ON A RELATION BETWEEN GROWTH ESTIMATES AND HARNACK INEQUALITIES FOR QUASILINEAR ELLIPTIC EQUATIONS WITH NONLINEAR LOWER ORDER TERMS

Kentaro Hirata

Communicated by Vicenţiu D. Rădulescu

Abstract. We investigate a relation between the Harnack inequalities and the (a priori) growth estimates for positive solutions of quasilinear elliptic equations with nonlinear terms involving the solution and its gradient in an arbitrary domain in $\mathbb{R}^N$.

Keywords: growth estimate, Harnack inequality, quasilinear elliptic equation.

Mathematics Subject Classification: 35J92, 35B09, 35B45.

1. INTRODUCTION

Let Ω be an arbitrary domain in $\mathbb{R}^N$ $(N \geq 2)$ with nonempty boundary $\partial\Omega$, and let $\delta_\Omega(x)$ stand for the distance from a point $x \in \Omega$ to $\partial\Omega$. Also, for $p > 1$, Δ_p denotes the p-Laplacian on $\mathbb{R}^N$ and $\Delta = \Delta_2$. This paper is concerned with the Harnack inequalities and the (a priori) growth estimates of the following form for positive solutions of quasilinear elliptic equations with nonlinear terms involving the solution and its gradient. First, let us recall two critical exponents for inequality $-\Delta_p u \geq |u|^{q-1}u$ and equation $-\Delta_p u = |u|^{q-1}u$:

$$p_* := \begin{cases} \dfrac{N(p-1)}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N, \end{cases} \quad \text{and} \quad p^* := \begin{cases} \dfrac{N(p-1)+p}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

In the case of $1 < q < 2^*$, Dancer [11] established the growth estimate

$$u(x) \leq C\delta_\Omega(x)^{-\frac{2}{q-1}} \quad \text{for all } x \in \Omega$$

for all positive solutions of the Lane–Emden equation $-\Delta u = u^q$ in Ω, using the scale invariant property of equation. In the comprehensive work due to Serrin–Zou [27], it was clarified that if $1 < p < N$, $p - 1 < q < p_*$ and $c_0 > 0$, then all positive solutions of

$$u^q - u^{p-1} \leq -\Delta_p u \leq c_0(u^q + 1) \quad \text{in } \Omega \tag{1.1}$$

satisfy the growth estimate

$$u(x) \le C \min\{\delta_\Omega(x), 1\}^{-\frac{p}{q-p+1}} \quad \text{for all } x \in \Omega, \tag{1.2}$$

where a constant C depends only on p, q, c_0 and N. Moreover, all positive solutions of $u^q \le -\Delta_p u \le c_0 u^q$ in Ω satisfy the stronger estimate

$$u(x) \le C\delta_\Omega(x)^{-\frac{p}{q-p+1}} \quad \text{for all } x \in \Omega. \tag{1.3}$$

They first derived the inequalities in Lemma 3.1 below from the weak formulation of (1.1) to have the Harnack inequality (Lemma 3.2) from Serrin's classical result [26], and then applied those to obtain (1.2). Poláčik–Quittner–Souplet [24] verified (1.2) for positive solutions of $-\Delta_p u = f(u)$ with continuous function f satisfying that $t^{-q} f(t)$ has a positive limit for some $q \in (p-1, p^*)$ as $t \to +\infty$. Their proof is based on the rescaling method, a new tool named doubling lemma and the Liouville type theorem by Serrin–Zou, and is entirely independent of the Harnack inequality. Also, in the case of $p > 1$, $q > \max\{p-1, 1\}$, $s \ge pq/(q+1)$ and $M > 0$, pointwise gradient estimates for positive solutions of

$$-\Delta_p u = u^q + M|\nabla u|^s \quad \text{in } \Omega \tag{1.4}$$

were established by Bidaut-Véron–Garcua-Huidobro–Véron [8] ($p = 2$) and Filippucci–Sun–Zheng [14]. For instance, when $s = pq/(q+1)$ and M is sufficiently large, it holds that

$$|\nabla u(x)| \le C\delta_\Omega(x)^{-\frac{q+1}{q-p+1}} \quad \text{for all } x \in \Omega. \tag{1.5}$$

Their proofs based on the Bernstein method are entirely independent of the Harnack inequality. The Harnack inequality for (1.4) was obtained by Ruiz [25] in the case of $p - 1 < q < p_*$ and $p - 1 < s < pq/(q+1)$, by extending the proof in [27]. One of the importance of the growth estimates (1.3) and (1.5) with constant C independent of Ω is that it leads to the Liouville type theorem. See also [1] for extensions to generalized equations of (1.4), and [5–7, 9] for pointwise gradient estimates and the Liouville type theorems for the (stationary) Hamilton–Jacobi equation $-\Delta_p u = |\nabla u|^s$ and its general one $-\Delta_p u = u^\alpha |\nabla u|^\beta$. Baldelli–Filippucci [2, 3] developed the argument in [24] to obtain

$$u(x) + |\nabla u(x)|^{\frac{p}{q+1}} \le C\left\{1 + \delta_\Omega(x)^{-\frac{p}{q-p+1}}\right\} \quad \text{for all } x \in \Omega \tag{1.6}$$

for positive solutions of $-\Delta_p u = a(x)u^q + b(x)u^\alpha |\nabla u|^\beta$ or more general equations of this type. Moreover, they applied (1.6) to obtain an a priori uniform estimate for positive solutions with zero Dirichlet boundary values and derived the existence result in a bounded smooth domain. Also, it is worth mentioning that the weak Harnack inequality (3.2) with nonrestricted radius in the whole space $\mathbb{R}^N$ also derives the Liouville theorem for nonnegative solutions of $-\Delta_p u \ge u^q$ or more general differential inequalities, as proved by D'Ambrosio–Mitidieri [10]. See also [21].

Given the importance of the above growth estimates, it would be interesting to find equivalent conditions under which such estimates hold for positive solutions of equations with general nonlinearities. For work in this direction, we refer to Baldelli–Filippucci [4], where they considered

$$-\Delta_p u = f(x,u,\nabla u) \text{ satisfying } 0 < a(x)t^\alpha |y|^\beta \le f(x,t,y) \le C(1+t^q+t^\alpha|y|^\beta),$$

and proved the dichotomy: either a growth estimate (1.6) with exponents appropriately changed holds, or there exists a sequence of triples consisting of domains, points and positive solutions with explosive properties related to the growth estimate.

In this paper, inspired by the work of Serrin–Zou, we aim to clarify a relation between the growth estimate (1.2) (or (1.3)) and the Harnack inequality for positive solutions of somewhat general equations

$$-\Delta_p u = a(x)u^{p-1} + b(x)u^q + c(x)|\nabla u|^{p-1} + d(x)|\nabla u|^s + e(x) \quad \text{in } \Omega, \tag{1.7}$$

where $a(x), b(x), c(x), d(x), e(x) \in L^\infty(\Omega)$ satisfy $e(x) \ge 0$ on Ω and $\inf_\Omega b(x) > 0$. The generalized Lane–Emden equation $-\Delta_p u = |u|^{q-1}u$ and (1.4) are special cases of (1.7). Also, our results can be applied to $-\Delta_p u = a(x)u^q + b(x)u^\alpha|\nabla u|^\beta$ through the Young inequality $u^\alpha|\nabla u|^\beta \le C(u^r + |\nabla u|^s)$ for an appropriate pair (r,s). The analytical difficulties lie in the indefinite sign of $-\Delta_p u$ and in finding stuff that connects the growth estimate with the Harnack inequality. To address these, we consider solutions of (1.7) as supersolutions of some appropriate equation to which the weak Harnack inequality and the Wolff potential estimate are applicable, and analyze the integral mean of the gradient of a solution.

2. MAIN RESULTS

2.1. STRUCTURE CONDITIONS OF EQUATIONS

Unless otherwise stated explicitly, we always suppose throughout this paper that:

(i) Ω is an arbitrary domain in $\mathbb{R}^N$ $(N \ge 2)$ with $\partial\Omega \ne \emptyset$;

(ii) $p > 1$, $q > p-1$ (no upper constraint is imposed), $s \in (p-1, q_\#]$, where

$$q_\# := \frac{pq}{q+1}; \tag{2.1}$$

(iii) $\sigma \in (p-1, q_\#)$ satisfies

$$\sigma < p - 1 + \frac{q_\#}{N} \qquad \text{if } p \le N, \tag{2.2}$$

$$\sigma \le p - 1 + \frac{N(q-p+1)}{p(q+1)} \quad \text{if } p > N; \tag{2.3}$$

(iv) $\mathscr{A} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $\mathscr{B} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are measurable functions satisfying the following conditions: Let a_1 and b_1' be positive constants, let a_i $(i = 0, 2)$, b_i' $(i = 2, 3, 4)$ and b_i $(i = 1, 2, 3, 4, 5)$ be nonnegative constants, let ω be a nonempty open subset of Ω and let χ_ω be the indicator function of ω. Then, for all $(x,t,y) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$,

(A1) $\mathscr{A}(x,t,y)\cdot y \geq |y|^p - a_0|t|^p$,
(A2) $|\mathscr{A}(x,t,y)| \leq a_1|y|^{p-1} + a_2|t|^{p-1}$,
(B1) $\mathscr{B}(x,t,y) \geq b_1'\chi_\omega(x)|t|^q - b_2'|t|^{p-1} - b_3'|y|^{p-1} - b_4'|y|^\sigma$,
(B2) $\mathscr{B}(x,t,y) \leq b_1|t|^q + b_2|t|^{p-1} + b_3|y|^{p-1} + b_4|y|^s + b_5$.

Note that $\sigma < p$, $s < p$ and

$$q_\# \leq p - 1 + \frac{q_\#}{N} \iff q \leq p_* \quad \text{when } p < N.$$

By a *weak solution* of

$$-\operatorname{div}\mathscr{A}(x,u,\nabla u) = \mathscr{B}(x,u,\nabla u) \quad \text{in } \Omega, \tag{2.4}$$

we mean a lower semicontinuous and p-finely continuous function $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^q_{\mathrm{loc}}(\Omega)$ satisfying

$$\int_\Omega \mathscr{A}(x,u,\nabla u)\cdot\nabla\phi\,dx = \int_\Omega \mathscr{B}(x,u,\nabla u)\phi\,dx \tag{2.5}$$

for all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω. Note that every positive weak solution of (2.5) in the ordinary sense has a lower semicontinuous and p-finely continuous representative (see [20, Theorem 4.8] together with the proof of Lemma 3.4), and that by restricting ordinary weak solutions to those representatives with lower semicontinuity and p-finely continuity, we see that the pointwise estimates (1.2), (1.3) and properties described in the next section are well-defined. By $\mathscr{U}(\Omega)$, we denote a collection of positive (lower semicontinuous and p-finely continuous) weak solutions of (2.4).

2.2. DEFINITIONS OF PROPERTIES

By $B(x,r)$, we denote the open ball of center $x \in \mathbb{R}^N$ and radius $r > 0$. The symbol $⨍_B$ stands for the integral average over a ball B.

Let $\mathscr{F}(\Omega)$ be a collection of positive measurable functions on Ω. We use the following terminologies and abbreviations:

(1) Let $C_1 > 1$ and $0 < r_1 \leq +\infty$. We say that $\mathscr{F}(\Omega)$ enjoys the (C_1, r_1)-*Harnack inequality* (abbreviated to HI) if every $u \in \mathscr{F}(\Omega)$ satisfies that

$$\sup_{B(x,r)} u \leq C_1 \inf_{B(x,r)} u,$$

whenever $B(x,6r) \subset \Omega$ and $0 < r < r_1$.

(2) Let $\tau > 0$, $C_2 > 0$ and $0 < r_2 \leq +\infty$. We say that $\mathscr{F}(\Omega)$ has the (τ, C_2, r_2)-*submean value property* if every $u \in \mathscr{F}(\Omega)$ satisfies that

$$u(x)^\tau \leq C_2 ⨍_{B(x,r)} u(y)^\tau\,dy,$$

whenever $B(x,6r) \subset \Omega$ and $0 < r < r_2$.

(3) Let $C_3 > 0$. We say that $\mathscr{F}(\Omega)$ enjoys the C_3-*growth estimate* (abbreviated to GE) if every $u \in \mathscr{F}(\Omega)$ satisfies that

$$u(x) \le C_3 \delta_\Omega(x)^{-\frac{p}{q-p+1}} \quad \text{for all } x \in \Omega. \tag{2.6}$$

Also, we say that $\mathscr{F}(\Omega)$ enjoys the C_3-*modified growth estimate* (abbreviated to MGE) if every $u \in \mathscr{F}(\Omega)$ satisfies that

$$u(x) \le C_3 \min\{\delta_\Omega(x), 1\}^{-\frac{p}{q-p+1}} \quad \text{for all } x \in \Omega. \tag{2.7}$$

(4) Let $\alpha > 0$, $C_4 > 0$ and $r_3 > 0$. We say that $\mathscr{F}(\Omega)$ enjoys the (α, C_4, r_3)-*average estimate for gradient* (abbreviated to AEG) if every $u \in \mathscr{F}(\Omega)$ satisfies that

$$\left(\fint_{B(x,r)} |\nabla u(y)|^\alpha \, dy \right)^{\frac{1}{\alpha}} \le \frac{C_4}{r} u(x),$$

whenever $B(x, 6r) \subset \Omega$ and $0 < r < r_3$.

2.3. STATEMENTS OF MAIN THEOREMS

For simplicity, we write $\mathscr{F}(\Omega) + 1 := \{u + 1 : u \in \mathscr{F}(\Omega)\}$. Note that $\mathscr{F}(\Omega) + 1$ satisfies the MGE if and only if $\mathscr{F}(\Omega)$ satisfies the MGE. The main theorem is as follows.

Theorem 2.1. *Let τ and r_1 be positive and finite. Assume either*

(1) $b_4' = 0$ *and* $\{x \in \Omega : \delta_\Omega(x) < r_4\} \cup (\Omega \setminus B) \subset \omega$ *for some* $r_4 > 0$ *and ball* $B \subset \mathbb{R}^N$, *or*

(2) $\omega = \Omega$.

Then the following statements hold:

(i) *If* $p \le N$, *then the following* (a), (b), (c) *for* $\mathscr{F}(\Omega) = \mathscr{U}(\Omega) + 1$ *are equivalent:*
 (a) $\mathscr{F}(\Omega)$ *satisfies the* C_3-MGE *and the* $(\max\{s, \sigma\}, C_4, r_1)$-AEG.
 (b) $\mathscr{F}(\Omega)$ *satisfies the* (C_1, r_1)-HI.
 (c) $\mathscr{F}(\Omega)$ *has the* (τ, C_2, r_1)-*submean value property.*
 The constants C_1, C_2, C_3, C_4 *depend on each other and also on* τ, r_1, r_4, a_i, b_i, b_i', p, q, s, σ, B, Ω *and* N *at most. Moreover, if* $b_5 = 0$, *then the above equivalence holds for* $\mathscr{F}(\Omega) = \mathscr{U}(\Omega)$.

(ii) *If* $p > N$, *then each of the above* (a), (b), (c) *holds for* $\mathscr{F}(\Omega) = \mathscr{U}(\Omega)$.

(iii) *If* $\mathscr{F}(\Omega) = \mathscr{U}(\Omega) + 1$ *satisfies one of the above* (a), (b), (c), *then every* $u \in \mathscr{U}(\Omega)$ *satisfies that*

$$\left(\fint_{B(x,r)} |\nabla u|^p \, dy \right)^{\frac{1}{p}} \le \frac{C}{r} \left\{ \left(\fint_{B(x,\rho)} u^\tau \, dy \right)^{\frac{1}{\tau}} + 1 \right\}$$

for all $x \in \Omega$ and $r, \rho \in (0, \min\{\delta_\Omega(x)/6, r_1\})$, where a constant C is independent of u, x, r and ρ. Moreover, when $\mathscr{F}(\Omega) = \mathscr{U}(\Omega)$, we have

$$\left(\fint_{B(x,r)} |\nabla u|^p \, dy \right)^{\frac{1}{p}} \leq \frac{C}{r} \left(\fint_{B(x,\rho)} u^\tau \, dy \right)^{\frac{1}{\tau}}. \tag{2.8}$$

Remark 2.2. If ω is located away from $\partial\Omega \cup \{\infty\}$, then the growth rate near $\partial\Omega$ and the decay rate at infinity of $u \in \mathscr{U}(\Omega)$ would be independent of q. This is the reason for imposing $\{x \in \Omega : \delta_\Omega(x) < r_4\} \cup (\Omega \setminus B) \subset \omega$ to preserve the effect of the term u^q near $\partial\Omega \cup \{\infty\}$.

The following version of nonrestricted r is available for equations like

$$-\Delta_p u = a(x)u^q + b(x)|\nabla u|^{q_\#}$$

with $a(x), b(x) \in L^\infty(\Omega)$ being nonnegative and $\inf_\Omega a(x) > 0$.

Theorem 2.3. *Assume $a_0 = a_2 = b_2 = b_2' = b_3 = b_3' = b_4' = b_5 = 0$, $s = q_\#$ and $\{x \in \Omega : \delta_\Omega(x) < r_4\} \cup (\Omega \setminus B) \subset \omega$ for some $r_4 > 0$ and ball $B \subset \mathbb{R}^N$. Let $\tau > 0$. Then the following statements are equivalent:*

(a) *$\mathscr{U}(\Omega)$ satisfies the C_3-GE and the $(q_\#, C_4, +\infty)$-AEG.*
(b) *$\mathscr{U}(\Omega)$ satisfies the $(C_1, +\infty)$-HI.*
(c) *$\mathscr{U}(\Omega)$ has the $(\tau, C_2, +\infty)$-submean value property.*

The constants C_1, C_2, C_3, C_4 depend on each other and also on τ, a_1, b_1, b_1', b_4, p, q, Ω and N at most. Moreover, the following assertions hold:

(i) *If $p > N$, then each of the above (a), (b), (c) holds.*
(ii) *If one of the above (a), (b), (c) holds, then (2.8) holds for all $x \in \Omega$ and $r, \rho \in (0, \delta_\Omega(x)/6)$.*

Remark 2.4. If $\omega = \Omega$, then the constants C_1, C_2, C_3, C_4 in Theorems 2.1 and 2.3 are independent of Ω. Also, as we will mention in Remark 4.4, the AEG can be removed from the statements in (a) of Theorems 2.1 and 2.3 if q satisfies (4.4). Notice that (2.8) is a reverse Poincaré inequality. A weak version was obtained in [10, Corollary 2.2].

The above results are new even for $p = 2$. Moreover, these complement the result of Serrin–Zou [27], since they gave the proof of the implication HI $\Longrightarrow$ MGE by using Lemma 3.1, but the converse remains unknown. We also provide another proof for the implication HI $\Longrightarrow$ MGE. Our proofs are based on integral estimates (different from theirs) derived from the weak formulation (2.5), the weak Harnack inequality by Trudinger [28] and the Wolff potential estimates by Kilpeläinen–Malý [16, 17] and Malý–Ziemer [20].

This paper is organized as follows. Some known results and integral estimates are collected in Section 3. Proofs of Theorems 2.1 and 2.3 are given in Section 4. As a byproduct, Section 5 provides a pointwise gradient estimate in a special case, using Lemma 4.3 and the results by Duzaar–Mingione [12, 13], Kuusi–Mingione [18] and Nguyen–Phuc [22].

3. PRELIMINARY MATERIALS

The letter C stands for a generic positive constant whose value may vary at each occurrence, while $C_1, C_2, \ldots$ denote specific constants. The notation $C = C(a, b, \ldots)$ means that the constant C depends on the parameters $a, b, \ldots$ at most.

The following two lemmas can be shown by repeating the same argument as in Serrin–Zou [27] and Ruiz [25]. Recall that

$$q_\# := \frac{pq}{q+1} \quad \text{and} \quad p_* := \begin{cases} \dfrac{N(p-1)}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

Lemma 3.1. *Let $\nu \in (0, q)$ and $\mu \in (0, q_\#)$. Then there exists*

$$C = C(\nu, \mu, a_i, b_i', p, q, \sigma, N) > 0$$

such that

$$\left(\fint_{B(x,r)} u^\nu \, dy \right)^{\frac{1}{\nu}} \leq C \min\{r, 1\}^{-\frac{p}{q-p+1}}$$

and

$$\left(\fint_{B(x,r)} |\nabla u|^\mu \, dy \right)^{\frac{1}{\mu}} \leq C \min\{r, 1\}^{-\frac{q+1}{q-p+1}},$$

whenever $B(x, 2r) \subset \omega$ and $u \in \mathscr{U}(\Omega)$.

Lemma 3.2. *Assume $p \leq N$ and $q < p_*$. Let $r_0 > 0$. Then there exists*

$$C = C(r_0, a_i, b_i, b_i', p, q, s, \sigma, N) > 0$$

such that

$$\sup_{B(x,r)} u \leq C \left(\inf_{B(x,r)} u + b_5 r^p \right),$$

whenever $B(x, 6r) \subset \omega$, $0 < r \leq r_0$ and $u \in \mathscr{U}(\Omega)$.

Remark 3.3. Lemma 3.2 is not used in our proofs of Theorems 2.1 and 2.3. We should note that the conclusion of Lemma 3.2 does not follow from Serrin's classical result [26] and Lemma 3.1, when $q \geq p_*$. Thus, for such q, Theorem 2.1 provides a means of deriving the HI from the MGE for $\mathscr{U}(\Omega) + 1$. On the other hand, if $\omega = \Omega$, $p \leq N$ and $q < p_*$, then $\mathscr{U}(\Omega) + 1$ (equivalently, $\mathscr{U}(\Omega)$) satisfies the C_3-MGE with C_3 being independent of Ω by Theorem 2.1 and Lemma 3.2, and therefore, by expanding Ω to $\mathbb{R}^N$, we see that all elements in $\mathscr{U}(\mathbb{R}^N)$ are bounded by $C_3 + 1$ on the whole of $\mathbb{R}^N$. See [8, 14] for equation $-\Delta_p u = u^q + M|\nabla u|^s$.

In [28], Trudinger established the following weak Harnack inequality (3.2) for positive supersolutions of (2.4) with $b_1 = b_1' = b_4 = b_4' = b_5 = 0$, and extended it to the case that b_2, b_2', b_3, b_3' are functions belonging to $L^{N,\lambda}_{\mathrm{uloc}}(\Omega)$ for some $\lambda > 0$ if $p \le N$, and belonging to $L^{p,0}_{\mathrm{uloc}}(\Omega)$ if $p > N$. Here $b \in L^{p,\lambda}_{\mathrm{uloc}}(\Omega)$ means that there exist constants $C_b > 0$ and $r_b > 0$ such that

$$\|b\|_{L^p(B(x,r))} \le C_b r^\lambda \quad \text{whenever } B(x,2r) \subset \Omega \text{ and } 0 < r \le r_b. \tag{3.1}$$

Note in his result that a constant C in (3.2) also depends on C_b and r_b ($b = b_2, b_2', b_3, b_3'$). With the help of Lemma 3.1, we can extend the weak Harnack inequality to our case as follows.

Lemma 3.4. *Let $r_0 > 0$ and $\alpha \in (0, p_*)$. Then there exists*

$$C = C(r_0, \alpha, a_i, b_i', p, q, \sigma, N) > 0$$

such that

(i) *whenever $B(x, 6r) \subset \omega$, $0 < r \le r_0$ and $u \in \mathscr{U}(\Omega)$, we have*

$$\left(\fint_{B(x,2r)} u^\alpha \, dy \right)^{\frac{1}{\alpha}} \le C \inf_{B(x,r)} u; \tag{3.2}$$

(ii) *if $p > N$, then every $u \in \mathscr{U}(\Omega)$ satisfies that*

$$\sup_{B(x,r)} u \le C \inf_{B(x,r)} u,$$

whenever $B(x, 6r) \subset \omega$ and $0 < r \le r_0$.

Moreover, the following statements hold:

(iii) *If $b_4' = 0$, then the above statements* (i), (ii) *hold on Ω in place of ω.*
(iv) *If $a_0 = a_2 = b_2' = b_3' = b_4' = 0$, then the restriction $r \le r_0$ can be removed and C is independent of r_0 in the statements* (i), (ii).

Proof. We need to check the condition of coefficients in the case $b_4' \ne 0$. Fix $u \in \mathscr{U}(\Omega)$ and let

$$\underline{\mathscr{B}}(x, t, y) := b_2'|t|^{p-1} + b(x)|y|^{p-1} \quad \text{with} \quad b(x) := b_3' + b_4'|\nabla u(x)|^{\sigma-p+1}. \tag{3.3}$$

Using (2.2), (2.3) and Lemma 3.1, we can show that

$$b \in \begin{cases} L^{N,\lambda}_{\mathrm{uloc}}(\omega) & \text{if } p \le N, \\ L^{p,0}_{\mathrm{uloc}}(\omega) & \text{if } p > N, \end{cases}$$

where $\lambda := (pq - (q+1)\sigma)/(q - p + 1) > 0$ by $\sigma < q_\#$, and we can take $r_b = r_0$ and $C_b = C(r_0, a_i, b_i', p, q, \sigma, N)$ in (3.1). Moreover, (B1) implies that u is a weak solution of

$$-\operatorname{div} \mathscr{A}(x, u, \nabla u) + \underline{\mathscr{B}}(x, u, \nabla u) \ge 0 \quad \text{in } \omega. \tag{3.4}$$

Therefore, the assertions (i), (ii) follow from [28, Section 5] (see also [20, Theorem 3.13]). Also, (iii) and (iv) can be obtained from [15, Theorem 3.59] or [20, Lemma 2.113]. □

Remark 3.5. When $b'_4 = 0$, there is no need to use Lemma 3.1, and so Lemma 3.4 (iii), (iv) hold under the weaker assumption (B1) with $b'_1 = 0$.

For $B(x,r) \subset \Omega$, we denote the Wolff potential associated with $u \in \mathscr{U}(\Omega)$ relative to (3.4) by

$$\mathrm{W}_p^u(x,r) := \int_0^r \left(t^{p-N} \int_{B(x,t)} \overline{\mathscr{B}}(y,u(y),\nabla u(y))\,dy \right)^{\frac{1}{p-1}} \frac{dt}{t}, \tag{3.5}$$

where

$$\overline{\mathscr{B}}(x,t,y) := \mathscr{B}(x,t,y) + b'_2|t|^{p-1} + b'_3|y|^{p-1} + b'_4|y|^{\sigma}. \tag{3.6}$$

Note that $\overline{\mathscr{B}}(x,t,y) \geq 0$ by (B1). The Wolff potential estimate (3.7) for weak solutions of $-\operatorname{div}\mathscr{A}(x,\nabla u) = \mu$ with a measure μ was first established by Kilpeläinen–Malý [16, 17]. Moreover, its extension to $-\operatorname{div}\mathscr{A}(x,u,\nabla u) + \underline{\mathscr{B}}(x,u,\nabla u) = \mu$ can be found in [20, Theorem 4.27]. See also [19]. To apply these results, we take β and ε as follows: If $p < N$, then noting $p < q_{\#}/(\sigma - p + 1)$ by $0 < p - 1 < \sigma < q_{\#}$, we take

$$\beta \in \left(p, \min\left\{ \frac{Np(q-p+1)}{N(q-p+1) - p\{pq - (q+1)\sigma\}}, \frac{q_{\#}}{\sigma - p + 1} \right\} \right)$$

and

$$\varepsilon := (N-p)\left(\frac{\beta}{p} - 1\right).$$

If $p = N$, then $N - 1 < \sigma < q_{\#}$ implies $1 < (q - N + 1)/(q+1)(\sigma - N + 1)$, so we take

$$\beta \in \left(N, \frac{N(q-N+1)}{(q+1)(\sigma-N+1)} \right) \quad \text{and} \quad \varepsilon := N - \frac{\beta(q+1)(\sigma-N+1)}{q-N+1}.$$

Then, using

$$\beta < \frac{Np(q-p+1)}{N(q-p+1) - p\{pq-(q+1)\sigma\}} < \frac{Np}{N-p} \quad \text{when } p < N,$$

we observe that $\varepsilon \in (0,p)$ and

$$(p-1)\cdot\frac{\beta p}{p(p-1)+\beta} < p_*.$$

The last one is used when applying Lemma 3.4. Let $u \in \mathscr{U}(\Omega)$ and let $b(x)$ and $\underline{\mathscr{B}}(x,t,y)$ be as in (3.3). Since $p(\sigma - p + 1) < \beta(\sigma - p + 1) < q_{\#}$, it follows from Lemma 3.1 that there exists $C > 0$ such that

$$\int_{B(x,r)} b^p\,dy \leq Cr^{N-p+\varepsilon} \quad \text{and} \quad \int_{B(x,r)} b^{\beta}\,dy \leq Cr^{N-p+\varepsilon},$$

whenever $B(x,2r) \subset \omega$ and $0 < r \le r_0$. Also, u is a weak solution of

$$-\operatorname{div} \mathscr{A}(x,u,\nabla u) + \underline{\mathscr{B}}(x,u,\nabla u) = \mu \quad \text{in } \omega$$

with $d\mu = \{\mathscr{B}(x,u,\nabla u) + \underline{\mathscr{B}}(x,u,\nabla u)\}\,dx = \overline{\mathscr{B}}(x,u,\nabla u)\,dx$. Thus, we can obtain the following estimate from the above mentioned results and Lemma 3.4.

Lemma 3.6. *Assume $p \le N$. Let $r_0 > 0$. Then there exists*

$$C = C(r_0, a_i, b_i', p, q, \sigma, N) > 0$$

such that

$$u(x) \le C \left\{ \inf_{B(x,r)} u + \mathrm{W}_p^u(x,2r) \right\}, \tag{3.7}$$

whenever $B(x,6r) \subset \omega$, $0 < r \le r_0$ and $u \in \mathscr{U}(\Omega)$. Moreover, the following statements hold:

(i) *If $b_4' = 0$, then the above statement holds on Ω in place of ω.*
(ii) *If $a_0 = a_2 = b_2' = b_3' = b_4' = 0$, then the restriction $r \le r_0$ can be removed and C is independent of r_0 in the above statement.*

Remark 3.7. When $b_4' = 0$, there is no need to use Lemma 3.1, and so Lemma 3.6 (i), (ii) hold under the weaker assumption (B1) with $b_1' = 0$.

In some lemmas below, we adopt the following condition weaker than (B1):

(B1w) there exist constants $b_2' \ge 0$, $b_3' \ge 0$ and $b_4' \ge 0$ such that

$$\mathscr{B}(x,t,y) \ge -b_2'|t|^{p-1} - b_3'|y|^{p-1} - b_4'|y|^{q_\#} \quad \text{for all } (x,t,y) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N,$$

where $q_\#$ is given by (2.1). Even if (B1) is changed to (B1w), the weak formulation (2.5) is well-defined, and we use the same notation $u \in \mathscr{U}(\Omega)$ to denote that u is a positive (lower semicontinuous and p-finely continuous) weak solution of (2.4).

Lemma 3.8. *Assume* (B1w) *instead of* (B1). *Let $\beta < 0$. Then there exists*

$$C = C(\beta, a_i, b_i', p, q) > 0$$

such that

$$\int_{B(x,r)} u^{\beta-1}|\nabla u|^p\,dy \le C \left(\min\{r,1\}^{-p} \int_{B(x,2r)} u^{p+\beta-1}\,dy + \int_{B(x,2r)} u^{q+\beta}\,dy \right),$$

whenever $B(x,2r) \subset \Omega$ and $u \in \mathscr{U}(\Omega)$. Moreover, the following statements hold:

(i) *If $a_0 = a_2 = b_2' = b_3' = 0$, then we can replace* $\min\{r,1\}$ *by r.*
(ii) *If $b_4' = 0$, then the term $\int_{B(x,2r)} u^{q+\beta}\,dy$ can be removed.*

Proof. Take $\psi \in C_c^\infty(B(x,2r))$ with $0 \le \psi \le 1$, $|\nabla\psi| \le 2/r$ on $B(x,2r)$ and $\psi = 1$ on $B(x,r)$. Since u has a positive lower bound on the support of ψ by lower semicontinuity, we see that $\phi := u^\beta\psi^p \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω. Using (A1), (A2) and (B1w), let us estimate (2.5). Then

$$-\beta \int_\Omega |\nabla u|^p u^{\beta-1}\psi^p \, dy \le \int_\Omega \big\{(-\beta a_0 + b_2')u^{p+\beta-1}\psi^p + a_1 p|\nabla u|^{p-1}u^\beta\psi^{p-1}|\nabla\psi| + a_2 p u^{p+\beta-1}\psi^{p-1}|\nabla\psi| + b_3'|\nabla u|^{p-1}u^\beta\psi^p + b_4'|\nabla u|^{q\#}u^\beta\psi^p\big\}\, dy. \tag{3.8}$$

By the Young inequality,

$$\begin{aligned} a_1 p|\nabla u|^{p-1}u^\beta\psi^{p-1}|\nabla\psi| &\le \frac{-\beta}{4}|\nabla u|^p u^{\beta-1}\psi^p + Cu^{p+\beta-1}|\nabla\psi|^p, \\ b_3'|\nabla u|^{p-1}u^\beta\psi^p &\le \frac{-\beta}{4}|\nabla u|^p u^{\beta-1}\psi^p + Cu^{p+\beta-1}\psi^p, \\ b_4'|\nabla u|^{q\#}u^\beta\psi^p &\le \frac{-\beta}{4}|\nabla u|^p u^{\beta-1}\psi^p + Cu^{q+\beta}\psi^p, \\ p\psi^{p-1}|\nabla\psi| &\le (p-1)\psi^p + |\nabla\psi|^p. \end{aligned} \tag{3.9}$$

Substituting these into (3.8) and shifting all the terms including $|\nabla u|^p u^{\beta-1}\psi^p$ to the left hand side, we have

$$\frac{-\beta}{4}\int_\Omega |\nabla u|^p u^{\beta-1}\psi^p \, dy \le \int_\Omega \big\{ (-\beta a_0 + a_2(p-1) + b_2' + C)\, u^{p+\beta-1}\psi^p + (a_2 + C)u^{p+\beta-1}|\nabla\psi|^p + Cu^{q+\beta}\psi^p\big\}\, dy.$$

Therefore, the desired inequality follows from the property of ψ. □

Lemma 3.9. *Assume* (B1w) *instead of* (B1). *Let* $\alpha \in (0,p]$. *Then there exists* $C = C(\alpha, a_i, b_i', p, q, N) > 0$ *such that*

$$⨍_{B(x,r)} \left(\frac{|\nabla u|}{u}\right)^\alpha dy \le C\left\{\min\{r,1\}^{-\alpha} + \left(⨍_{B(x,2r)} u^{q-p+1}\, dy\right)^{\frac{\alpha}{p}}\right\},$$

whenever $B(x,2r) \subset \Omega$ *and* $u \in \mathscr{U}(\Omega)$. *Moreover, the following statements hold:*

(i) *If* $a_0 = a_2 = b_2' = b_3' = 0$, *then we can replace* $\min\{r,1\}$ *by* r.
(ii) *If* $b_4' = 0$, *then the term* $(⨍_{B(x,2r)} u^{q-p+1}\, dy)^{\alpha/p}$ *can be removed.*

Proof. By Lemma 3.8 with $\beta = 1-p$, we have

$$⨍_{B(x,r)} \left(\frac{|\nabla u|}{u}\right)^p dy \le C\left\{\min\{r,1\}^{-p} + ⨍_{B(x,2r)} u^{q-p+1}\, dy\right\}.$$

The conclusion follows after squaring both sides to the (α/p)-th power and applying the Jensen inequality to the left hand side. □

Lemma 3.10. *Let $\beta \in (0,1]$. Then there exists $C = C(\beta, a_i, b_i, p, q) > 0$ such that*

$$\int_{B(x,r)} u^{\beta-1}|\nabla u|^p\,dy \le C\left(\min\{r,1\}^{-p}\int_{B(x,2r)} u^{p+\beta-1}\,dy + \int_{B(x,2r)} u^{q+\beta}\,dy + \int_{B(x,2r)} u^{\beta}\,dy\right),$$

whenever $B(x,2r) \subset \Omega$ and $u \in \mathscr{U}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$. Moreover, the following statements hold:

(i) *If $a_0 = a_2 = b_2 = b_3 = 0$ and $s = q_\#$, then we can replace $\min\{r,1\}$ by r.*
(ii) *If $b_5 = 0$, then the term $\int_{B(x,2r)} u^\beta\,dy$ can be removed.*

Proof. We may assume that (B2) is satisfied for $s = q_\#$ by replacing b_3 with $b_3 + b_4$. Take $\psi \in C_c^\infty(B(x,2r))$ so that $0 \le \psi \le 1$, $|\nabla\psi| \le 2/r$ on $B(x,2r)$ and $\psi = 1$ on $B(x,r)$. Then $\phi := u^\beta\psi^p \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω, since we have assumption $u \in L^\infty_{\rm loc}(\Omega)$ this time. As in the proof of Lemma 3.8, conditions (A1), (A2) and (B2) yield that

$$\begin{aligned}\beta\int_\Omega |\nabla u|^p u^{\beta-1}\psi^p\,dy \le \int_\Omega \big\{&(\beta a_0 + b_2)u^{p+\beta-1}\psi^p + a_1 p|\nabla u|^{p-1}u^\beta\psi^{p-1}|\nabla\psi| \\ &+ a_2 p u^{p+\beta-1}\psi^{p-1}|\nabla\psi| + b_1 u^{q+\beta}\psi^p + b_3|\nabla u|^{p-1}u^\beta\psi^p \\ &+ b_4|\nabla u|^{q_\#}u^\beta\psi^p + b_5 u^\beta\psi^p\big\}\,dy.\end{aligned}$$

Applying the similar inequalities to (3.9) and shifting all the terms including $|\nabla u|^p u^{\beta-1}\psi^p$ to the left hand side, we have

$$\int_\Omega |\nabla u|^p u^{\beta-1}\psi^p\,dy \le C\int_\Omega \big\{u^{p+\beta-1}\psi^p + u^{p+\beta-1}|\nabla\psi|^p + u^{q+\beta}\psi^p + u^\beta\psi^p\big\}\,dy.$$

Therefore, the desired inequality follows from the property of ψ. □

Lemma 3.11. *Let $\beta \in (0,1)$ and $B := B(z,r)$. If v is a nonnegative bounded measurable function on B for which there exists $C_5 > 0$ such that*

$$v(y) \le C_5 \fint_{B(y,\delta_B(y)/2)} v\,dx \quad \text{for all } y \in B, \tag{3.10}$$

then there exists $C_6 = C(\beta, C_5, N) > 0$ such that

$$v(z)^\beta \le C_6 \fint_B v^\beta\,dx.$$

Proof. An idea of the proof is based on Pavlovic [23]. Let

$$M := \sup\left\{\delta_B(y)^N v(y)^\beta : y \in B\right\}.$$

Then M is finite by $v \in L^\infty(B)$. We may let $M > 0$. Let $y \in B$. Multiplying both sides of (3.10) by $\delta_B(y)^{N/\beta}$ and using $\delta_B(y) \le 2\delta_B(x)$ for all $x \in B(y, \delta_B(y)/2)$, we have

$$\begin{aligned}\delta_B(y)^{\frac{N}{\beta}} v(y) &\le C_5 \frac{2^N}{\nu_N} \delta_B(y)^{\frac{N(1-\beta)}{\beta}} \int\limits_{B(y,\delta_B(y)/2)} v(x)\,dx \\ &\le C_5 \frac{2^{\frac{N}{\beta}}}{\nu_N} \int\limits_B v(x)^\beta \left\{\delta_B(x)^N v(x)^\beta\right\}^{\frac{1-\beta}{\beta}} dx,\end{aligned}$$

where ν_N is the volume of the unit ball in $\mathbb{R}^N$. Taking the supremum in both sides, we have

$$M^{\frac{1}{\beta}} \le C_5 \frac{2^{\frac{N}{\beta}}}{\nu_N} M^{\frac{1-\beta}{\beta}} \int\limits_B v^\beta\,dx,$$

which gives the desired estimate with $C_6 = 2^{\frac{N}{\beta}} C_5$, since $r^N v(z)^\beta \le M$. □

4. PROOFS OF THEOREMS 2.1 AND 2.3

Since Lemma 3.4 guarantees that the (C_1, r_1)-HI for $\mathscr{U}(\Omega)$ always holds true under $\omega = \Omega$ or $b_4' = 0$ (without additional assumptions below) when $p > N$, we focus on the case $p \le N$ in the following result. Recall that s and σ are the exponents in (B1) and (B2).

Proposition 4.1. *Assume $p \le N$ and either $\omega = \Omega$ or $b_4' = 0$ in* (B1). *Then the following statements hold:*

(i) *Let $r_1 > 0$. For $\mathscr{U}(\Omega) + 1$, we have the implication*

$$C_3\text{-MGE } \textit{and } (\max\{s,\sigma\}, C_4, r_3)\text{-AEG} \implies (C_1, r_1)\text{-HI}. \tag{4.1}$$

(ii) *If $b_5 = 0$, then the implication* (4.1) *holds for $\mathscr{U}(\Omega)$.*

(iii) *If $a_0 = a_2 = b_2 = b_2' = b_3 = b_3' = b_4' = b_5 = 0$ and $s = q_\#$, then the following implication holds for $\mathscr{U}(\Omega)$:*

$$C_3\text{-GE } \textit{and } (q_\#, C_4, +\infty)\text{-AEG} \implies (C_1, +\infty)\text{-HI}.$$

In (i)–(iii), *the constant C_1 depends only on r_1, r_3, C_3, C_4, a_i, b_i, b_i', p, q, s, σ and N at most.*

Proof. We give a proof only for (i), since (ii) and (iii) can be obtained by a simple modification of the argument below. For example, when proving (ii), we use the following inequality instead of (4.2):

$$\begin{aligned}\left(\frac{C_4u(z)}{r}\right)^{p-1}+\left(\frac{C_4u(z)}{r}\right)^{s}&\le 2\left(\frac{C_4u(z)}{r}\right)^{p-1}+\left(\frac{C_4u(z)}{r}\right)^{q_\#}\\&\le C\Big(r^{1-p}\min\{\delta_\Omega(z),1\}^{-1}\\&\qquad+r^{-q_\#}\min\{\delta_\Omega(z),1\}^{-\frac{p}{q+1}}\Big)u(z)^{p-1}\end{aligned}$$

by $1\le\min\{\delta_\Omega(z),1\}^{-1}$ and (2.7). When proving (iii), we may define $\delta_*(z):=\kappa\delta_\Omega(z)$.

Note by assumption that (3.2) and (3.7) hold for $\omega=\Omega$. Also, we may assume $\sigma\le s$ by considering $\max\{s,\sigma\}$. Then, by (B1) and (B2),

$$0\le\overline{\mathscr{B}}(x,t,y)\le C\left(|t|^q+|t|^{p-1}+|y|^{p-1}+|y|^s+1\right)\quad\text{for all }(x,t,y)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N,$$

where $\overline{\mathscr{B}}(x,t,y)$ is given by (3.6).

Let $u\in\mathscr{U}(\Omega)$, let $z\in\Omega$ and let r, ρ satisfy

$$0<r/2\le\rho\le\delta_*(z):=\kappa\min\{\delta_\Omega(z),1,r_3\},$$

where $\kappa\in(0,1/12]$ is chosen later. By the assumptions in (4.1) and the Jensen inequality, we have

$$u(x)^q\le C\min\{\delta_\Omega(x),1\}^{-p}u(x)^{p-1}\le C\min\{\delta_\Omega(z),1\}^{-p}u(x)^{p-1}\quad\text{for all }x\in B(z,2\rho),$$

and

$$\left(\fint_{B(z,r)}|\nabla u|^{p-1}\,dx\right)^{\frac{1}{p-1}}\le\left(\fint_{B(z,r)}|\nabla u|^{s}\,dx\right)^{\frac{1}{s}}\le\frac{C_4}{r}\{u(z)+1\}.$$

Without loss of generality, we may assume $C_4\ge1$. Since $p-1<s\le q_\#$ and $r\le1$, we have

$$1\le\left(\frac{C_4\{u(z)+1\}}{r}\right)^{p-1}\le\left(\frac{C_4\{u(z)+1\}}{r}\right)^{s}\le\left(\frac{C_4\{u(z)+1\}}{r}\right)^{q_\#}.\qquad(4.2)$$

Using the above estimates together with Lemma 3.4 and (2.7), we have

$$\begin{aligned}\fint_{B(z,r)}\overline{\mathscr{B}}(x,u,\nabla u)\,dx&\le C\fint_{B(z,r)}\left(u^q+u^{p-1}+|\nabla u|^{p-1}+|\nabla u|^s+1\right)dx\\&\le C\left\{\min\{\delta_\Omega(z),1\}^{-p}\fint_{B(z,r)}u^{p-1}\,dx+\left(\frac{u(z)+1}{r}\right)^{q_\#}\right\}\\&\le C\Big(\min\{\delta_\Omega(z),1\}^{-p}+r^{-q_\#}\min\{\delta_\Omega(z),1\}^{-\frac{p}{q+1}}\Big)\{u(z)+1\}^{p-1}.\end{aligned}$$

Therefore, the Wolff potential defined by (3.5) is estimated by

$$\begin{aligned}\mathrm{W}_p^u(z,2\rho) &\le C\left\{\left(\frac{\rho}{\min\{\delta_\Omega(z),1\}}\right)^{\frac{p}{p-1}}+\left(\frac{\rho}{\min\{\delta_\Omega(z),1\}}\right)^{\frac{p}{(p-1)(q+1)}}\right\}\{u(z)+1\}\\ &\le C\kappa^{\frac{p}{(p-1)(q+1)}}\{u(z)+1\}.\end{aligned}$$

Taking $\kappa=\kappa(C_3,C_4,a_i,b_i,b_i',p,q,s,\sigma,N)>0$ small enough, we obtain from Lemma 3.6 that

$$\frac{1}{2}u(z)\le C\left(\inf_{B(z,\rho)}u+1\right).$$

Note that this is valid for any pair of $z\in\Omega$ and $\rho\in(0,\delta_*(z))$.

Let $w\in\Omega$ and $\rho\in(0,\delta_*(w))$. If $y\in B(w,\rho/2)$, then $\rho/2<\delta_*(y)$, and so

$$u(y)\le C\left(\inf_{B(y,\rho/2)}u+1\right)\le C\{u(w)+1\}\le C\left(\inf_{B(w,\rho)}u+1\right)\le C\left(\inf_{B(w,\rho/2)}u+1\right).$$

Therefore,

$$\sup_{B(w,\rho/2)}(u+1)\le C\inf_{B(w,\rho/2)}(u+1). \tag{4.3}$$

Finally, let $B(x,6r)\subset\Omega$ and $0<r<r_1$. After covering $B(x,r)$ by finitely many balls $B(x_j,\delta)$ $(j=1,\dots,m)$ with $x_j\in B(x,r)$, $\delta:=\kappa\min\{r,1,r_3\}/2$ and $m=m(\kappa,r_1,r_3,N)\in\mathbb{N}$, we can repeat the use of (4.3) with $w=x_j$ and $\rho=2\delta$ to obtain $\sup_{B(x,r)}(u+1)\le C\inf_{B(x,r)}(u+1)$, since $r<\delta_\Omega(x_j)$ implies $2\delta<\delta_*(x_j)$. This completes the proof. □

Remark 4.2. If $b_4'=0$, then Proposition 4.1 holds true even when $b_1'=0$, in which case the validity of Lemmas 3.1 and 3.2 is unknown (since those proofs rely on estimates for the term of b_1').

After the following lemma, we make a remark that the MGE implies the AEG in a special case.

Lemma 4.3. *Assume either $\omega=\Omega$ or $b_4'=0$ in* (B1) *and that $\mathscr{U}(\Omega)$ enjoys the* C_3-MGE. *Let $\alpha\in(0,\min\{p,p_*\})$ and $r_3>0$. Then there exists*

$$C_4=C(\alpha,r_3,C_3,a_i,b_i,b_i',p,q,\sigma,N)>0$$

*such that $\mathscr{U}(\Omega)+1$ satisfies the (α,C_4,r_3)-*AEG. *Moreover, the following assertions hold:*

(i) *If $b_5=0$, then $\mathscr{U}(\Omega)$ satisfies the (α,C_4,r_3)-*AEG.

(ii) *If $a_0=a_2=b_2=b_2'=b_3=b_3'=b_4'=b_5=0$, $s=q_\#$ and $\mathscr{U}(\Omega)$ satisfies the C_3-*GE, *then $\mathscr{U}(\Omega)$ satisfies the $(\alpha,C_4,+\infty)$-*AEG, *where C_4 is independent of r_3.*

Proof. Note that (2.7) implies $u \in L^\infty_{\mathrm{loc}}(\Omega)$. In view of the Jensen inequality, we may assume $\alpha > p-1$. Let $\beta := \alpha + 1 - p$. Then $0 < \beta < 1$ and

$$\beta < p + \beta - 1 = \frac{\alpha(1-\beta)}{p-\alpha} = \alpha < p_*.$$

Let $B(x, 6r) \subset \Omega$ and $0 < r < r_3$. Since

$$u(y)^{q+\beta} = u(y)^{q-p+1} u(y)^{p+\beta-1} \le C \min\{r,1\}^{-p} u(y)^{p+\beta-1} \quad \text{for all } y \in B(x, 2r)$$

by (2.7), it follows from the Hölder inequality, Lemma 3.10 and Lemma 3.4 that

$$\begin{aligned}
\fint_{B(x,r)} |\nabla u|^\alpha \, dy &\le \left(\fint_{B(x,r)} u^{\beta-1} |\nabla u|^p \, dy \right)^{\frac{\alpha}{p}} \left(\fint_{B(x,r)} u^{\frac{\alpha(1-\beta)}{p-\alpha}} \, dy \right)^{\frac{p-\alpha}{p}} \\
&\le C \left(\min\{r,1\}^{-p} \fint_{B(x,2r)} u^{p+\beta-1} \, dy + \fint_{B(x,2r)} u^\beta \, dy \right)^{\frac{\alpha}{p}} \\
&\quad \times \left(\fint_{B(x,r)} u^{\frac{\alpha(1-\beta)}{p-\alpha}} \, dy \right)^{\frac{p-\alpha}{p}} \\
&\le C \left(\min\{r,1\}^{-p} u(x)^{p+\beta-1} + u(x)^\beta \right)^{\frac{\alpha}{p}} u(x)^{\frac{\alpha(1-\beta)}{p}} \\
&\le C \left(\min\{r,1\}^{-\alpha} u(x)^\alpha + u(x)^{\frac{\alpha}{p}} \right) \\
&\le C \min\{r,1\}^{-\alpha} (u(x)+1)^\alpha \le C \left(\frac{u(x)+1}{r} \right)^\alpha,
\end{aligned}$$

since $r < r_3$. Therefore, $\mathscr{U}(\Omega) + 1$ satisfies the (α, C, r_3)-AEG. □

Remark 4.4. Obviously, $q_\# < p$. Also, if

$$p - 1 < q < \begin{cases} \dfrac{N(p-1)}{N-p^2} & \text{if } p < \sqrt{N}, \\ +\infty & \text{if } p \ge \sqrt{N}, \end{cases} \tag{4.4}$$

then $q_\# < p_*$. Thus, the conclusion of Lemma 4.3 with $\alpha = q_\#$ holds for such q.

The next result is the converse of Proposition 4.1. Note that the upper range of p is not restricted.

Proposition 4.5. *Assume* $\{x \in \Omega : \delta_\Omega(x) < r_4\} \cup (\Omega \setminus B(0, r_5)) \subset \omega$ *for some* $r_4 > 0$ *and* $r_5 > 0$*. Let* $r_1 > 0$*. Then the following statements hold:*

(i) *For* $\mathscr{U}(\Omega) + 1$*, we have the implication*

$$(C_1, r_1)\text{-HI} \implies C_3\text{-MGE } \textit{and } (p, C_4, r_1)\text{-AEG}. \tag{4.5}$$

(ii) *The implication* (4.5) *holds for* $\mathscr{U}(\Omega)$*.*

(iii) *If* $a_0 = a_2 = b'_2 = b'_3 = b'_4 = 0$*, then the following implication holds for* $\mathscr{U}(\Omega)$*:*

$$(C_1, +\infty)\text{-HI} \implies C_3\text{-GE } \textit{and } (p, C_4, +\infty)\text{-AEG}.$$

In (i)–(iii)*, the constants* C_3 *and* C_4 *depend only on* C_1*,* r_1*,* r_4*,* r_5*,* a_i*,* b'_i*,* p*,* q*,* σ*,* N *and* Ω *at most. When* $\omega = \Omega$*, these constants are independent of* Ω*.*

Proof. When $\omega = \Omega$, the MGE can be verified in the same manner as in Serrin–Zou [27] using Lemma 3.1, but the AEG does not follow from their results. Here we give another proof which does not rely on Lemma 3.1. A proof is given only for (i), since (ii) and (iii) can be obtained by a simple modification of the argument below. Particularly, for (ii) and (iii), there is no need to separate into two cases $C_7 \le 2C_1$ and $C_7 > 2C_1$.

Without loss of generality, we may assume $0 \in \partial\Omega$ by retaking r_5 large enough. To show the MGE, we first consider a point $x \in \omega$ such that $B(x, r) \subset \omega$, where $r := \min\{\delta_\Omega(x), r_1\}/12$. Let $u \in \mathscr{U}(\Omega)$ and

$$C_7 := \sup_{B(x,2r)} u.$$

Then $0 < C_7 < +\infty$ by $0 < u \in L^p_{\mathrm{loc}}(\Omega)$ and the HI. If $C_7 \le 2C_1$, then

$$u(x) \le C_7 \le 2C_1 \min\{\delta_\Omega(x), 1\}^{-\frac{p}{q-p+1}}.$$

We consider the case $C_7 > 2C_1$. Take $\phi \in C_c^\infty(B(x, r))$ so that $0 \le \phi \le 1$, $|\nabla\phi| \le 4/r$ on $B(x, r)$ and $\phi = 1$ on $B(x, r/2)$. From (A2) and (B1), we see that

$$\begin{aligned} \int_{B(x,r)} b'_1 u^q \phi \, dy \le \int_{B(x,r)} \Big\{ &\left(b'_2 u^{p-1} + b'_3 |\nabla u|^{p-1} + b'_4 |\nabla u|^\sigma\right) \phi \\ &+ \left(a_1 |\nabla u|^{p-1} + a_2 u^{p-1}\right) |\nabla\phi| \Big\} \, dy. \end{aligned} \tag{4.6}$$

Since $\sigma < q_\# < q$, we apply the Young inequality with exponents $\frac{q-p+1}{\sigma-p+1}$ and its conjugate to obtain

$$b'_4 |\nabla u|^\sigma = b'_4 u^{\frac{q(\sigma-p+1)}{q-p+1}} \cdot u^{-\frac{q(\sigma-p+1)}{q-p+1}} |\nabla u|^\sigma \le \frac{b'_1}{2} u^q + C \left(\frac{|\nabla u|}{u}\right)^{\frac{\sigma(q-p+1)}{q-\sigma}} u^{p-1}.$$

Substituting this into (4.6), we have

$$\begin{aligned} \frac{b'_1}{2} \int_{B(x,r)} u^q \phi \, dy \le C_7^{p-1} \int_{B(x,r)} \Bigg\{ &b'_2 + \frac{4a_2}{r} + \left(b'_3 + \frac{4a_1}{r}\right) \left(\frac{|\nabla u|}{u}\right)^{p-1} \\ &+ C \left(\frac{|\nabla u|}{u}\right)^{\frac{\sigma(q-p+1)}{q-\sigma}} \Bigg\} \, dy. \end{aligned} \tag{4.7}$$

On the other hand, the HI gives

$$C_7 \le \sup_{B(x,2r)} (u+1) \le C_1 \inf_{B(x,2r)} (u+1) \le C_1 \inf_{B(x,r/2)} u + C_1,$$

and so

$$\int_{B(x,r)} u^q \phi \, dy \ge \left(\frac{C_7 - C_1}{C_1} \right)^q |B(x,r/2)| \ge \left(\frac{C_7}{2C_1} \right)^q |B(x,r/2)|. \tag{4.8}$$

Note by $\sigma < q_\# < q$ that

$$\frac{\sigma(q-p+1)}{q-\sigma} < p,$$

and that $\mathscr{B}$ satisfies (B1w), described just behind Remark 3.7, by replacing b_3' with $b_3' + b_4'$ if necessary. Therefore, by (4.7), (4.8) and Lemma 3.9, we have

$$\begin{aligned} C_7^{q-p+1} \le C \Bigg\{ & \left(1 + \frac{1}{r}\right) \left(1 + \min\{r,1\}^{-(p-1)}\right) + \min\{r,1\}^{-\frac{\sigma(q-p+1)}{q-\sigma}} \\ & + \left(1 + \frac{1}{r}\right) \left(\fint_{B(x,2r)} u^{q-p+1} \, dy \right)^{\frac{p-1}{p}} + \left(\fint_{B(x,2r)} u^{q-p+1} \, dy \right)^{\frac{\sigma(q-p+1)}{p(q-\sigma)}} \Bigg\}. \end{aligned}$$

By the Young inequality, it follows that for sufficiently small $\varepsilon > 0$,

$$\left(1 + \frac{1}{r}\right) \left(\fint_{B(x,2r)} u^{q-p+1} \, dy \right)^{\frac{p-1}{p}} \le \left(1 + \frac{1}{r}\right) C_7^{\frac{(p-1)(q-p+1)}{p}} \le \varepsilon C_7^{q-p+1} + C \left(1 + \frac{1}{r}\right)^p$$

and

$$\left(\fint_{B(x,2r)} u^{q-p+1} \, dy \right)^{\frac{\sigma(q-p+1)}{p(q-\sigma)}} \le C_7^{\frac{\sigma(q-p+1)(q-p+1)}{p(q-\sigma)}} \le \varepsilon C_7^{q-p+1} + C.$$

Therefore, $C_7^{q-p+1} \le C \min\{r,1\}^{-p}$, from which we obtain

$$u(x) \le C_7 \le C \min\{r,1\}^{-\frac{p}{q-p+1}} \le C \min\{\delta_\Omega(x), 1\}^{-\frac{p}{q-p+1}}. \tag{4.9}$$

If $\omega = \Omega$, then the proof of the MGE is done and the constant C is independent of Ω. Otherwise, (4.9) holds for all $x \in D \coloneqq \{x \in \Omega : \delta_\Omega(x) \le r_4/2\} \cup (\Omega \setminus B(0, 2r_5))$, since $|x| \ge 2r_5$ and $y \in B(x,r)$ imply that

$$|y| \ge |x| - r \ge |x| - \frac{1}{12}\delta_\Omega(x) \ge \frac{11}{12}|x| \ge \frac{11}{6} r_5 > r_5.$$

Consider the case $\omega \neq \Omega$. Take any $x \in \Omega \setminus D$ and a point $z \in \Omega$ so that $\delta_\Omega(z) = C(r_4, \Omega) \leq r_4/2$. Since the closure of $\Omega \setminus D$ is compact in Ω, we derive from the HI and the above result that there exists $C = C(C_1, r_4, r_5, \Omega, N) > 0$ such that

$$u(x) \leq C\{u(z) + 1\} \leq C \min\{\delta_\Omega(z), 1\}^{-\frac{q}{q-p+1}} \leq C \leq C \min\{\delta_\Omega(x), 1\}^{-\frac{q}{q-p+1}}$$

(when proving (iii), we use $\delta_\Omega(x) \leq |x| \leq 2r_5$ in the last inequality). This together with (4.9) for $x \in D$ concludes that the MGE holds on the whole of Ω.

To show the AEG, let $B(x, 6r) \subset \Omega$ and $0 < r < r_1$. Then (2.7) gives

$$\fint_{B(x,2r)} u^{q-p+1}\, dy \leq C \min\{r, 1\}^{-p} \leq Cr^{-p}.$$

By Lemma 3.9 with $\alpha = p$ and the HI, we obtain

$$\fint_{B(x,r)} |\nabla u|^p\, dy \leq C_7^p \fint_{B(x,r)} \left(\frac{|\nabla u|}{u}\right)^p dy \leq C \left(\frac{C_7}{r}\right)^p \leq C \left(\frac{C_1\{u(x) + 1\}}{r}\right)^p.$$

This completes the proof. □

Remark 4.6. If $p > N$ and either $\omega = \Omega$ or $b_4' = 0$ in (B1), then $\mathscr{U}(\Omega)$ enjoys the (C_1, r_1)-HI by Lemma 3.4. Therefore, in this case, $\mathscr{U}(\Omega)$ satisfies the MGE and AEG if ω satisfies the assumption in Proposition 4.5.

Proof of Theorem 2.1. (i) (a)$\Longleftrightarrow$(b) This follows from Propositions 4.1 and 4.5, since $\max\{s, \sigma\} \leq q_\# < p$.

(b)$\Longrightarrow$(c) This is easy.

(c)$\Longrightarrow$ (b) Observe from Lemma 3.11 that $\mathscr{U}(\Omega) + 1$ has the (τ, C_2, r_1)-submean value property with $\tau \in (0, p_*)$ by retaking C_2 large enough. Let $B(x, 25r) \subset \Omega$ and $0 < r < r_1/4$. Then, for all $y \in B(x, r)$, we have $B(y, 24r) \subset \Omega$ and $B(x, r) \subset B(y, 2r)$, and so Lemma 3.4 gives

$$\begin{aligned} u(y) + 1 &\leq C \left(\fint_{B(y,4r)} (u + 1)^\tau\, dz\right)^{\frac{1}{\tau}} \leq C \left\{\left(\fint_{B(y,4r)} u^\tau\, dz\right)^{\frac{1}{\tau}} + 1\right\} \\ &\leq C \left(\inf_{B(y,2r)} u + 1\right) \leq C \left(\inf_{B(x,r)} u + 1\right). \end{aligned}$$

Therefore,

$$\sup_{B(x,r)} (u + 1) \leq C \inf_{B(x,r)} (u + 1).$$

To extend this to the case $B(x, 6r) \subset \Omega$ and $0 < r < r_1$, we need only repeat the same argument as in the last paragraph of the proof of Proposition 4.1.

(ii) See Remark 4.6.

(iii) This follows from Proposition 4.5, the (p, C, r_1)-AEG and the submean value property. This completes the proof. □

Proof of Theorem 2.3. We can prove Theorem 2.3 in the same manner as the proof of Theorem 2.1. □

5. ADDITIONAL ESTIMATES

The following pointwise gradient estimate is an interesting consequence of Lemma 4.3 and the results by Duzaar–Mingione [12, 13], Kuusi–Mingione [18] and Nguyen–Phuc [22]. The Δ_p denotes the p-Laplacian on $\mathbb{R}^N$.

Corollary 5.1. *Assume $p > 2N/(N+1)$ and $b_3 = b_3' = b_4 = b_4' = 0$. Then there exist $C_8 > 0$, $C_9 \geq 0$ and $C > 0$, all depending only on C_3, b_i, b_i', p, q, N, such that*

$$|\nabla u(x)| \leq \frac{C_8 u(x) + C_9}{\min\{\delta_\Omega(x), 1\}} \leq C \min\{\delta_\Omega(x), 1\}^{-\frac{q+1}{q-p+1}} \quad \text{for all } x \in \Omega, \tag{5.1}$$

whenever $u \in C^1(\Omega)$ is a positive weak solution of

$$-\Delta_p u = \mathscr{B}(x, u, \nabla u) \quad \text{in } \Omega \tag{5.2}$$

satisfying the C_3-MGE (2.7). Moreover, the following statements hold:

(i) *If $b_5 = 0$, then we can take $C_9 = 0$ in (5.1).*
(ii) *If $b_2 = b_2' = b_5 = 0$ and u satisfies the C_3-GE (2.6), then $\min\{\delta_\Omega(x), 1\}$ can be replaced by $\delta_\Omega(x)$ in (5.1).*
(iii) *If $p > N$, then (5.1) holds for all positive weak solutions $u \in C^1(\Omega)$ of (5.2).*

Proof. Let $x \in \Omega$ and $r := \min\{\delta_\Omega(x), 1\}/6$. Observe from [12, 13, 18, 22] that there exist $C = C(p, N) > 0$ and $\gamma = \gamma(p, N) \in (0, 1]$ such that

$$|\nabla u(x)| \leq C \left\{ \left(\int_0^r \left(\frac{\mu_u(B(x,t))}{t^{N-1}} \right)^\gamma \frac{dt}{t} \right)^{\frac{1}{\gamma(p-1)}} + \fint_{B(x,r)} |\nabla u| \, dy \right\}, \tag{5.3}$$

where

$$\mu_u(B(x,t)) = \int_{B(x,t)} |\mathscr{B}(y, u, \nabla u)| \, dy$$

and $|\mathscr{B}(y, u, \nabla u)| \leq C(u^q + u^{p-1} + 1)$ in this corollary. By (2.7) and Lemma 3.4, we have for all $t \in (0, r]$,

$$\begin{aligned} \mu_u(B(x,t)) &\leq Ct^N \left(\min\{\delta_\Omega(x), 1\}^{-p} u(x)^{p-1} + 1 \right) \\ &\leq Ct^N \min\{\delta_\Omega(x), 1\}^{-p} \{u(x) + 1\}^{p-1}, \end{aligned}$$

and so

$$\begin{aligned} \left(\int_0^r \left(\frac{\mu_u(B(x,t))}{t^{N-1}} \right)^\gamma \frac{dt}{t} \right)^{\frac{1}{\gamma(p-1)}} &\leq Cr^{\frac{1}{p-1}} \min\{\delta_\Omega(x), 1\}^{-\frac{p}{p-1}} \{u(x) + 1\} \\ &\leq \frac{C}{\min\{\delta_\Omega(x), 1\}} \{u(x) + 1\}. \end{aligned} \tag{5.4}$$

By the way, since $p_* > 1$ if $p > 2N/(N+1)$, it follows from Lemma 4.3 that

$$\fint_{B(x,r)} |\nabla u| \, dy \le \frac{C}{\min\{\delta_\Omega(x), 1\}} \{u(x) + 1\}. \tag{5.5}$$

Substituting (5.4) and (5.5) into (5.3), we obtain the first inequality in (5.1). The second inequality in (5.1) follows immediately from (2.7). □

Corollary 5.2. *Assume $p \in (2N/(N+1), 2)$, $s < 1$ and $b_4' = 0$. Then the first assertion* (5.1) *of Corollary 5.1 holds, where the constants C_8, C_9 and C also depend on s. Moreover, the following statements hold:*

(i) *If $b_5 = 0$, then we can take $C_9 = 0$ in* (5.1).
(ii) *If $b_2 = b_2' = b_3 = b_3' = b_5 = 0$, $s = q_\# < 1$ and a positive weak solution $u \in C^1(\Omega)$ of* (5.2) *satisfies* (2.6), *then $\min\{\delta_\Omega(x), 1\}$ can be replaced by $\delta_\Omega(x)$ in* (5.1).

Proof. We use the same notation as in the proof of Corollary 5.1 and need to estimate the terms involving $|\nabla u|^{p-1}$ and $|\nabla u|^s$. Let $\tau \in \{p-1, s\}$. Then $0 < \tau < 1$. By Lemma 4.3 and (2.7), we have for all $t \in (0, r]$,

$$\begin{aligned} I(t) := \int_{B(x,t)} |\nabla u|^\tau \, dy &\le C t^{N-\tau} \{u(x) + 1\}^\tau \\ &\le C t^{N-\tau} \min\{\delta_\Omega(x), 1\}^{-\frac{p(\tau-p+1)}{q-p+1}} \{u(x) + 1\}^{p-1} \\ &\le C t^{N-\tau} \min\{\delta_\Omega(x), 1\}^{\tau-p} \{u(x) + 1\}^{p-1}, \end{aligned}$$

since $s \le q_\#$ (is always assumed) implies

$$\frac{p(\tau - p + 1)}{q - p + 1} \le p - \tau.$$

Therefore,

$$\begin{aligned} \left(\int_0^r \left(\frac{I(t)}{t^{N-1}} \right)^\gamma \frac{dt}{t} \right)^{\frac{1}{\gamma(p-1)}} &\le C r^{\frac{1-\tau}{p-1}} \min\{\delta_\Omega(x), 1\}^{\frac{\tau-p}{p-1}} \{u(x) + 1\} \\ &\le \frac{C}{\min\{\delta_\Omega(x), 1\}} \{u(x) + 1\}. \end{aligned}$$

This and the proof of Corollary 5.1 complete the proof. □

Remark 5.3. Let $p > 2N/(N+1)$. Thanks to Serrin–Zou [27, Theorem IV] and Poláčik–Quittner–Souplet [24], all positive weak solutions $u \in C^1(\Omega)$ of (5.2) with $q \in (p-1, p_*)$ or of equation $-\Delta_p u = b_1 u^q + b_2 u^{p-1}$ with $q \in (p-1, p^*)$ satisfy (2.7), and so they enjoy (5.1). For the equation, only the estimate $|\nabla u(x)| \le C \min\{\delta_\Omega(x), 1\}^{-\frac{q+1}{q-p+1}}$ is known (see [24]).

Acknowledgements

This work was supported in part by JSPS KAKENHI Grant Number JP23K03149. I would like to thank the referee for his/her comments which have helped improve this manuscript.

REFERENCES

[1] Y. Bai, Z. Zhang, Z. Zhang, *A Liouville-type theorem and one-dimensional symmetry of solutions for elliptic equations with general gradient nonlinearity*, J. Math. Anal. Appl. **537** (2024), 128286.

[2] L. Baldelli, R. Filippucci, *A priori estimates for elliptic problems via Liouville type theorems*, Discrete Contin. Dyn. Syst. Ser. S **13** (2020), 1883–1898.

[3] L. Baldelli, R. Filippucci, *Existence results for elliptic problems with gradient terms via a priori estimates*, Nonlinear Anal. **198** (2020), 111894.

[4] L. Baldelli, R. Filippucci, *A priori estimates for convective quasilinear equations and systems*, Rend. Istit. Mat. Univ. Trieste **57** (2025), Art. No. 16.

[5] M.F. Bidaut-Véron, *Liouville results and asymptotics of solutions of a quasilinear elliptic equation with supercritical source gradient term*, Adv. Nonlinear Stud. **21** (2021), 57–76.

[6] M.F. Bidaut-Véron, M. Garcia-Huidobro, L. Véron, *Local and global properties of solutions of quasilinear Hamilton-Jacobi equations*, J. Funct. Anal. **267** (2014), 3294–3331.

[7] M.F. Bidaut-Véron, M. García-Huidobro, L. Véron, *Estimates of solutions of elliptic equations with a source reaction term involving the product of the function and its gradient*, Duke Math. J. **168** (2019), 1487–1537.

[8] M.F. Bidaut-Véron, M. Garcia-Huidobro, L. Véron, *A priori estimates for elliptic equations with reaction terms involving the function and its gradient*, Math. Ann. **378** (2020), 13–56.

[9] C. Chang, B. Hu, Z. Zhang, *Liouville-type theorems and existence of solutions for quasilinear elliptic equations with nonlinear gradient terms*, Nonlinear Anal. **220** (2022), 112873.

[10] L. D'Ambrosio, E. Mitidieri, *A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities*, Adv. Math. **224** (2010), 967–1020.

[11] E.N. Dancer, *Superlinear problems on domains with holes of asymptotic shape and exterior problems*, Math. Z. **229** (1998), 475–491.

[12] F. Duzaar, G. Mingione, *Gradient estimates via linear and nonlinear potentials*, J. Funct. Anal. **259** (2010), 2961–2998.

[13] F. Duzaar, G. Mingione, *Gradient estimates via non-linear potentials*, Amer. J. Math. **133** (2011), 1093–1149.

[14] R. Filippucci, Y. Sun, Y. Zheng, *A priori estimates and Liouville type results for quasilinear elliptic equations involving gradient terms*, J. Anal. Math. **153** (2024), 367–400.

[15] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Dover Publications, Inc., Mineola, NY, 2006.

[16] T. Kilpeläinen, J. Malý, *Degenerate elliptic equations with measure data and nonlinear potentials*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **19** (1992), 591–613.

[17] T. Kilpeläinen, J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137–161.

[18] T. Kuusi, G. Mingione, *Linear potentials in nonlinear potential theory*, Arch. Ration. Mech. Anal. **207** (2013), 215–246.

[19] J. Malý, *Pointwise estimates of nonnegative subsolutions of quasilinear elliptic equations at irregular boundary points*, Comment. Math. Univ. Carolin. **37** (1996), 23–42.

[20] J. Malý, W.P. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Mathematical Surveys and Monographs, vol. 51, American Mathematical Society, Providence, RI, 1997.

[21] E. Mitidieri, *A view on Liouville theorems in PDEs*, Anal. Geom. Metr. Spaces **12** (2024), 20240008.

[22] Q. Nguyen, N. Phuc, *Pointwise gradient estimates for a class of singular quasilinear equations with measure data*, J. Funct. Anal. **278** (2020), 108391.

[23] M. Pavlović, *Inequalities for the gradient of eigenfunctions of the invariant Laplacian in the unit ball*, Indag. Math. (N.S.) **2** (1991), 89–98.

[24] P. Poláčik, P. Quittner, P. Souplet, *Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems*, Duke Math. J. **139** (2007), 555–579.

[25] D. Ruiz, *A priori estimates and existence of positive solutions for strongly nonlinear problems*, J. Differential Equations **199** (2004), 96–114.

[26] J. Serrin, *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), 247–302.

[27] J. Serrin, H. Zou, *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math. **189** (2002), 79–142.

[28] N.S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.

Kentaro Hirata
hiratake@hiroshima-u.ac.jp

Hiroshima University
Department of Mathematics
Graduate School of Advanced Science and Engineering
Higashi-Hiroshima 739-8526, Japan

Received: September 29, 2025.
Revised: March 25, 2026.
Accepted: March 27, 2026.
Published online: May 7, 2026.