$. Putting $\Sigma = \{\alpha > \varepsilon_*\}$ and using (20) we estimate:

$$\begin{aligned}|\varphi|^2_{\theta,\alpha} &\geq \left(\min_i \theta_i\right)\left(\|\nabla\varphi\|^2_{L^2} + \varepsilon_* \cdot \int\limits_{\{\alpha>\varepsilon_*\}} \varphi^2\, d\sigma\right) \geq \\ &\geq \left(\min_i \theta_i\right)\cdot \varepsilon_* \cdot \left(\|\nabla\varphi\|^2_{L^2} + \int\limits_{\{\alpha>\varepsilon_*\}} \varphi^2\, d\sigma\right) \geq \\ &\geq \varepsilon_* \cdot \left(\min_i \theta_i\right)\cdot \left(\|\nabla\varphi\|^2_{L^2} + \frac{1}{\sigma(\Sigma)}\int\limits_{\Sigma} \varphi^2\, d\sigma\right) = \\ &= \varepsilon_* \cdot \left(\min_i \theta_i\right)\cdot |\varphi|^2_\Sigma \geq \varepsilon_* \cdot \left(\min_i \theta_i\right)\cdot c^2_\Sigma \cdot \|\varphi\|^2_{H^1(\Omega)}. \qquad \square\end{aligned}$$

Lemma 1. *Consider $\Sigma \in \{\sigma > 0\}$. Then the functional* (21) *is an admissible norm in the Sobolev space $H^1(\Omega)$.*

Proof. Of course the semi-norm (21) is $H^1(\Omega)$-continuous. In other words, the norm (19) is stronger then the semi-norm (21).

Suppose $|\varphi|_\Sigma = 0$ for a fixed $\varphi \in H^1(\Omega)$. Then $\|\nabla\varphi\|_{L^2} = \int_\Sigma \varphi\, d\sigma = 0$. But Ω is connected, so, be the generalized du Bois-Reymond Lemma, $\varphi \equiv c$ for some $c \in \mathbb{R}$. Consequently,

$$0 = \int\limits_\Sigma \varphi\, d\sigma = c \cdot \sigma(\Sigma)$$

and $c = 0$, $\varphi \equiv 0$. Thus, $|\cdot|_\Sigma$ is a norm.

In those circumstences, it is sufficient to show (thanks to the Banach open maping theorem) that the norm (21) is complete, i.e. the pair $(H^1(\Omega), |\cdot|_\Sigma)$ is a Banach space. And so, assume that a sequence $(\varphi_\nu) \in (H^1(\Omega))^{\mathbb{N}}$ is regular in the sense of $|\cdot|_\Sigma$. Then the sequences $(\nabla\varphi_\nu)$ and $(\int_\Sigma \varphi_\nu\, d\sigma)$ are regular in the Hilbert spaces $L^2 := L^2(\Omega, \mathbb{R}^n)$ (of square summable vector fields) and $\mathbb{R}$, respectively. Therefore, there are $F \in L^2$ and $r \in \mathbb{R}$, such that

$$\lim_{\nu\to\infty} \|\nabla\varphi_\nu - F\|_{L^2} = 0 \quad \text{and} \quad \lim_{\nu\to\infty} \int\limits_\Sigma \varphi_\nu\, d\sigma = r.$$

In particular, by the definition (12), for any solenoidal test vector field $\psi\colon \Omega \to \mathbb{R}^n$ (i.e. $\psi_1, \ldots, \psi_n \in \mathfrak{D}(\Omega)$, $\mathrm{div}\psi = \sum_{i=1}^n \frac{\partial\psi_i}{\partial x_i} \equiv 0$) we have:

$$(F \mid \psi)_{L^2} \overset{\infty\leftarrow\nu}{\longleftarrow} (\nabla\varphi_\nu \mid \psi)_{L^2} = -\left(\varphi_\nu \mid \mathrm{div}\psi\right)_{L^2(\Omega)} = 0.$$

In view of the arbitrariness of ψ, the regular distribution

$$[F]\colon \mathfrak{D}(\Omega) \ni \lambda \overset{df}{\longrightarrow} \int\limits_\Omega \lambda(x)F(x)\, dx \in \mathbb{R}^n$$

is *conservative*, i.e. there exists a scalar distribution $V \in \mathfrak{D}'(\Omega)$ for which

$$[F_i] = -\frac{\partial V}{\partial x_i} \quad \text{for any} \quad 1 \leq i \leq n. \tag{22}$$

It follows from the de Rham theorem (see [12]). Simultaneously, $\partial\Omega$ fulfils the Lipschitz condition, thus, V is a regular "square – summable" distribution, i.e. there is a function $v \in L^2(\Omega)$ such that $V = [v]$ (see [13, 11], or [10]). By (22), $v \in H^1(\Omega)$ and $\forall\, 1 \leq i \leq n$ $\frac{\partial v}{\partial x_i} = F_i$ in the sense of the definition (12). Finally, the function

$$\underbrace{v}_{\in H^1(\Omega)} + \underbrace{\frac{1}{\sigma(\Sigma)}\left(r - \int\limits_{\Sigma} v d\sigma\right)}_{\text{constant function}} \qquad \big(\in H^1(\Omega)\big)$$

is the limit of (φ_ν) in the normed space $(H^1(\Omega), |\cdot|_\Sigma)$. Then, the norm (21) is complete. □

In the same way we can prove the following analogons

Lemma 2. *The functional*

$$H^1(\Omega) \ni \varphi \to |\varphi|_{\mathcal{P}} := \sqrt{\int\limits_{\Omega} |\nabla_x \varphi|^2 \, dx + \left(\int\limits_{\Omega} \varphi(x)\, dx\right)^2} \in \mathbb{R}_+$$

is an admissible norm in $H^1(\Omega)$.

In particular, there is a constant $C \in \mathbb{R}_+$, dependent only on Ω, such that

$$\|\varphi\|_{L^2(\Omega)} \leq C\, \|\varphi\|_{\mathcal{P}} \quad \text{for any} \quad \varphi \in H^1(\Omega). \tag{23}$$

It is one of two well-known Poincaré inequalities.

We say that a sequence $(E_N)_{N\in\mathbb{N}}$ of finite-dimensional linear subspaces of a normable space E is an *inner approximation* of E if for every point $e \in E$

$$\lim_{N\to\infty} \operatorname{dist}(e, E_N) = 0, \tag{24}$$

where $\operatorname{dist}(e, E_N) := \inf\{|e - \varphi| : \varphi \in E_N\}$ and $|\cdot|$ is an admissible norm in E (of course, the condition (24) does not depend on choice of such a norm). Fixing a norm $|\cdot|_E$ in E we may consider the Banach space $M_2(E)$ of all bilinear continuous forms on $E \times E$, with the norm

$$\|\gamma\|_{M_2(E)} := \sup\left\{|\gamma(e_1, e_2)| : \; |e_1|_E \leq 1, \; |e_2|_E \leq 1\right\}.$$

Now, we will prove a theorem on the existence, uniqueness and continuous dependence of the solution of (3)–(6) on the data. The solution T will be constructed as a limit of the sequence $(T_N)_{N=0}^{\infty}$ in $H^1(\Omega)$.

Theorem 1. *Suppose* $\sigma(\{\alpha \neq 0\} \cup \partial_I \Omega) > 0$. *Then, for arbitrary given data* a)–d) *(from Definition 1), there exists the only solution* T *of the problem* (3)–(6). *Moreover, if:*

$$\overline{b} \in H^1(\Omega) \text{ is any extension of } b \text{ (i.e. } b = \overline{b}_{|\partial_1\Omega}),$$

$$(\Phi_N)_{N\in\mathbb{N}} \quad \text{is an inner approximation of} \quad \Phi,$$

and:

$$\overset{N}{\theta} \overset{N\to\infty}{\longrightarrow} \theta \text{ in } \mathbb{R}^s,$$

$$\alpha_N \overset{N\to\infty}{\longrightarrow} \alpha \text{ in } \begin{cases} L^{n-1}(\partial_{III}\Omega), & \text{when} \quad n > 2 \\ L^{1+\varepsilon}(\partial_{III}\Omega), & \text{when} \quad n = 2, \end{cases}$$

$$\overline{b}_N \overset{N\to\infty}{\longrightarrow} \overline{b} \text{ in } H^1(\Omega),$$

$$\xi_N \overset{N\to\infty}{\longrightarrow} \xi \text{ in } \Phi',$$

then:

a) $\forall!\, N \in \mathbb{N} \quad \exists!\, u_N \in \Phi_N:$

$$a_{\overset{N}{\theta},\alpha_N}(u_N, \varphi) = \xi_N(\varphi) - a_{\overset{N}{\theta},\alpha_N}\left(\overline{b}_N, \varphi\right) \quad \text{for} \quad \varphi \in \Phi_N; \tag{25}$$

b) $T_N := u_N + \overline{b}_N \overset{N\to\infty}{\longrightarrow} T \quad \text{in} \quad H^1(\Omega).$

The symbols "$\forall!$" and "$\exists!$" above denote "for sufficiently large" and "there exists the unique", respectively (see, at once the comment (27)).

Proof. Existence. In agreement with Proposition 1, the bilinear form $a_{\theta,\alpha}$ (in restriction to $\Phi \times \Phi$) is an admissible scalar product in Φ. Let $u \in \Phi$ mean the Riesz representation of the functional

$$l\colon \Phi \ni \varphi \overset{df}{\longrightarrow} \xi(\varphi) - a_{\theta,\alpha}(\overline{b}, \varphi) \in \mathbb{R}$$

in the sense of the scalar product $a_{\theta,\alpha}$. Then

$$T := u + \overline{b}$$

is a solution of the problem (3)–(6) because $T_{|\partial_I\Omega} = \overline{b}_{|\partial_I\Omega} = b$ and for any $\varphi \in \Phi$:

$$a_{\theta,\alpha}(T, \varphi) = \underbrace{a_{\theta,\alpha}(u, \varphi)}_{l(\varphi)} + a_{\theta,\alpha}(\overline{b}, \varphi) = \xi(\varphi).$$

Uniqueness. Let $\widetilde{T}$ be a solution of (3)–(6). Then $T - \widetilde{T} \in \Phi$ and for each $\varphi \in \Phi$:

$$a_{\theta,\alpha}(T - \widetilde{T}, \varphi) = a_{\theta,\alpha}(T, \varphi) - a_{\theta,\alpha}(\widetilde{T}, \varphi) = \xi(\varphi) - \xi(\varphi) = 0.$$

Putting $\varphi = T - \widetilde{T}$ we get: $T - \widetilde{T} = 0$.

Approximation. For a fixed exponent $r \in [1, \infty[$ consider the Sobolev space

$$W^{1,r}(\Omega) := \Big\{ \varphi \in L^r(\Omega)\colon \quad \underbrace{\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n}}_{\text{in the sense of definition (12)}} \quad \in L^r(\Omega) \Big\}.$$

It is a Banach space with the norm

$$\|\varphi\|_{W^{1,r}(\Omega)} := \left(\int_\Omega | \varphi(x) |^r \ dx + \sum_{i=1}^n \int_\Omega \left| \frac{\partial \varphi}{\partial x_i}(x) \right|^r \right)^{\frac{1}{r}}.$$

Using fundamental facts from the trace theory and the Sobolev embedding theorems (see [10, 9]) one can prove that the restriction operator

$$W^{1,r}(\Omega) \overset{\text{restr.}}{\longrightarrow} L^{\frac{(n-1)r}{n-r}}(\partial\Omega) \tag{26}$$

is well defined and continuous.

Hence, in particular,

$$a_{\underset{\theta,\alpha_N}{N}} \overset{N\to\infty}{\longrightarrow} a_{\theta,\alpha} \quad \text{in} \quad M_2\left(H^1(\Omega)\right).$$

Consequently, the sequence $(l_N)_{N\in\mathbb{N}}$ of functionals

$$l_N\colon \Phi \ni \varphi \overset{df}{\longrightarrow} \xi_N(\varphi) - a_{\underset{\theta,\alpha_N}{N}}(\overline{b}_N, \varphi) \in \mathbb{R}$$

converges to l in the Banach space Φ', as $N \to \infty$. By Lemma 3,

$$u_N \overset{N\to\infty}{\longrightarrow} u \quad \text{in} \quad \Phi.$$

Finally,

$$T_N = u_N + \overline{b}_N \overset{N\to\infty}{\longrightarrow} u + \overline{b} = T \quad \text{in} \quad H^1(\Omega).$$ □

The after-mentioned assertion, originating mainly from Ritz and Riesz, is a result of great weight in the discretization problems. Namely, modifying slightly the proof of so-colled *projective theorem* (see [14]) we obtain the following

Lemma 3. *Let $(E_N)_{N\in\mathbb{N}}$ be an inner approximation of a Hilbert space E. Consider a functional $l \in E'$ and a bilinear continuous form $a\colon E \times E \to \mathbb{R}$ satisfying the condition*

$$\inf_{\varphi \in E\setminus\{0\}} \frac{a(\varphi, \varphi)}{|\varphi|_E^2} > 0$$

(where $|\cdot|_E$ is the norm of E). Consider additionally sequences

$$(l_N) \in (E')^{\mathbb{N}} \quad \text{and} \quad (a_N \in M_2(E)^{\mathbb{N}})$$

convergent to l and a, respectively (in the Banach spaces E' and $M_2(E)$, respectively).

Then:

a) $\forall!\, N \in \mathbb{N} \quad \exists!\, u_N \in E_N$:

$$a_N(u_N, \varphi) = l_N(\varphi) \quad \textit{for any} \quad \varphi \in E_N;$$

b) $u_N \overset{N\to\infty}{\longrightarrow} u$ *in* E,
where u *is the only vector of* E *such that* $a(u, \cdot) = l$.

Certainly, if a is additionally symmetric, then a is an admissible scalar product in E, while the vector u is the Riesz representation of l in the Hilbert space (E, a). Remark that there exists such large integer $N_0 \in \mathbb{N}$ as the form a_N is coercitive for every $N \geq N_0$. Then the part a) of the above proposition takes the form "$\forall\, N \geq \mathbb{N}\, \exists! u_N \in E_N \ldots$". This concerns also Theorem 1. In particular,

$$\text{if } \overset{N}{\theta} \in\,]0, \infty[^s \text{ and } \alpha_N \geq 0 \text{ for every } N \in \mathbb{N}, \text{ than we can put } N_0 = 0 \tag{27}$$

(i.e. the part a) of proposition of Theorem 1 takes the form "$\forall\, N \in \mathbb{N}\, \exists!\, u_N \in \Phi \ldots$". The assumption "$\forall\, N \in \mathbb{N}$: $\overset{N}{\theta} \in\,]0, \infty[^s$" is natural whereas the approximation α_N do not have to be non-negative for some first N. For example, it may happen when α_N is the finite expansion of α in a Fourier series.

The pure Neuman boundary value problem needs an individual treatment, because in the case of $\alpha = 0$ and $\partial_I \Omega = \emptyset$ we can not use Proposition 1.

Theorem 2. *Assume* $\alpha = 0$ *and* $\partial_I \Omega = \emptyset$. *Consider arbitrary data* $\theta = (\theta_1, \ldots, \theta_s) \in\,]0, \infty[^s$ *and* $\xi \in \big(H^1(\Omega)\big)'$ *(see a), c) from Definition 1).*

1) *Then the following conditions are equivalent:*
   (i) *there exists a solution of the problem* (3)–(6)*;*
   (ii) $\xi(\mathbf{1}) = 0$, *where* $\mathbf{1}(\in H^1(\Omega))$ *is the function equal identically to* $1\ (\in \mathbb{R})$.
2) *Suppose that* $\xi(\mathbf{1}) = 0$ *and consider an arbitrary number* $h \in \mathbb{R}$. *Then there exists the unique solution* T *of* (3)–(6) *such that*

$$\int_\Omega T(x)dx = h. \tag{28}$$

Furthermore, if $(E_N)_{N\in\mathbb{N}}$ *is an inner approximation of the Hilbert space* $H^1(\Omega) \cap \big\{\int = 0\big\}$ *and:*

$$h_N \overset{N\to\infty}{\longrightarrow} h \quad \textit{in} \quad \mathbb{R},$$
$$\overset{N}{\theta} \overset{N\to\infty}{\longrightarrow} \theta \quad \textit{in} \quad]0, \infty[^s,$$
$$\xi_N \overset{N\to\infty}{\longrightarrow} \xi \quad \textit{in} \quad \big(H^1(\Omega)\big)',$$

then:

a) $\forall N \in \mathbb{N}\, \exists!\, u_N \in E_N$:

$$a_{\substack{N\\ \theta,0}}(u_N, \varphi) = \xi_N(\varphi) \quad for \quad \varphi \in E_N; \tag{29}$$

b) $T_N := u_N + \frac{1}{m(\Omega)} h_N \overset{N\to\infty}{\longrightarrow} T \quad in \quad H^1(\Omega)$.

The linear subspace

$$H^{1,0}(\Omega) := \left\{ \varphi \in H^1(\Omega) \colon \int\limits_{\Omega} \varphi(x) dx = 0 \right\}$$

of $H^1(\Omega)$ is closed in the Sobolev space $H^1(\Omega)$, so it is a Hilbert space with the inner product (13) *(in fact, it is a closed hyperplane in $H^1(\Omega)$).*

Proof. Since for any $\varphi \in H^1(\Omega)$

$$a_{\theta,0}(\varphi, \varphi) = \sum_{j=1}^{s} \theta_j \int\limits_{\Omega} |\nabla_x \varphi|^2 \, dx \geq \left(\min_{1 \leq i \leq s} \theta_i \right) \int\limits_{\Omega} |\nabla_x \varphi|^2 \, dx, \tag{30}$$

the restriction of the bilinear form $a_{\theta,0}$ to $\left(H^{1,0}(\Omega)\right)^2$ is a scalar product inducing the norm (18) which, on the strenght of Lemma 2, is equivalent to the original norm (19).

Suppose T is a solution of (3)–(6) fulfilling the condition (28). Then the centring-function

$$u := T - \frac{1}{m(\Omega)} h$$

is in the hyperplane $H^{1,0}(\Omega)$ and for every $\varphi \in H^1(\Omega)$:

$$a_{\theta,0}(u, \varphi) = a_{\theta,0}\left(u + \frac{1}{m(\Omega)} h, \varphi \right) = a_{\theta,0}(T, \varphi) = \xi(\varphi) \tag{31}$$

because of the obvious equality

$$a_{\theta,0}(1, \varphi) = 0. \tag{32}$$

Ad 1) (i)⇒(ii). We suppose there is a solution T of (3)–(6). In accordance with (31),

$$\xi(\mathbf{1}) = a_{\theta,0}(T, \mathbf{1}) = 0.$$

(ii)⇒(i). Let $u \in H^{1,0}(\Omega)$ stand for the Riesz representation of the functional $\xi_{|H^{1,0}(\Omega)}$ in the sense of scalar product $a_{\theta,0|(H^{1,0}(\Omega))^2}$. Then the function

$$T := u + \frac{1}{m(\Omega)} h \tag{33}$$

is a solution of (3)–(6) satisfying (28). Indeed, remembering (32), we have:

$$a_{\theta,0}(T,\varphi) = a_{\theta,0}(u,\varphi) = a_{\theta,0}\Bigg(u, \underbrace{\varphi - \frac{1}{m(\Omega)}\int_{\Omega}\varphi dm}_{\in H^{1,0}(\Omega)}\Bigg) =$$

$$= \xi\left(\varphi - \frac{1}{m(\Omega)}\int_{\Omega}\varphi dm\right) = \xi(\varphi)$$

for every $\varphi \in H^1(\Omega)$.

Ad 2) Assume $\xi(\mathbf{1}) = 0$ and consider $h \in \mathbb{R}$. In the previous step we have showed that the function (33) is a solution of (3)–(6) satisfying (28).

Uniqueness. Let $\widetilde{T}$ be a solution of the problem (3)–(6), (28). Then the difference $T - \widetilde{T}$ belongs to $H^{1,0}(\Omega)$ and

$$a_{\theta,0}(T - \widetilde{T}, \varphi) = a_{\theta,0}(T,\varphi) - a_{\theta,0}(\widetilde{T},\varphi) = \xi(\varphi) - \xi(\varphi) = 0$$

for each $\varphi \in H^1(\Omega)$. Putting $\varphi = T - \widetilde{T}$ we get: $T - \widetilde{T} = 0$.

Approximation. By Lemma 3, there holds part a) of the proposition and $u_N \overset{N\to\infty}{\longrightarrow} u$ in the Hilbert space $H^{1,0}(\Omega)$. Finally,

$$T_N = u_N + \frac{1}{m(\Omega)}h_N \quad \overset{N\to\infty}{\longrightarrow} \quad u + \frac{1}{m(\Omega)}h = T \quad \text{in} \quad H^1(\Omega). \qquad \square$$

In the heat theory the above number h is interpreted as the *total heat* of the medium.

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