

## COMPARISON THEOREMS FOR PROPERTY (B) OF THE THIRD-ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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**Abstract.** The aim of this paper is to introduce a new comparison theorem (in both delayed and advanced cases) that allows us to investigate the properties of third-order differential equations with quasi-derivatives

$$\left( r_1(t) (r_2(t)y'(t))' \right)' - p(t)y(\tau(t)) = 0$$

using the following simpler differential equations

$$\left( r(t) (r(t)z'(t))' \right)' - p(t)z(\tau(t)) = 0$$

and

$$y'''(t) - q(t)y(\sigma(t)) = 0.$$

The obtained comparison principles allow for the immediate transcription of the oscillatory results known for the simpler equations into studied equation with quasi-derivatives. The progress achieved will be illustrated through several examples.

**Keywords:** canonical equation, comparison theorem, property (B).

**Mathematics Subject Classification:** 34K11, 34C10.

### 1. INTRODUCTION

Consider the third-order functional differential equation with deviating argument of the form

$$\left( r_1(t) (r_2(t)y'(t))' \right)' - p(t)y(\tau(t)) = 0, \tag{E}$$

where we assume throughout that the following conditions hold:

- (H<sub>1</sub>)  $p(t) \in C([t_0, \infty))$ ,  $p(t) > 0$ ,
- (H<sub>2</sub>)  $r_i(t) \in C^1([t_0, \infty))$ ,  $r_i(t) > 0$ , for  $i = 1, 2$ ,
- (H<sub>3</sub>)  $\tau(t) \in C^1([t_0, \infty))$ ,  $\tau'(t) > 0$  and  $\lim_{t \rightarrow \infty} \tau(t) = \infty$ .

Let  $\tau_*(t) = \min\{t, \tau(t)\}$  for  $t \geq t_0 \geq 0$ , and  $\tau_0 = \min\{\tau_*(t) : t \geq t_0\}$ . A continuous function  $y : [\tau_0, \infty) \rightarrow (-\infty, \infty)$  is said to be a proper solution of (E) if it satisfies (E) almost everywhere on  $[t_0, \infty)$  and  $\sup\{|y(t)| : t \geq T\} > 0$  for all  $T \geq t_0$ . A proper solution of (E) is called oscillatory if it has a sequence of zeros tending to  $\infty$ . Otherwise, it is called nonoscillatory.

There are many works and ways devoted to the study of properties for (E), see e.g., [1]–[15]. In this paper, we intend to establish a suitable comparison technique that allows us to investigate (E) using simpler differential equations

$$\left(r(t)(r(t)z'(t))'\right)' - p(t)z(\tau(t)) = 0 \quad (E_1)$$

and

$$y'''(t) - q(t)y(\sigma(t)) = 0. \quad (E_2)$$

Comparison theorems greatly simplify the investigation of more complex differential equations because, instead of adapting techniques known for simpler differential equations, they allow for the immediate rewriting of already known oscillatory results.

Let us discuss which properties of (E) are investigated in oscillation theory. We must emphasize that we consider the canonical case of (E), that is

$$R_i(t) = \int_{t_0}^t \frac{1}{r_i(s)} ds \uparrow \infty \quad \text{as } t \uparrow \infty \quad \text{for } i = 1, 2.$$

In this case (according to classical result of Kiguradze [9, Lemma 1.1]), the set  $\mathcal{N}$  of all nonoscillatory solutions of (E) has the following decomposition:

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3,$$

where an eventually positive solution  $y(t)$  of (E) satisfies

$$\begin{aligned} y(t) \in \mathcal{N}_1 &\Leftrightarrow y'(t) > 0, (r_2(t)y'(t))' < 0, \left(r_1(t)(r_2(t)y'(t))'\right)' > 0, \\ y(t) \in \mathcal{N}_3 &\Leftrightarrow y'(t) > 0, (r_2(t)y'(t))' > 0, \left(r_1(t)(r_2(t)y'(t))'\right)' > 0, \end{aligned}$$

eventually, say for  $t \geq t_0$ . As usual, a solution from the class  $\mathcal{N}_1$  (or  $\mathcal{N}_3$ ) is said to be of degree 1 (or 3).

Following Kusano and Naito [13], Eq. (E) is said to have property (B) if every its nonoscillatory solution  $y(t)$  is strongly increasing in the sense that

$$\frac{|y(t)|}{I[r_2, r_1](t)} \uparrow \infty \quad \text{as } t \uparrow \infty, \quad (1.1)$$

where

$$I[r_2, r_1](t) = \int_{t_0}^t \frac{1}{r_2(s_2)} \int_{t_0}^{s_2} \frac{1}{r_1(s_1)} ds_1 ds_2.$$

It is useful to note that condition (1.1) is equivalent to

$$\left| r_1(t) (r_2(t)y'(t))' \right| \uparrow \infty \quad \text{as } t \uparrow \infty. \quad (1.2)$$

A closer look shows that property (B) requires two conditions: the class  $\mathcal{N}_1 = \emptyset$  and, at the same time, every solution  $y(t) \in \mathcal{N}_3$  satisfies (1.1).

Various techniques have been introduced to obtain criteria for property (B). Let us recall some of them. Grigorian [8] used the Riccati substitution method to study the asymptotic properties of third-order linear ordinary differential equations. Graef *et al.* [7] studied oscillation of third-order neutral delay differential equations by applying the integral averaging method. In [6], Grace provided, via comparison with first-order equations oscillatory criteria for third-order differential equations of the form

$$(a(t)(x''(t))^\alpha)' = q(t)x^\alpha(\tau(t)) + p(t)x^\alpha(\omega(t)).$$

Koplatadze *et al.* [11] employed an integral-type technique to obtain a very useful result for  $(E_2)$ :

**Theorem 1.1.** *Assume that either  $\sigma(t) \leq t$  and*

$$\limsup_{t \rightarrow \infty} \left\{ \sigma(t) \int_t^\infty sq(s) ds + \int_{\sigma(t)}^t s\sigma(s)q(s) ds + \frac{1}{\sigma(t)} \int_0^{\sigma(t)} s^2\sigma(s)q(s) ds \right\} > 2 \quad (1.3)$$

*or  $\sigma(t) \geq t$  and*

$$\limsup_{t \rightarrow \infty} \left\{ \sigma(t) \int_{\sigma(t)}^\infty sq(s) ds + \int_t^{\sigma(t)} s\sigma(s)q(s) ds + \frac{1}{\sigma(t)} \int_0^t s^2\sigma(s)q(s) ds \right\} > 2.$$

*Then equation  $(E_2)$  has property (B).*

Our main goal is to establish a new comparison principle for property (B) between equations  $(E)$  and  $(E_2)$ . This will allow us to expand all known criteria for  $(E_2)$  to the more general equation  $(E)$ . We will show this process in detail in Theorem 1.1.

Our article is organized as follows. In the first part, we establish comparison theorem for transferring property (B) from  $(E_1)$  to  $(E)$ . In this step, we improve the following comparison result due to Kusano and Naito (see [13, Theorem 1]).

**Theorem 1.2.** *Let  $r(t) \in C([t_0, \infty))$ ,  $r(t) > 0$ , and  $\int^\infty 1/r(s) ds = \infty$ . If*

$$r(t) \geq r_i(t) \quad \text{for } t \in [t_0, \infty) \quad i = 1, 2, \quad (1.4)$$

*then equation  $(E)$  has property (B) if equation  $(E_1)$  has property (B).*

If the functions  $r_1(t)$  and  $r_2(t)$  are of different degrees such as  $r_1(t) = t^\alpha$ ,  $r_2(t) = t^\beta$ ,  $\alpha \neq \beta$ , then condition (1.4) is very strong. It is natural to expect that the best choice of  $r(t)$  is the geometric mean (up to positive constant) of  $r_1(t), r_2(t)$ , i.e.,  $r(t) = \sqrt{r_1(t)r_2(t)}$ .

In the second stage, we provide a comparison theorem between (E) and (E<sub>2</sub>), and we extend Theorem 1.1 to (E). Moreover, we can now show the benefits of our new approach, because if we apply existing Theorem 1.2, condition (1.4) requires

$$r_i(t) \leq 1 \quad \text{for } i = 1, 2,$$

which significantly narrows down the options for functions  $r_i(t)$ .

On the other hand, we compare our results with the latest criteria for property (B). The present authors (see Theorem 3.1 in [3]) presented a sufficient condition for property (B) in the partial case when  $\tau(t) \geq t$ . Later Chatzarakis *et al.* contributed to the subject (see Theorem 2 in [15]) and presented the following result:

**Theorem 1.3.** *Assume that  $\tau(t) \geq t$  and*

$$\liminf_{t \rightarrow \infty} \int_t^{\tau(t)} \frac{1}{r_2(s)} \int_s^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty p(s_2) ds_2 ds_1 ds > \frac{1}{e},$$

*then equation (E) has property (B).*

Our one of contributions is that we also investigate the opposite case, namely when  $\tau(t) \leq t$ . Further progress will be presented in Example 3.2.

## 2. MAIN RESULTS

In the first step, we find a condition guaranteeing that every solution  $y(t) \in \mathcal{N}_3$  satisfies (1.1).

**Lemma 2.1.** *Assume that*

$$\int_{t_0}^\infty p(s) I[r_2, r_1](\tau(s)) ds = \infty. \quad (2.1)$$

*Then every solution  $y(t) \in \mathcal{N}_3$  of (E) satisfies (1.1).*

*Proof.* Assume that  $y(t) \in \mathcal{N}_3$  is a positive solution of (E). We shall show that  $y(t)$  satisfies (1.2). Suppose, on the contrary, that there exists there is  $c > 0$  such that

$$r_1(t) (r_2(t)y'(t))' \uparrow c \quad \text{as } t \uparrow \infty.$$

Integrating (E) from  $t_0$  to  $\infty$  yields

$$c - r_1 (r_2 y')'(t_0) = \int_{t_0}^\infty p(s) y(\tau(s)) ds. \quad (2.2)$$

On the other hand, since  $r_1(t) (r_2(t)y'(t))' > 0$  is increasing, there is a constant  $c_1 > 0$  such that

$$r_1(t) (r_2(t)y'(t))' \geq c_1.$$

Integrating the above inequality twice from  $t_0$  to  $t$ , we obtain

$$y(t) \geq c_1 I[r_2, r_1](t). \quad (2.3)$$

Substituting (2.3) into (2.2) leads to a contradiction with (2.1). This completes the proof.  $\square$

We will investigate two mutually exclusive cases

$$\frac{r_1(t)}{r_2(t)} \downarrow, \quad (2.4)$$

and

$$\frac{r_2(t)}{r_1(t)} \downarrow. \quad (2.5)$$

We are now ready to state the first comparison theorem.

**Theorem 2.2.** *Assume that (2.1), (2.4) hold and*

$$r(t) = \sqrt{r_1(t)r_2(t)}. \quad (2.6)$$

*Then equation (E) has property (B) if equation (E<sub>1</sub>) has property (B).*

*Proof.* Based on the definition of property (B) and Lemma 2.1, it suffices to show that  $\mathcal{N}_1 = \emptyset$ . Suppose, on the contrary, that (E) possesses a positive solution  $y(t) \in \mathcal{N}_1$ . An integration of (E) from  $t$  to  $\infty$  yields

$$-(r_2(t)y'(t))' = \frac{1}{r_1(t)} \int_t^\infty p(s)y(\tau(s)) ds = \frac{1}{r_1(t)} P(t). \quad (2.7)$$

Integrating once more from  $t$  to  $\infty$ , we obtain

$$y'(t) \geq \frac{1}{r_2(t)} \int_t^\infty \frac{1}{r_1(s_1)} P(s_1) ds_1.$$

Integrating from  $t_0$  to  $t$ , we get

$$y(t) \geq \int_{t_0}^t \frac{1}{r_2(s_2)} \int_{s_2}^\infty \frac{1}{r_1(s_1)} P(s_1) ds_1 ds_2. \quad (2.8)$$

We claim that, for  $t \geq t_0$ , we have

$$\int_{t_0}^t \frac{1}{r_2(s_2)} \int_{s_2}^\infty \frac{1}{r_1(s_1)} P(s_1) ds_1 ds_2 \geq \int_{t_0}^t \frac{1}{r(s_2)} \int_{s_2}^\infty \frac{1}{r(s_1)} P(s_1) ds_1 ds_2.$$

To verify this, we use the auxiliary function

$$F(t) = \int_{t_0}^t \frac{1}{r_2(s_2)} \int_{s_2}^{\infty} \frac{1}{r_1(s_1)} P(s_1) \, ds_1 \, ds_2 - \int_{t_0}^t \frac{1}{r(s_2)} \int_{s_2}^{\infty} \frac{1}{r(s_1)} P(s_1) \, ds_1 \, ds_2.$$

Since  $F(t_0) = 0$ , we shall show that  $F'(t) \geq 0$  for  $t \geq t_0$ . Because

$$F'(t) = \frac{1}{r_2(t)} \int_t^{\infty} \frac{1}{r_1(s_1)} P(s_1) \, ds_1 - \frac{1}{r(t)} \int_t^{\infty} \frac{1}{r(s_1)} P(s_1) \, ds_1, \quad (2.9)$$

it is sufficient to verify that the function

$$G_1(t) = \frac{r(t)}{r_2(t)} \int_t^{\infty} \frac{1}{r_1(s_1)} P(s_1) \, ds_1 - \int_t^{\infty} \frac{1}{r(s_1)} P(s_1) \, ds_1 \geq 0, \quad t \geq t_0.$$

Taking (2.6) and (2.4) into account, we see that  $G_1(\infty) = 0$ , and it follows that it is enough to show that  $G_1'(t) \leq 0$ . On the other hand, (2.4) implies that

$$\begin{aligned} G_1'(t) &= \left( \frac{r(t)}{r_2(t)} \right)' \int_t^{\infty} \frac{P(s_1)}{r_1(s_1)} \, ds_1 - \frac{r(t)}{r_2(t)r_1(t)} P(t) + \frac{P(t)}{r(t)} \\ &\leq \frac{P(t)}{r(t)} \left( 1 - \frac{r^2(t)}{r_2(t)r_1(t)} \right) = 0. \end{aligned}$$

We have just verified that  $F(t) > 0$ , which together with (2.8) yields

$$y(t) \geq \int_{t_0}^t \frac{1}{r(s_2)} \int_{s_2}^{\infty} \frac{1}{r(s_1)} P(s_1) \, ds_1 \, ds_2. \quad (2.10)$$

Let us denote the right-hand side of (2.10) by  $z(t)$ . Then  $0 < z(t) \leq y(t)$  and

$$r(t)z'(t) > 0, \quad r(t)(r(t)z'(t))' < 0.$$

But this means that  $z(t)$  is of degree 1, and moreover  $z(t)$  is a solution of the differential inequality

$$\left( r(t)(r(t)z'(t))' \right)' - p(t)z(\tau(t)) \geq 0.$$

On the other hand, by Theorem 2 in [13] the corresponding differential equation  $(E_1)$  also has a solution of degree 1. This contradicts the assumption that  $(E_1)$  has property (B). The proof is finished.  $\square$

We next investigate the properties of  $(E)$  and  $(E_2)$ .

**Theorem 2.3.** *Let (2.1), (2.4) and (2.6) hold. Assume that*

$$R(t) = \int_{t_0}^t \frac{1}{r(s)} ds, \quad (2.11)$$

and

$$q(t) = p(R^{-1}(t)) r(R^{-1}(t)), \quad \sigma(t) = (R(\tau(R^{-1}(t))))). \quad (2.12)$$

Then equation (E) has property (B) if equation (E<sub>2</sub>) has property (B).

*Proof.* It is sufficient to show that  $\mathcal{N}_1 = \emptyset$  for (E). If we assume the opposite, namely that (E) possesses a solution of degree 1, then proceeding in the same way as in the proof of Theorem 2.2, we find that (E<sub>1</sub>) also has a positive solution  $z(t) > 0$  that is of degree 1. On the other hand, simple calculations show that function

$$y(t) = z(R^{-1}(t)) \quad (2.13)$$

satisfies

$$y'(t) = r(s)z'(s) \Big|_{s=R^{-1}(t)}, \quad y''(t) = r(s)(r(s)z'(s))' \Big|_{s=R^{-1}(t)}$$

and

$$y'''(t) = r(s) \left( r(s)(r(s)z'(s))' \right)' \Big|_{s=R^{-1}(t)}. \quad (2.14)$$

But this means that  $y(t)$  is also a function of degree 1. Moreover, since (E<sub>1</sub>) can be rewritten in the form

$$r(s) \left( r(s)(r(s)z'(s))' \right)' \Big|_{s=R^{-1}(t)} - r(R^{-1}(t)) p(R^{-1}(t)) z(\tau(R^{-1}(t))) = 0,$$

in view of (2.13) and (2.14), we conclude that  $y(t)$  is a solution of (E<sub>2</sub>). This contradicts the assumption that (E<sub>2</sub>) has property (B). The proof is complete.  $\square$

Let us turn our attention to the case when (2.5) holds. As we will see later, the situation in this case is more complicated, and some preliminary results are needed.

**Lemma 2.4.** *Assume that  $y(t) \in \mathcal{N}_1$  is a positive solution of (E). Then*

$$y(t) \uparrow \quad \text{and} \quad \frac{y(t)}{R_2(t)} \downarrow. \quad (2.15)$$

*Proof.* For a positive solution  $y(t)$  of degree 1, the first part of (2.15) is obvious. On the other hand, the function  $r_2(t)y'(t)$  is decreasing and therefore

$$y(t) \geq \int_{t_0}^t r_2(s)y'(s) \frac{1}{r_2(s)} ds \geq r_2(t)y'(t)R_2(t).$$

But this implies that

$$\left( \frac{y(t)}{R_2(t)} \right)' \leq 0.$$

The proof is complete.  $\square$

The following considerations serve to improve the monotonicity (2.15). We assume that there exist positive constants  $\omega_1$  and  $\omega_2$  for the delay and advanced case of (E), respectively:

$$\frac{1}{R_2(t)} \int_{t_0}^t \frac{R_2(s_1)R_2(\tau(s_1))}{r_1(s_1)} \int_{s_1}^{\infty} p(s) \, ds \, ds_1 \geq \omega_1$$

and

$$\frac{1}{R_2(t)} \int_{t_0}^t \frac{R_2^2(s_1)}{r_1(s_1)} \int_{s_1}^{\infty} p(s) \, ds \, ds_1 \geq \omega_2.$$

**Lemma 2.5.** *Assume that  $y(t) \in \mathcal{N}_1$  is a positive solution of (E). Then*

$$\frac{y(t)}{R_2^{1-\omega_i}(t)} \downarrow, \quad (2.16)$$

with  $i = 1$  for  $\tau(t) \leq t$  and  $i = 2$  for  $\tau(t) \geq t$ .

*Proof.* Assume that  $y(t) \in \mathcal{N}_1$  is a positive solution of (E). Then (2.7) and (2.15) imply that

$$-(r_2(t)y'(t))' \geq \frac{y(\tau(t))}{r_1(t)} \int_t^{\infty} p(s) \, ds.$$

Hence,

$$(y(t) - R_2(t)r_2(t)y'(t))' = -R_2(t) (r_2(t)y'(t))' \geq \frac{y(\tau(t))R_2(t)}{r_1(t)} \int_t^{\infty} p(s) \, ds.$$

Integrating the last inequality from  $t_0$  to  $t$ , we obtain

$$y(t) - R_2(t)r_2(t)y'(t) \geq \int_{t_0}^t \frac{y(\tau(s_1))R_2(s_1)}{r_1(s_1)} \int_{s_1}^{\infty} p(s) \, ds \, ds_1. \quad (2.17)$$

For  $\tau(t) \leq t$ , (2.17) together with (2.15) implies

$$y(t) - R_2(t)r_2(t)y'(t) \geq \frac{y(t)}{R_2(t)} \int_{t_0}^t \frac{R_2(s_1)R_2(\tau(s_1))}{r_1(s_1)} \int_{s_1}^{\infty} p(s) \, ds \, ds_1 \geq \omega_1 y(t),$$

while for  $\tau(t) \geq t$ , we obtain

$$y(t) - R_2(t)r_2(t)y'(t) \geq \frac{y(t)}{R_2(t)} \int_{t_0}^t \frac{R_2^2(s_1)}{r_1(s_1)} \int_{s_1}^{\infty} p(s) \, ds \, ds_1 \geq \omega_2 y(t).$$

In both cases,

$$y(t)(1 - \omega_i) - R_2(t)r_2(t)y'(t) \geq 0,$$

which implies (2.16).  $\square$

Using relation (2.16), we establish a new monotonicity for the function  $P(t)$ . Let us denote

$$V_i(t) = \int_t^\infty p(s)R_2^{1-\omega_i}(\tau(s)) ds,$$

where  $i = 1$  for  $\tau(t) \leq t$  and  $i = 2$  for  $\tau(t) \geq t$ .

**Lemma 2.6.** *Assume that  $y(t) \in \mathcal{N}_1$  is a positive solution of (E). Then*

$$\frac{P(t)}{V_i(t)} \downarrow, \tag{2.18}$$

where  $i = 1$  for  $\tau(t) \leq t$  and  $i = 2$  for  $\tau(t) \geq t$ .

*Proof.* Using (2.16), it is easy to verify that

$$P(t) = \int_t^\infty p(s)y(\tau(s)) ds \leq \frac{y(\tau(t))}{R_2^{1-\omega_i}(\tau(t))} \int_t^\infty p(s)R_2^{1-\omega_i}(\tau(s)) ds. \tag{2.19}$$

Since

$$P'(t) + p(t)y(\tau(t)) = 0,$$

it follows from (2.19) that

$$P'(t) \int_t^\infty p(s)R_2^{1-\omega_i}(\tau(s)) ds + P(t)p(t)R_2^{1-\omega_i}(\tau(t)) \leq 0.$$

Therefore,

$$\left( \frac{P(t)}{\int_t^\infty p(s)R_2^{1-\omega_i}(\tau(s)) ds} \right)' \leq 0.$$

The proof is complete. □

Employing a suitable constant, we adjust the choice of function  $r(t)$ . In what follows, we shall assume that

$$\int_{t_0}^\infty \frac{V_i(s)}{\sqrt{r_1(s)r_2(s)}} ds < \infty. \tag{2.20}$$

Let there exist a couple of positive constants  $\kappa_i$  such that

$$-\frac{r_1(t)}{V_i(t)} \left( \sqrt{\frac{r_2(t)}{r_1(t)}} \right)' \int_t^\infty \frac{V_i(s)}{\sqrt{r_1(s)r_2(s)}} ds \leq \kappa_i. \tag{2.21}$$

Then we choose positive constants  $\ell_i$  as follows

$$\ell_i = \frac{1}{1 + \kappa_i}, \quad i = 1, 2. \tag{2.22}$$

**Theorem 2.7.** *Assume that (2.1), (2.5), (2.20), (2.21), (2.22) hold and*

$$r(t) = \frac{\sqrt{r_1(t)r_2(t)}}{\sqrt{\ell_i}}, \quad (2.23)$$

*with  $i = 1$  for  $\tau(t) \leq t$  and  $i = 2$  for  $\tau(t) \geq t$ . Then equation (E) has property (B) if equation (E<sub>1</sub>) has property (B).*

*Proof.* We shall prove that  $\mathcal{N}_1 = \emptyset$  for (E). Proceeding in the same way as in the proof of Theorem 2.2, we arrive at (2.9). To verify that  $F(t) > 0$ , it suffices to show that

$$G_2(t) = \int_t^\infty \frac{1}{r_1(s_1)} P(s_1) ds_1 - \frac{r_2(t)}{r(t)} \int_t^\infty \frac{1}{r(s_1)} P(s_1) ds_1 \geq 0, \quad t \geq t_0.$$

On the other hand, (2.5) together with (2.23) imply  $G_2(\infty) = 0$ , and therefore it is enough to verify that  $G_2'(t) \leq 0$ . Indeed, taking into account (2.23) and (2.18), one can see that

$$\begin{aligned} G_2'(t) &= -\frac{P(t)}{r_1(t)} - \left(\frac{r_2(t)}{r(t)}\right)' \int_t^\infty \frac{P(s_1)}{r(s_1)} ds_1 + \frac{r_2(t)}{r^2(t)} P(t) \\ &\leq -\frac{P(t)}{r_1(t)} - \frac{P(t)}{V_i(t)} \left(\frac{r_2(t)}{r(t)}\right)' \int_t^\infty \frac{V_i(s_1)}{r(s_1)} ds_1 + \frac{r_2(t)}{r^2(t)} P(t) \\ &= \frac{P(t)}{r_1(t)} \left( -1 - \ell_i \frac{r_1(t)}{V_i(t)} \left(\sqrt{\frac{r_2(t)}{r_1(t)}}\right)' \int_t^\infty \frac{V_i(s_1)}{\sqrt{r_1(s_1)r_2(s_1)}} ds_1 + \ell_i \right) \\ &\leq \frac{P(t)}{r_1(t)} (-1 + \ell_i \kappa_i + \ell_i) = 0. \end{aligned}$$

The remaining part of the proof is identical to that of Theorem 2.2.  $\square$

Now we can establish a comparison result for (E) and (E<sub>2</sub>) under condition (2.5).

**Theorem 2.8.** *Let (2.1), (2.5) and (2.23) hold. Assume that  $R(t)$ ,  $q(t)$  and  $\sigma(t)$  are defined by (2.11) and (2.12). Then equation (E) has property (B) if equation (E<sub>2</sub>) has property (B).*

The proof is identical to the proof of Theorem 2.3, so it is omitted.

### 3. APPLICATIONS

The obtained comparison results allow immediate extension of known results for (E<sub>2</sub>) to (E). Our technique works regardless of whether we consider delay, advanced, or equations without deviating argument, and also for both mutually exclusive conditions (2.4) and (2.5). As an illustrative example, we show the extension of Theorem 1.1 to equation (E).

**Corollary 3.1.** *Let (2.1) hold. Assume that either (2.4), (2.6) or (2.5), (2.22), (2.23) are satisfied. Suppose that  $R(t)$  is defined by (2.11). If either  $\tau(t) \leq t$  and*

$$\limsup_{t \rightarrow \infty} \left\{ R(\tau(t)) \int_t^{\infty} R(x)p(x) \, dx + \int_{\tau(t)}^t R(x)R(\tau(x))p(x) \, dx + \frac{1}{R(\tau(t))} \int_0^{\tau(t)} R^2(x)R(\tau(x))p(x) \, dx \right\} > 2 \quad (3.1)$$

or  $\tau(t) \geq t$  and

$$\limsup_{t \rightarrow \infty} \left\{ R(\tau(t)) \int_{\tau(t)}^{\infty} R(x)p(x) \, dx + \int_t^{\tau(t)} R(x)R(\tau(x))p(x) \, dx + \frac{1}{R(\tau(t))} \int_0^t R^2(x)R(\tau(x))p(x) \, dx \right\} > 2, \quad (3.2)$$

then equation (E) has property (B).

*Proof.* Condition (1.3) guarantees property (B) of  $(E_2)$ . Taking into account that in view of (2.12) condition (1.3) is equivalent to (3.1), the assertion follows immediately from Theorems 2.3 and 2.8. The rest of the proof is done analogously.  $\square$

We support our criteria by the following illustrative example.

**Example 3.2.** Consider the Euler differential equation with deviating argument

$$\left( t^\alpha (t^\beta y'(t))' \right)' - \frac{p_0}{t^{3-\alpha-\beta}} y(\lambda t) = 0, \quad p_0 > 0, \quad \lambda > 0, \quad \beta < 1, \quad \alpha < 1. \quad (E_u)$$

It is easy to verify that (2.1) is true. We will examine two cases.

Case  $\alpha \leq \beta$  (i.e.,  $\frac{r_1(t)}{r_2(t)} \downarrow$ ). A simple calculation reveals that

$$r(t) = t^{\frac{\alpha+\beta}{2}} \quad \text{and} \quad R(t) = \frac{2}{2-\alpha-\beta} t^{1-\frac{\alpha+\beta}{2}}.$$

*Subcase*  $0 < \lambda < 1$ . It is easy to verify that condition (3.1) reduces to

$$\frac{2p_0\lambda^{\frac{2-\alpha-\beta}{2}}}{(2-\alpha-\beta)^3} [4 - (2-\alpha-\beta)\ln\lambda] > 1.$$

But then, by Corollary 3.1, it follows that equation  $(E_u)$  enjoys property (B).

*Subcase*  $\lambda > 1$ . In this case, condition (3.2) takes the form

$$\frac{2p_0}{(2-\alpha-\beta)^3} \left[ 4 + (2-\alpha-\beta)\lambda^{\frac{2-\alpha-\beta}{2}} \ln\lambda \right] > 1. \quad (3.3)$$

Now, thanks to Corollary 3.1, we see that  $(E_u)$  enjoys property (B). On the other hand, for this subcase Theorem 3.1 in [3] requires

$$\frac{p_0}{(1-\beta)^2(2-\alpha-\beta)} [2 + (1-\beta)\ln\lambda] > 1. \quad (3.4)$$

Conditions (3.3) and (3.4) are independent, nevertheless, for  $\alpha = 0.4$ ,  $\beta = 0.5$  and  $\lambda = 4$  criteria (3.3) and (3.4) reduce to  $p_0 > 0.0916$  and  $p_0 > 0.1021$ , respectively. On the other hand, Theorem 1.3 guarantees property (B) provided that

$$\frac{p_0}{(1-\beta)(2-\alpha-\beta)} \ln\lambda > \frac{1}{e}$$

which, for  $\alpha = 0.4$ ,  $\beta = 0.5$  and  $\lambda = 4$ , takes the form  $p_0 > 0.1460$ .

*Case*  $\alpha > \beta$  (i.e.,  $\frac{r_2(t)}{r_1(t)} \downarrow$ ). It is easy to verify that

$$R_2(t) = \frac{t^{1-\beta}}{1-\beta}, \quad \omega_1 = \frac{p_0\lambda^{1-\beta}}{(1-\beta)^2(2-\alpha-\beta)}, \quad \omega_2 = \frac{p_0}{(1-\beta)^2(2-\alpha-\beta)}$$

and also that

$$V_i(t) = \left( \frac{\lambda}{1-\beta} \right)^{1-\omega_i} \frac{p_0 t^{\alpha+\beta-2+(1-\beta)(1-\omega_i)}}{2-\alpha-\beta-(1-\beta)(1-\omega_i)}.$$

On the other hand, restrictive condition (2.20) requires

$$\alpha < \beta + 2\omega_i(1-\beta),$$

but then

$$\kappa_i = \frac{\alpha-\beta}{2-\alpha-\beta-2(1-\beta)(1-\omega_i)}, \quad \ell_i = \frac{1}{1+\kappa_i},$$

and finally we have

$$r(t) = \frac{t^{\frac{\alpha+\beta}{2}}}{\sqrt{\ell_i}}, \quad \text{and} \quad R(t) = 2\sqrt{\ell_i} \frac{t^{\frac{2-\alpha-\beta}{2}}}{2-\alpha-\beta},$$

with  $i = 1$  for  $\tau(t) \leq t$  and  $i = 2$  for  $\tau(t) \geq t$ . Proceeding similarly as above, we arrive at the following criteria: if either  $0 < \lambda < 1$

$$\frac{2p_0\lambda^{\frac{2-\alpha-\beta}{2}}}{(2-\alpha-\beta)^3} [4 - (2-\alpha-\beta) \ln \lambda] > \frac{1}{\ell_1}$$

or  $\lambda > 1$  and

$$\frac{2p_0}{(2-\alpha-\beta)^3} \left[ 4 + (2-\alpha-\beta)\lambda^{\frac{2-\alpha-\beta}{2}} \ln \lambda \right] > \frac{1}{\ell_2},$$

then  $(E_u)$  has property (B).

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