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Dedicated to Professor Giuseppe Marino
on the occasion of his retirement

SOLUTIONS TO SECOND-ORDER NONLOCAL EVOLUTION EQUATIONS GOVERNED BY NON-AUTONOMOUS FORMS

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Abstract. Our main contributions include proving sufficient conditions for the existence of solution to a second order problem with nonzero nonlocal initial conditions, and providing a comprehensive analysis using fundamental solutions and fixed-point techniques. The theoretical results are illustrated through applications to partial differential equations, including vibrating viscoelastic membranes with time-dependent material properties and nonlocal memory effects.

Keywords: non-autonomous evolution equations, fundamental solution, nonlocal conditions, sesquilinear forms, fixed point.

Mathematics Subject Classification: 35L15, 35B65, 47H10, 35R09.

1. INTRODUCTION

Second-order evolution equations with time-dependent operators constitute an important class of problems within both theoretical and physical frameworks, particularly in modeling dynamic systems where the underlying medium properties vary over time.

Indeed, these equations naturally arise in describing vibrations and wave propagation phenomena in non-homogeneous media, where spatial differential operators exhibit temporal dependence due to changing material properties, boundary conditions, or external influences. For instance, the motion of a string or beam with time-varying stiffness, or a wave traveling through a non-homogeneous elastic medium, naturally leads to abstract wave equations of the form

$$\ddot{u}(t) + A(t)u(t) = f(t, u(t)),$$

where $u(t)$ represents the state of the system at time t, $A(t)$ is a time-dependent linear operator modeling the spatial part of the evolution, and $f(t, u(t))$ captures nonlinear effects and external forcing terms.

If the system is subject to time-dependent damping (e.g., due to friction or control feedback), the model includes a damping term $B(t)\dot{u}(t)$, yielding

$$\ddot{u}(t) + B(t)\dot{u}(t) + A(t)u(t) = f(t, u(t)).$$

The operator $A(t)$ typically represents spatial differential operators such as the Laplacian with variable coefficients, describing phenomena like heat conduction with temperature-dependent conductivity or elasticity problems with spatially varying material properties. The damping operator $B(t)$ models dissipative mechanisms that may themselves depend on time, such as viscous damping in fluid-structure interactions or feedback control systems with time-varying gains.

The term $f(t, u(t))$ represents a nonlinear source or forcing term, and the initial conditions

$$u(0) = g(u), \quad \dot{u}(0) = h(u),$$

are nonlocal in nature, meaning that the initial state depends on the entire solution trajectory over the time interval $[0, T]$. Such nonlocal conditions arise naturally in systems with memory effects, hereditary phenomena, or global constraints, and include as special cases multipoint conditions, integral average conditions, and periodic boundary conditions in time. These conditions can also be expressed as a dependency on the entire function u over the interval $[0, T]$. We point out that in the above cases, the operators $A(t)$ and $B(t)$ can be modeled via non-autonomous sesquilinear forms on a Hilbert space V densely embedded into a second one H. This functional analytic framework, first introduced by Lions [33], provides a natural setting for studying evolution equations with variable coefficients. The sesquilinear form approach allows us to handle operators that may not be densely defined in the classical sense, while still maintaining the essential spectral and regularity properties needed for the analysis.

More precisely, we assume that $A(t)$ and $B(t)$ are associated with sesquilinear forms $a : [0, T] \times V \times V \to \mathbb{C}$ and $b : [0, T] \times V \times V \to \mathbb{C}$, respectively, such that

$$a(t, u, v) = \langle A(t)u, v\rangle \quad \text{and} \quad b(t, u, v) = \langle B(t)u, v\rangle,$$

where $\langle \cdot, \cdot\rangle$ denotes the inner product in H and V, respectively. The forms are assumed to satisfy suitable conditions of boundedness (uniform control in the operator norm), coercivity (ensuring ellipticity and well-posedness), and temporal regularity (continuity or measurability in time).

As already mentioned, the study of non-autonomous evolution equations governed by sesquilinear forms dates back to the seminal work of J.-L. Lions [33], who introduced the concept of maximal regularity in the dual space V' for first-order problems. More precisely, Lions established that under appropriate assumptions on the sesquilinear form $a(t, \cdot, \cdot)$ – namely measurability in time, uniform boundedness, and coercivity – the first-order evolution equation admits a unique solution in the maximal regularity space, which provides optimal smoothness in both time and space directions. Lions showed that if $u_0 \in H$ and $f \in L^2(0, T; H)$, then under suitable assumptions on the form $a(t, \cdot, \cdot)$, the problem

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0$$

admits a unique solution $u \in MR(V, V') := H^1(0,T;V') \cap L^2(0,T;V)$. This result was groundbreaking as it established the existence of solutions with optimal regularity without requiring the operators to be autonomous or to have nice spectral properties. Further research, including [1, 4, 6, 10, 23, 24, 28, 38, 41], has extended this theory to L^p-maximal regularity for first-order problems using functional analytic and operator-theoretic techniques.

The extension to second-order evolution equations presents significantly greater challenges, due to the need of handling both position and velocity variables simultaneously. Second-order non-autonomous evolution equations, however, remain less explored. Chill and Srivastava [11, 20] and Arendt and Chill [7] investigated the L^p-maximal regularity for second-order Cauchy problems, typically under zero initial conditions. Their approach relied on reducing the second-order problem to a first-order system and applying vectorial maximal regularity theory, though this required additional structural assumptions on the operators. Dier and Ouhabaz [25] later established L^2-maximal regularity for damped wave equations with non-autonomous forms, using Lions' representation theorem. More recently, Achache [2] extended their result to arbitrary $p \in (1, \infty)$ and improved the treatment of regularity in H.

On the other hand, nonlocal initial conditions represent a rapidly growing area of research in evolution equations, motivated by their remarkable ability to model complex physical phenomena that exhibit memory effects, hereditary behavior, or spatial averaging constraints. Unlike classical initial value problems where the initial state is explicitly prescribed, nonlocal formulations define the initial datum through an implicit condition, such as

$$u(0) = g(u),$$

where g is an operator representing a functional dependence, e.g., on the history or spatial profile of u. This formulation is broad enough to include, as special cases, classical multipoint initial specifications, integral (average) conditions, and periodicity requirements. For instance, a multipoint condition can be written as

$$u(0,x) + g\big(t_1, \ldots, t_m; u(\cdot, x)\big) = u_0(x), \tag{1.1}$$

with $0 < t_1 < \ldots < t_m \leq T$, extending the standard Cauchy initial condition by incorporating the solution's values at intermediate times $t_1, \ldots, t_m$, as in the seminal work [18]. Since the above cited paper by Byszewski, a variety of existence results have been obtained over the years by different methods and under diverse hypotheses. Early works often assumed compactness or contraction conditions to deal with the nonlocal term. For instance, Boucherif and Precup [15] established the existence of mild solutions for a semilinear Cauchy problem with a multipoint initial condition, assuming the linear operator generates a compact semigroup. In a similar spirit, other authors studied mild and strong solutions under nonlocal conditions; for example, Paicu and Vrabie [39] investigated an abstract semilinear equation with an initial condition of type (1.1). We also mention that numerous contributions by Ntouyas and collaborators have expanded the theory of nonlocal Cauchy problems (see, for instance, Ntouyas [37] for a survey of various existence techniques). For recent advances on nonlocal problems, we refer to the works [3, 12–14, 18, 21, 32, 36, 42–44].

In this work, we investigate the existence of strong solutions through L^2-maximal regularity for the following semilinear, non-autonomous abstract wave equations:

$$\begin{cases} \ddot{u}(t) + A(t)u(t) = f(t, u(t)), & \text{for a.e. } t \in [0, T], \\ u(0) = g(u), \quad \dot{u}(0) = h(u), \end{cases} \tag{1.2}$$

and

$$\begin{cases} \ddot{u}(t) + B(t)\dot{u}(t) + A(t)u(t) = f(t, u(t)), & \text{for a.e. } t \in [0, T], \\ u(0) = g(u), \quad \dot{u}(0) = h(u), \end{cases}$$

where the operators $A(t)$ and $B(t)$ are associated with forms on V, satisfying coercivity, boundedness, and appropriate time regularity. The functions f, g, and h are assumed to satisfy some mild continuity assumptions and suitable growth conditions, such as transversality conditions, boundedness constraints, or sublinear growth properties, which will be specified later in the paper.

Our techniques will rely on the theory of fundamental solutions to non-autonomous second-order problems, as well as on Schauder-type fixed-point arguments and maximal L^2-regularity. The architecture of the paper is as follows. In Section 2, we introduce the functional setting, notation, and assumptions on the sesquilinear forms and nonlinearities. Section 3 is devoted to the analysis of undamped wave equations, where we establish the fundamental solution framework and prove our main existence result using finite-dimensional approximations and compactness arguments. In Section 4, we analyse the damped wave equations case.

Finally, in Section 5, we provide examples and applications to PDEs, such as non-autonomous wave equations with Robin boundary conditions and variable damping. In particular, we present a detailed analysis of a vibrating viscoelastic membrane problem, showing explicitly how each hypothesis of our main theorem is verified, and we introduce additional applications, including controlled wave systems and memory-dependent diffusion processes.

2. PRELIMINARIES

In this section, we establish the functional analytic framework for our study of second-order evolution equations with time-dependent operators. We introduce the function spaces, assumptions on the sesquilinear forms, and the fundamental solution theory that will be essential for our main results.

Throughout this paper, we denote by V a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_V$ and norm $\| \cdot \|_V$, and by H a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$. We assume that V is continuously embedded in H, i.e., there exists a constant $C > 0$ such that

$$\|u\|_H \leq C\|u\|_V \quad \text{for all } u \in V.$$

Furthermore, we assume that the embedding $V \hookrightarrow H$ is compact, which is essential for our compactness arguments and applications to partial differential equations.

We denote by V' the dual space of V and by $\mathcal{L}(X,Y)$ the space of bounded linear operators from a Banach space X to a Banach space Y. The space of bounded linear operators on X is denoted by $\mathcal{L}(X)$.

We now introduce the key assumptions on the sesquilinear forms that will govern our evolution equations. These conditions ensure the existence of strong solution through maximal regularity for the associated non-autonomous problems.

We assume that the following conditions hold for the sesquilinear form $a : [0,T] \times V \times V \to \mathbb{C}$:

(A_1) $a(\cdot,u,v) : [0,T] \to \mathbb{C}$ is strongly measurable for any $u, v \in V$,
(A_2) $a(t,\cdot,\cdot)$ is uniformly bounded, that is $|a(t,u,v)| \le C\|u\|_V\|v\|_V$ for $C \ge 0$, $t \in [0,T]$ and $u, v \in V$,
(A_3) $a(t,\cdot,\cdot)$ is coercive: Re $a(t,u,u) \ge \alpha\|u\|_V^2$ for $\alpha > 0, t \in [0,T]$ and $u \in V$,
(A_4) $|a(t,u,v) - a(s,u,v)| \le \omega(|t-s|)\|u\|_V\|v\|_V$, for some nondecreasing function $\omega : [0,T] \to [0,+\infty)$ which satisfies

$$\int_0^T t^{-3/2}\omega(t)dt < \infty \quad \text{and} \quad \int_0^T t^{-2}\omega(t)^2 dt < \infty.$$

In the foundational work of J.L. Lions [33], the concept of maximal regularity was introduced for evolution equations governed by non-autonomous sesquilinear forms. Specifically, Lions considered the abstract Cauchy problem

$$\begin{cases} \dot{u}(t) + \mathcal{A}(t)u(t) = f(t), & t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $\mathcal{A}(t) : V \to V'$ is the operator associated to $a(t,\cdot,\cdot)$ via

$$\langle \mathcal{A}(t)u, v\rangle_{V' \times V} = a(t,u,v), \quad \forall u, v \in V.$$

Lions proved that if $u_0 \in H$ and $f \in L^2(0,T;V')$, then under suitable assumptions on a, there exists a unique solution u in the so-called maximal regularity space

$$MR(V,V') := H^1(0,T;V') \cap L^2(0,T;V),$$

such that u solves (2.1) in the sense of distributions.

In this article, we adopt the framework (see [6, 8, 9]), where the realization of $A(t)$ in H is defined via the a sesquilinear form through:

$$\langle A(t)u, v\rangle_H = a(t,u,v), \quad u \in D(A(t)),\ v \in V,$$

with

$$D(A(t)) := \{u \in V : a(t,u,v) \text{ is } H\text{-valued for all } v \in V\}.$$

A central question, raised by Lions and further developed by Arendt and collaborators (see [6, 8, 9]), is under which conditions on the form a and the initial data u_0 the solution u actually belongs to the stronger space

$$MR(V,H) := H^1(0,T;H) \cap L^2(0,T;V).$$

This is particularly relevant for applications to boundary value problems.

We summarize the main function spaces and notations: $L^2(0,T;H)$ is the space of square-integrable functions on $[0,T]$ with values in H, $H^k(0,T;H)$ is the Sobolev space of H-valued functions with weak derivatives up to order k in $L^2(0,T;H)$, and $MR[0,T] := H^2(0,T;H) \cap H^1(0,T;V)$ is the maximal regularity space for second-order problems.

The *trace space* is defined as

$$Tr := \{(u(0), \dot{u}(0)) : u \in MR[0,T]\},$$

with the norm

$$\|(x,y)\|_{Tr} := \inf\{\|u\|_{MR[0,T]} : u(0) = x,\ \dot{u}(0) = y\}.$$

For further properties of these spaces, we refer to [20].

We also set

$$\Delta = \{(t,s) \in [0,T]^2 : s \le t\}.$$

A key assumption is the so-called square root property:

(A_5) $D(A(t)^{1/2}) = V$ for all $t \in [0,T]$,

which ensures that the domain of the square root of $A(t)$ coincides with V. Under this and related assumptions, one can obtain L^2-maximal regularity in H for (2.1). In particular, we cite the following result:

Theorem 2.1 ([38])**.** *Let a satisfy the conditions (A_1)–(A_4) and let $A(t)$ be the realization of $a(t,\cdot,\cdot)$ in H. Assume that the square root property $D(\sqrt{A(0)}) = V$ holds. Then for every $f \in L^2(0,T;H)$ and $u_0 \in V$, there exists a unique solution $u \in MR(V,H)$ to the problem (2.1).*

The extension of maximal regularity theory to second-order non-autonomous evolution equations is more subtle and less developed. For the second-order abstract Cauchy problem

$$\begin{cases} \ddot{u}(t) + B(t)\dot{u}(t) + A(t)u(t) = f(t), & t \in [0,T], \\ u(0) = u_0 \in V, \quad \dot{u}(0) = u_1 \in V, \end{cases} \tag{2.2}$$

where $A(t), B(t) \in \mathcal{L}(V,V')$ are associated to sesquilinear forms a and b as above, we introduce the following:

Definition 2.2 ([33])**.** For every $f \in L^2(0,T;H)$ and all (u_0,u_1) in the trace space Tr, we say that problem (2.2) has L^2-maximal regularity in H if there exists a unique $u \in MR[0,T]$ solving (2.2).

The existence of maximal regularity in H has been established under additional hypotheses (see [2, 7, 8, 11, 20, 25]). In particular, Ouhabaz and Dier [25] proved L^2-maximal regularity for (2.2) using Lions' representation theorem, while Achache [2] extended these results to L^p-maximal regularity for all $p \in (1,\infty)$.

To start our investigation, we need to introduce the concept of fundamental solutions for first and second-order non-autonomous Cauchy problems. In this context, consider the following homogeneous equations:

$$\begin{aligned} &\dot{u}(t) + A(t)u(t) = 0, \quad t \in [0,T], \\ &\ddot{u}(t) + A(t)u(t) = 0, \quad t \in [0,T]. \end{aligned} \tag{2.3}$$

Definition 2.3 ([40, Chapter 5])**.** The two parameter family of bounded linear operator $\{E(t,s)\}_{t,s\in\Delta}$ on H is called evolution family associated to $\{A(t)\}_{t\in[0,T]}$ if it satisfies the following properties:

(i) $E(t,t) = I$, $E(t,s) = E(t,r)E(r,s)$ for all $0 \le s \le r \le t \le T$,
(ii) the mapping $\Delta \ni (t,s) \mapsto E(t,s)$ is strongly continuous for all $0 \le s \le t \le T$.

If $E(t,s)$ is the evolution family associated to $A(t)$, then we can express the solution of the inhomogeneous problem

$$\begin{cases} \dot{u}(t) + A(t)u(t) = f(t), & t \in [0,T], \\ u(0) = u_0 \in V \end{cases}$$

by

$$u(t) = E(t,0)u_0 + \int_0^t E(t,s)f(s)ds.$$

Definition 2.4 ([17, 30])**.** A fundamental solution to (2.3) associated with $A(t)$ is a family of bounded linear operators $\{\mathcal{S}(t,s)\}_{(t,s)\in\Delta}$ on H satisfying the following conditions:

(S1) (a) $\mathcal{S}(t,t) = 0$ for all $t \in [0,T]$.
(b) The mapping $\Delta \ni (t,s) \mapsto \mathcal{S}(t,s)$ is strongly continuous on H.
(c) For all $x \in H$ and $s \in [0,T]$, the mapping $[s,T] \ni t \mapsto \mathcal{S}(t,s)x$ is continuously differentiable, and $(t,s) \mapsto \frac{\partial}{\partial t}\mathcal{S}(t,s)x$ is continuous with $\frac{\partial}{\partial t}\mathcal{S}(t,s)x\big|_{t=s} = x$.
(d) For all $x \in D(A(t))$ and $t \in [0,T]$, the mapping $[0,t] \ni s \mapsto \mathcal{S}(t,s)x$ is continuously differentiable, and $(t,s) \mapsto \frac{\partial}{\partial s}\mathcal{S}(t,s)x$ is continuous with $\frac{\partial}{\partial s}\mathcal{S}(t,s)x\big|_{t=s} = -x$.

(S2) $\mathcal{S}(t,s)D(A(t)) \subseteq D(A(t))$ for all $(t,s) \in \Delta$. For $x \in D(A(t))$, the mapping $\Delta \ni (t,s) \mapsto \mathcal{S}(t,s)x$ is twice continuously differentiable, and:
(a) $\frac{\partial^2}{\partial t^2}\mathcal{S}(t,s)x = -A(t)\mathcal{S}(t,s)x$,
(b) $\frac{\partial^2}{\partial s^2}\mathcal{S}(t,s)x = -\mathcal{S}(t,s)A(s)x$,
(c) $\frac{\partial^2}{\partial t\partial s}\mathcal{S}(t,s)x\big|_{t=s} = 0$.

(S3) For all $(t,s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}\mathcal{S}(t,s)x \in D(A(t))$, and the second derivatives $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}\mathcal{S}(t,s)x$ and $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}\mathcal{S}(t,s)x$ exist. The following properties hold:
(a) $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}\mathcal{S}(t,s)x = -A(t)\frac{\partial}{\partial s}\mathcal{S}(t,s)x$,
(b) $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}\mathcal{S}(t,s)x = -\frac{\partial}{\partial t}\mathcal{S}(t,s)A(s)x$,
(c) The mapping $\Delta \ni (t,s) \mapsto A(t)\frac{\partial}{\partial s}\mathcal{S}(t,s)x$ is continuous.

Proposition 2.5 ([17, Lemma 2.10]). *Let $(\mathcal{S}(t,s))_{(t,s)\in\Delta}$ be a fundamental solution of* (2.3) *in H. Then $(\mathcal{S}(t,s))_{(t,s)\in\Delta}$ and $\left(\frac{\partial}{\partial t}\mathcal{S}(t,s)\right)_{(t,s)\in\Delta}$ are bounded in $\mathcal{L}(H)$.*

For simplicity, we introduce the operator $\mathcal{C}(t,s) : H \to H$ defined by

$$\mathcal{C}(t,s) = -\frac{\partial}{\partial s}\mathcal{S}(t,s).$$

We call a fundamental solution $(\mathcal{S}(t,s))_{(t,s)\in\Delta}$ evolutionary if additionally:

(S4) For all $(t,s),(s,r) \in \Delta$ and $x \in D(A(t))$, we have

$$\mathcal{C}(t,s)\mathcal{S}(s,r)x + \mathcal{S}(t,s)\frac{\partial}{\partial s}\mathcal{S}(s,r)x = \mathcal{S}(t,r)x.$$

Whenever the families $\{\mathcal{S}(t,s)\}_{(t,s)\in\Delta}$ and $\{\mathcal{C}(t,s)\}_{(t,s)\in\Delta}$ are uniformly bounded, the following inequalities hold for some constants C_1, M_1 (see [19]):

(S0) $\|\mathcal{S}(t_1,s) - \mathcal{S}(t_2,s)\|_{\mathcal{L}(H)} \le M_1|t_1 - t_2|, \quad \forall (t_1,s),(t_2,s) \in \Delta,$

(C0) $\|\mathcal{C}(t_1,s) - \mathcal{C}(t_2,s)\|_{\mathcal{L}(H)} \le C_1|t_1 - t_2|, \quad \forall (t_1,s),(t_2,s) \in \Delta.$

Now we recall the following definitions.

Definition 2.6. A function $u : [0,T] \to H$ is called a strong solution of (2.3) if u is twice differentiable a.e., $u(t) \in D(A(t))$ and satisfies (2.3) with $u(0) = u_0$, $\dot{u}(0) = u_1$.

If $\mathcal{S}(t,s)$ is the fundamental solution to (2.3) then according to Kozak [30] the solution of

$$\begin{cases} \ddot{u}(t) + A(t)u(t) = f(t), & t \in [0,T], \\ u(0) = u_0, \quad \dot{u}(0) = u_1 \end{cases}$$

is given by

$$u(t) = \mathcal{C}(t,0)u_0 + \mathcal{S}(t,0)u_1 + \int_0^t \mathcal{S}(t,s)f(s)ds.$$

Suppose that $D(A(t)) = D$ for all $t \in [0,T]$ and set $v = \dot{u}$ in (2.3). We get

$$\begin{cases} \dot{U}(t) + \mathbb{A}(t)U(t) = F(t), \\ U(0) = U_0, \end{cases} \tag{2.4}$$

where

$$\mathbb{A}(t) := \begin{pmatrix} 0 & -I \\ A(t) & 0 \end{pmatrix},\ \mathcal{D}(\mathbb{A}(t)) := \mathcal{D} := D \times V,\ U(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix},\ F(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Most recently, C. Budde and C. Seifert proved the following result.

Proposition 2.7 ([17]). *The following assertions are equivalent:*

(a) *there exists an evolutionary fundamental solution* $(\mathcal{S}(t,s))_{(t,s)\in\Delta}$ *on* H *of* (2.3) *associated to* $\{A(t)\}_{t\in[0,T]}$ *such that for all* $(t,s)\in\Delta$ *we have:*
   (i) $\mathcal{S}(t,s)H\subseteq V$, $\mathcal{S}(t,s)V\subseteq D(A(t))$, *the mapping* $(t,s)\mapsto\mathcal{S}(t,s)x\in V$ *is continuous for all* $x\in H$,
   (ii) $\frac{\partial}{\partial t}\mathcal{S}(t,s)V\subseteq V$, $\frac{\partial^2}{\partial t^2}\mathcal{S}(t,s)x$ *exists for all* $x\in V$, $\frac{\partial^2}{\partial t^2}\mathcal{S}(t,s)x=A(t)\mathcal{S}(t,s)x$,
   (iii) $\frac{\partial}{\partial s}\mathcal{S}(t,s)x$ *exists for all* $x\in V$, $\frac{\partial}{\partial s}\mathcal{S}(t,s)V\subseteq V$, *the mapping* $(t,s)\mapsto\frac{\partial}{\partial s}\mathcal{S}(t,s)x\in V$ *is continuous for all* $x\in V$,
   (iv) $\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathcal{S}(t,s)D(A(t))\subseteq V$, $\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathcal{S}(t,s)x$ *exists for all* $x\in V$ *and there exists* $C\geq 0$ *such that*

$$\left\|\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathcal{S}(t,s)x\right\|_V\leq C\|x\|_V\quad\text{for all } x\in V \text{ and } (t,s)\in\Delta,$$

(b) *there exists an evolution system* $(E(t,s))_{(t,s)\in\Delta}$ *on* $\mathcal{L}(\mathcal{Z}=V\times H)$ *of* (2.4) *associated to* $\{\mathbb{A}(t)\}_{t\in[0,T]}$.

Thanks to the above result, we can express the evolution family in the form of a fundamental solution and vice versa. In Section 3, we used the above Proposition to prove our preparatory lemmas.

In 2008, R. Chill and S. Srivastava connected the L^p-maximal regularity of the first and second-order Cauchy problems with zero initial data, by the following result.

Theorem 2.8 ([11]). *Assume that* $B(t), A(t)$ *are strongly measurable for all* $t\in[0,T]$, *and there exists* $h\in L^2(0,T)$ *such that* $\|A(t)\|\leq h(t)$ *for almost every* t. *Then the following hold:*

(a) *if the first order Cauchy problem*

$$\dot{u}+B(t)u=f\quad(t\in[a,b]),\quad u(a)=0$$

has L^2*-maximal regularity for each subinterval* (a,b) *of* $(0,T)$, *then the second order problem* (2.3) *with* $u(0)=\dot{u}(0)=0$ *has* L^2*-maximal regularity,*

(b) *if the second order problem*

$$\ddot{u}+B(t)\dot{u}+A(t)u=f\quad(t\in[a,b]),\quad u(a)=\dot{u}(a)=0.$$

has L^2*-maximal regularity for each subinterval* (a,b) *of* $(0,T)$, *then the first-order problem* $\dot{u}+B(t)u=f$, $u(0)=0$ *has* L^2*-maximal regularity.*

We extend this result to the case of nonzero initial conditions in Section 4.

Let $f:[0,T]\times X\to Y$ be a mapping, with X and Y being Banach spaces, and $N_f:L^p([0,T],X)\to L^q([0,T],Y)$ be the superposition operator defined by

$$N_f(u)(t):=f(t,u(t)).$$

We recall the following classical results, which will be used to prove our main result.

Theorem 2.9 ([34])**.** *If X and Y are separable and f is measurable in $[0,T]\times X$, then $N_f : L^p([0,T],X) \to L^q([0,T],Y)$ is well-defined if and only if there exists a constant $a > 0$ and a function $b \in L^q([0,T],\mathbb{R}_+)$ such that*

$$\|f(t,x)\|_Y \leq a\|x\|_X^{p/q} + b(t).$$

Moreover, N_f maps bounded subsets into bounded subsets.

Theorem 2.10 ([31])**.** *Let X be a Banach space, and $T : X \to X$ be a continuous and compact operator such that the set*

$$\{x \in X : x = \lambda T(x) \text{ for some } 0 \leq \lambda \leq 1\},$$

is bounded. Then T has a fixed point in X.

Theorem 2.11 (Aubin–Lions Lemma, [16])**.** *Let X_0, X, and X_1 be three Banach spaces. Suppose that X_0 is compactly embedded in X and X is continuously embedded in X_1. Suppose also that X_0 and X_1 are reflexive. Then for $0 < T < +\infty$ and $1 < r, s < \infty$, we have that $L^r([0,T],X_0) \cap W^{1,s}([0,T],X_1)$ is compactly embedded in $L^r([0,T],X)$.*

Definition 2.12 ([21])**.** Given two Banach spaces X and Y, a function $F : X \to Y$ is called *demicontinuous* if, for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ that strongly converges to $x \in X$, the sequence $\{F(x_n)\}_{n\in\mathbb{N}}$ weakly converges to $F(x)$. In other words, this means that

$$w - \lim_{n\to\infty} F(x_n) = F(x) \quad \text{whenever } x_n \to x.$$

Proposition 2.13 ([21])**.** *Suppose that $f : [0,T] \times H \to H$ satisfies:*

$(F1)$ *$f(\cdot, x)$ is measurable for any $x \in H$,*
$(F2)$ *$f(t, \cdot)$ is demicontinuous in H for any fixed $t \in [0,T]$,*
$(F3)$ *there exist $a > 0$ and $b \in L^2([0,T],\mathbb{R}_+)$ such that*

$$\|f(t,x)\|_H \leq a\|x\|_H + b(t).$$

Then the superposition operator $N_f : L^2([0,T],H) \to L^2([0,T],H)$ given by

$$N_f(u)(t) := f(t,u(t))$$

is well-defined and maps bounded sets into bounded sets; moreover, it is demicontinuous.

3. ABSTRACT WAVE EQUATION

In this section, we study the existence of solution through the L^2-maximal regularity of abstract wave equation (1.2). We consider $V = H^1(\Omega)$ and $H = L^2(\Omega)$, where Ω is a bounded domain with Lipschitz boundary. In the sequel, we will also denote $MR[0,T]$ by $MR[s,T]$. We first prove the following preparatory lemmas.

Lemma 3.1. *Assume that the equation*

$$\begin{cases} \ddot{u}(t) + A(t)u(t) = f(t), & t \in [0,T], \\ u(0) = 0, \ \dot{u}(0) = 0 \end{cases}$$

has L^2-maximal regularity for every $0 < t \le T$. Then for every $(x,y) \in Tr$ and every $s \in [0,T]$, there exists at least one $u \in MR[s,T]$ such that

$$\begin{cases} \ddot{u} + A(t)u = 0 & a.e. \ on \ [s,T], \\ u(s) = x, \ \dot{u}(s) = y. \end{cases} \tag{3.1}$$

Proof. For existence, let $w \in MR[0,T]$ be such that $w(0) = x, \dot{w}(0) = y$. Let $w_s(t) := w(t-s)$ for $t \in [s,T]$ and define

$$f_s(t) = \begin{cases} 0 & \text{if } 0 \le t < s, \\ -\ddot{w}_s(t) - A(t)w_s(t) & \text{if } s \le t \le T. \end{cases}$$

Let $v_s \in MR[0,T]$ be a solution of

$$\begin{cases} \ddot{v}_s + A(t)v_s = f_s & \text{a.e. on } [0,T], \\ v_s(0) = 0, \ \dot{v}_s(0) = 0, \end{cases}$$

and set $u_s(t) := v_s(t) + w_s(t)$ for $t \in [s,T]$. Note that $v_s(s) = 0$, because $A(t)$ has L^2-maximal regularity for every $t \in [0,s]$ and $f_s = 0$ on $[0,s]$. Thus, u_s solves (3.1). □

Lemma 3.2. *Assume that (A_1)–(A_5) hold, and let $\{\mathcal{S}(t,s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(H)$ be the fundamental solution to* (2.3) *generated by $A(t)$. Then*

$$u(t) := \mathcal{C}(t,0)u_0 + \mathcal{S}(t,0)u_1$$

belong to $MR[0,T]$ and solve the homogeneous Cauchy problem

$$\ddot{u}(t) + A(t)u(t) = 0 \quad a.e. \ on \ [0,T], \quad u(0) = u_0, \quad \dot{u}(0) = u_1. \tag{3.2}$$

Proof. By property (S1), we have $\mathcal{C}(0,0) = I$ and $\mathcal{S}(0,0) = 0$. Hence,

$$u(0) = \mathcal{C}(0,0)u_0 + \mathcal{S}(0,0)u_1 = (u_0) + 0u_1 = u_0.$$

Similarly, by (S1) and (S2), we obtain

$$\dot{u}(0) = \frac{\partial}{\partial t}\mathcal{C}(0,0)u_0 + \frac{\partial}{\partial t}\mathcal{S}(0,0)u_1 = 0u_0 + Iu_1 = u_1.$$

Next, using (S2)(a) and (S3)(b), we compute

$$\begin{aligned} \ddot{u}(t) + A(t)u(t) &= A(t)\frac{\partial}{\partial s}\mathcal{S}(t,0)u_0 - A(t)\mathcal{S}(t,0)u_1 \\ &\quad - A(t)\frac{\partial}{\partial s}\mathcal{S}(t,0)u_0 + A(t)\mathcal{S}(t,0)u_1 = 0. \end{aligned}$$

By Proposition 2.7, the fundamental solution ensures that

$$u(t) \in H^2([0,T],H) \cap H^1([0,T],V).$$

Setting $\dot{u} = v$ in (3.2), we get

$$\dot{U}(t) + \mathbb{A}(t)U(t) = 0, \quad U(0) = U_0.$$

Assume that $\mathbb{A}(t)$ has L^2-maximal regularity and $\{E(t,s)\}_{(t,s)}$ be an evolution family associated to $\mathbb{A}(t)$. Then, by Proposition 2.7, there exists a fundamental solution $\mathcal{S}(t,s)$ of (3.2) such that

$$\begin{aligned} u(t) = \pi_1 E(t,s)\begin{pmatrix} x \\ y \end{pmatrix} &= \pi_1 \begin{pmatrix} \mathcal{C}(t,s) & \mathcal{S}(t,s) \\ \frac{\partial}{\partial t}\mathcal{C}(t,s) & \frac{\partial}{\partial t}\mathcal{S}(t,s) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \pi_1 \begin{pmatrix} \mathcal{C}(t,s)x + \mathcal{S}(t,s)y \\ \frac{\partial}{\partial t}\mathcal{C}(t,s)x + \frac{\partial}{\partial t}\mathcal{S}(t,s)y \end{pmatrix} = \mathcal{C}(t,0)x + \mathcal{S}(t,0)y. \end{aligned}$$

Here, π_1 is the projection on the first element. By [7, Proposition 2.3], if $(x,y) \in Tr$, then $E(t,s)\begin{pmatrix} x \\ y \end{pmatrix} \in Tr$, which implies that

$$E(t,s)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathcal{C}(t,s)x + \mathcal{S}(t,s)y \\ \frac{\partial}{\partial t}\mathcal{C}(t,s)x + \frac{\partial}{\partial t}\mathcal{S}(t,s)y \end{pmatrix} = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} \in Tr.$$

Hence, $u \in MR[0,T]$. □

Lemma 3.3. *Assume that $A(t)$ has L^2-maximal regularity for every $t \in [0,T]$ and for every $f \in L^2(0,T;Tr)$. Then a solution u of the inhomogeneous problem*

$$\ddot{u}(t) + A(t)u(t) = f(t), \quad u(0) = \dot{u}(0) = 0 \tag{3.3}$$

is given by

$$u(t) = \int_0^t \mathcal{S}(t,s)f(s)\,ds.$$

Proof. By Lemma 3.1, if $(0,x) \in Tr$, then $\mathcal{S}(t,s)x \in L^2(\Delta,V)$ for all $(t,s) \in \Delta$. Then for every $f \in L^2([0,T],Tr)$, the function $(t,s) \to \mathcal{S}(t,s)f(s) \in L^2(\Delta,V)$. Set $v = \dot{u}$, we get

$$\dot{U}(t) + \mathbb{A}(t)U(t) = F(t), \quad U(0) = 0, \tag{3.4}$$

where $U(t) = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$, $F(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$ and $\mathbb{A}(t) = \begin{pmatrix} 0 & -I \\ A(t) & 0 \end{pmatrix}$.

By [7, Proposition 2.4] $U(t) = \int_0^t E(t,s)F(s)ds$ is a solution of (3.4). Hence, $u(t) = \int_0^t \mathcal{S}(t,s)f(s)$ is a solution of (3.3). □

Now we assume that $\{\Psi_m\}_{m\in\mathbb{N}}$ forms a Schauder basis for V, which is also a basis for H since V is densely embedded into H. For $n \in \mathbb{N}$, we denote by

$$\mathcal{P}_n : H \to \mathrm{span}_{\mathbb{C}}\{\Psi_1, \ldots, \Psi_n\}$$

the canonical projection, and the set $H_n := \mathcal{P}_n H$, $V_n := \mathcal{P}_n V$ endowed with the norms of H and V, respectively. From [21], we know that $\mathcal{P}_n$ is self-adjoint.

Given a sesquilinear form $a : [0,T] \times V \times V \to \mathbb{C}$ satisfying (A_1)–(A_4), we define its approximation

$$a_n(t,u,v) := a(t, \mathcal{P}_n u, \mathcal{P}_n v) + \alpha\langle (I-\mathcal{P}_n)u, (I-\mathcal{P}_n)v\rangle_V,$$

which also enjoys properties (A_1)–(A_4) (see [21, Remark 3.1]).

For each n, we associate an operator $A_n(t)$ with a_n by

$$a_n(t,u,v) = \langle A_n(t)u, v\rangle_H, \quad u,v \in V.$$

As for the domain of $(A_n(t))$, we consider

$$D(A_n(t)) = \{v \in V : A_n(t)v \in H\}.$$

It follows from [21, Remark 3.2] that

$$A_n(t) = \mathcal{P}_n A(t)\mathcal{P}_n + \alpha(I-\mathcal{P}_n)G(I-\mathcal{P}_n),$$

where $G : V \to H$ is associated with the inner product of V. In particular, for $u \in V_n$ one has $A_n(t)u = \mathcal{P}_n A(t)u$. We denote by $\mathcal{S}_n(t,s)$ the fundamental solution generated by $A_n(t)$ and $-\frac{\partial}{\partial_s}\mathcal{S}_n(t,s) = \mathcal{C}_n(t,s)$.

Lemma 3.4. *Let $\{\mathcal{S}_n(t,s)\}_{(t,s)\in\Delta} \subseteq \mathcal{L}(V)$ and $\{\mathcal{C}_n(t,s)\}_{(t,s)\in\Delta} \subseteq \mathcal{L}(V)$ be defined as above. Then, for any $u_0, u_1 \in V$, the solution of the homogeneous problem*

$$\begin{cases} \ddot{u}(t) + A_n(t)u(t) = 0 \quad a.e.\ t \in [0,T], \\ u(0) = u_0 \in V, \\ \dot{u}(0) = u_1 \in V \end{cases}$$

is given by

$$u(t) := \mathcal{C}_n(t,0)u_0 + \mathcal{S}_n(t,0)u_1 \in MR[0,\, T].$$

Furthermore, $u(t) \in V_n$ for any $t \in [0,T]$, if $u_0, u_1 \in V_n$.

Proof. We know that $A_n(t)$ satisfies (A_1)–(A_4). If $A_n(t)$ satisfies the square root property, then the existence of the fundamental solution directly follows from Lemma 3.2. The operator $A_n(t)$ generates a cosine function on H for fixed $t \in [0,T]$, since the first part $\mathcal{P}_n A(t)\mathcal{P}_n$ of $A_n(t)$ is bounded and the second part $\alpha(I-\mathcal{P}_n)G(I-\mathcal{P}_n)$ is symmetric.

The numerical range

$$W(A_n(t)) := \{\langle A_n(t)u, u\rangle_H \mid u \in V, \|u\|_H = 1\}$$

is therefore contained in a parabola. Finally, by [35, Theorems A and C], it follows that the square root property $\mathcal{D}(A_n(t)^{1/2}) = V$ holds. This completes the proof of the first part.

Now, by Proposition 2.7, there exists an evolution system $E_n(t,s)$ associated with $\mathbb{A}_n(t)$ such that if $x = (u_0, u_1) \in Z_n = V_n \times V_n$, then $E_n(t,s)x \in Z_n$ by [21, Lemma 3.3]. This implies that

$$u(t) = \pi_1 E_n(t,s)x := \mathcal{C}_n(t,s)u_0 + \mathcal{S}_n(t,s)u_1 \in V_n. \qquad \square$$

Lemma 3.5. *Suppose that $\{A(t)\}_{t\in[0,T]}$ and $\{A_n(t)\}_{t\in[0,T]}$ generate the families of fundamental solutions $\{\mathcal{S}(t,s)\}_{(t,s)\in\Delta}$ and $\{\mathcal{S}_n(t,s)\}_{(t,s)\in\Delta}$, respectively. Then, $\{\mathcal{S}_n(t,s)\mathcal{P}_n y\}_{n\in\mathbb{N}}$ converges uniformly on $t > s \in [0,T]$ to $\mathcal{S}(t,s)y$ in H for any fixed $y \in H$.*

Proof. Suppose that the fundamental solutions $\{\mathcal{S}(t,s)\}_{(t,s)\in\Delta}$, $\{\mathcal{S}_n(t,s)\}_{(t,s)\in\Delta}$ satisfy the conditions (i)–(iv) of Proposition (2.7)(a). Then there exists evolution systems $\{E(t,s)\}_{(t,s)\in\Delta}$ and $\{E_n(t,s)\}_{(t,s)\in\Delta}$ associated to $\mathbb{A}(t)$ and $\mathbb{A}_n(t)$, respectively. Then for any fixed $(0,y) \in V \times H$, the sequence $\left\{E_n(t,s)\binom{0}{\mathcal{P}_n y}\right\}_{n\in\mathbb{N}}$ converges uniformly to $E(t,s)y$ for $s < t$ in $[0,T]$. Then, by continuity of projection map π_1, it implies that $\left\{\pi_1 E_n(t,s)\binom{0}{\mathcal{P}_n y}\right\}_{n\in\mathbb{N}}$ converges to $\pi_1 E(t,s)y$. This proves that $\{\mathcal{S}_n(t,s)\mathcal{P}_n y\}_{n\in\mathbb{N}}$ converges in H to $\mathcal{S}(t,s)y$ uniformly on $t > s$ in $[0,T]$. The same is true for $\mathcal{C}(t,s)$. $\square$

Remark 3.6. The preceding Lemma remains true if $\mathcal{S}_n(t,s)$ and $\mathcal{S}(t,s)$ are replaced by their adjoint. Indeed, the following identity holds for the adjoint operators:

$$\mathcal{S}(t,s)^*x = \mathcal{S}_r(T-s, T-t)x, \quad \text{for any fixed } x \in H \text{ and } (t,s) \in \Delta.$$

where $\{\mathcal{S}_r(t,s)\}$ denotes the fundamental system corresponding to the operator $A_r(t)$ associated to the returned adjoint form $a_r^*(t,u,v) = \overline{a(T-t,v,u)}$, which also satisfies the properties (A_1)–(A_4). For further details, we refer the reader to [21, 22, 27].

Theorem 3.7. *Suppose that the following hypothesis holds:*

(i) *$V = H^1(\Omega)$ and $H = L^2(\Omega)$ are Hilbert spaces such that the embedding $V \hookrightarrow H$ is dense and compact, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary,*
(ii) *$\{A(t)\}_{t\in[0,T]}$ satisfies (A_1)–(A_5), which is associated with a non-autonomous sesquilinear form a,*
(iii) *the mapping $f : [0,T] \times H \to H$ satisfies the assumptions $(F1)$–$(F3)$ of Proposition 2.13,*
(iv) *$g, h : L^2([0,T]; H) \to H$ are demicontinuous and compact. Moreover, $\|g(u)\|_H \le r_1$, $\|h(u)\|_H \le r_2$ for every $u \in L^2([0,T], H)$.*

Then the problem

$$\begin{cases} \ddot{u}(t) + A(t)u(t) = f(t,u(t)), & for \;\; a.e \;\; t \in [0,T], \\ u(0) = g(u), \quad \dot{u}(0) = h(u) \end{cases} \tag{3.5}$$

admits at least one solution $u \in H^2([0,T], H) \cap H^1([0,T], V)$.

Proof. Let $n \in \mathbb{N}$ and for fixed $w \in L^2([0,T], H_n)$, the linear problem:

$$\begin{cases} \ddot{u}(t) + A_n(t)u(t) = \mathcal{P}_n f(t, w(t)), \\ u(0) = \mathcal{P}_n g(w), \\ \dot{u}(0) = \mathcal{P}_n h(w) \end{cases}$$

admits at least one solution in $H^2([0,T]; H_n) \cap H^1([0,T]; V)$, which can be represented by

$$u(t) = \mathcal{C}_n(t,0)\mathcal{P}_n g(w) + \mathcal{S}_n(t,0)\mathcal{P}_n h(w) + \int_0^t \mathcal{S}_n(t,s)\mathcal{P}_n N_f(w(s))ds,$$

by Lemmas 3.2, 3.3, 3.4 and Proposition 2.13, where $\mathcal{S}_n(t,s)$ is the fundamental solution associated to $A_n(t)$ and $N_f(w(t)) = f(t, w(t))$.

Now, consider the mapping

$$\mathcal{T} : C([0,T]; H_n) \longrightarrow C([0,T]; H_n),$$

defined by

$$\mathcal{T}(w)(t) := \mathcal{C}_n(t,0)\, \mathcal{P}_n g(w) + \mathcal{S}_n(t,0)\, \mathcal{P}_n h(w) + \int_0^t \mathcal{S}_n(t,s)\, \mathcal{P}_n f(s, w(s))\, ds.$$

To prove $\mathcal{T}$ is continuous, let $\{w_k\}_{k\in\mathbb{N}}$ be a sequence in $C([0,T]; H_n)$, such that $w_k \to w_0$ in $C([0,T]; H_n)$.

By the uniform boundedness of $\mathcal{S}_n(t,s)$, $\mathcal{C}_n(t,s)$, we have

$$\begin{aligned} &\|\mathcal{S}_n(t,0)\mathcal{P}_n h(w_k) - \mathcal{S}_n(t,0)\mathcal{P}_n h(w_0)\|_H \\ &\leq \|\mathcal{S}_n(t,0)\|_{\mathcal{L}(H)} \|\mathcal{P}_n(h(w_k) - \mathcal{P}_n h(w_0))\|_H \end{aligned}$$

and

$$\begin{aligned} &\|\mathcal{C}_n(t,0)\mathcal{P}_n g(w_k) - \mathcal{C}_n(t,0)\mathcal{P}_n g(w_0)\|_H \\ &\leq \|\mathcal{C}_n(t,0)\|_{\mathcal{L}(H)} \|\mathcal{P}_n g(w_k) - \mathcal{P}_n g(w_0))\|_H. \end{aligned}$$

This implies that

$$\|\mathcal{S}_n(\cdot,0)\mathcal{P}_n h(w_k) - \mathcal{S}_n(\cdot,0)\mathcal{P}_n h(w_0)\|_{C([0,T],H_n)} \leq M_1 \|\mathcal{P}_n h(w_k) - \mathcal{P}_n h(w_0)\|_H.$$

and

$$\|\mathcal{C}_n(\cdot,0)\mathcal{P}_n g(w_k) - \mathcal{C}_n(\cdot,0)\mathcal{P}_n g(w_0)\|_{C([0,T],H_n)} \leq M_1 \|\mathcal{P}_n g(w_k) - \mathcal{P}_n g(w_0)\|_H.$$

Since $\mathcal{P}_n$ is weak-to-strong continuous on bounded sequences, while g and h are demicontinuous, we derive that $\mathcal{P}_n g(w_k) \to \mathcal{P}_n g(w_0)$ and $\mathcal{P}_n h(w_k) \to \mathcal{P}_n h(w_0)$ as $k \to \infty$. We have then proved that

$$\|\mathcal{C}_n(\cdot,0)\mathcal{P}_n g(w_k) - \mathcal{C}_n(\cdot,0)\mathcal{P}_n g(w_0)\|_{C([0,T],H_n)} \to 0 \text{ as } k \to 0$$

and

$$\|\mathcal{S}_n(\cdot,0)\mathcal{P}_n h(w_k) - \mathcal{S}_n(\cdot,0)\mathcal{P}_n h(w_0)\|_{C([0,T],H_n)} \to 0 \text{ as } k \to 0.$$

Observe that $\mathcal{P}_n$ is of finite rank and hence weak-to-strong continuous, while N_f is demicontinuous. Therefore, $\mathcal{P}_n N_f(w_k)(s) \to \mathcal{P}_n N_f(w_0)(s)$ as $k \to \infty$ and it is dominated by $a\|w_0(s)\|_H + \varepsilon_0 + b(s)$ for a.e. $t \in [0,T]$. Then, by the dominated convergence theorem, we have

$$\int_0^{\cdot} \mathcal{S}_n(\cdot, s)\mathcal{P}_n N_f(w_k)(s)ds \to \int_0^{\cdot} \mathcal{S}_n(\cdot, s)\mathcal{P}_n N_f(w_0)(s)ds \tag{3.6}$$

in $C([0,T],H)$ as $k \to \infty$, and

$$\mathcal{T}(w_k) \to \mathcal{T}(w_0) \text{ in } C([0,T],H_n) \text{ as } k \to \infty.$$

This implies that $\mathcal{T}$ is continuous.

Now, let

$$D = \{u \in C([0,T],H_n) : \|u(\cdot)\|_\infty \leq R\}.$$

To prove that $\mathcal{T}(D)$ is bounded, let $u \in D$, then

$$\begin{aligned}\|\mathcal{T}(u)t\|_H \leq \|\mathcal{C}_n(t,0)\|_{\mathcal{L}(H)}\|\mathcal{P}_n g(w)\|_H + \|\mathcal{S}_n(t,0)\|_{\mathcal{L}(H)}\|\mathcal{P}_n h(w)\|_H \\ + \int_0^t \|\mathcal{S}_n(t,s)\|\|_{\mathcal{L}(H)} \|\mathcal{P}_n f(s,w(s))\|_H \, ds.\end{aligned}$$

By the uniform boundedness of $\mathcal{C}_n(t,s)$, $\mathcal{S}_n(t,s)$ and using (iii) and $(F3)$, we have

$$\begin{aligned}\|\mathcal{T}(u)t\|_H &\leq M_1 r_1 + M_2 r_2 + M_2 \int_0^t a\|w(s)\|_H + b(s)\, ds, \\ \|\mathcal{T}(u)t\|_H &\leq M_1 r_1 + M_2 r_2 + M_2(aRT + \|b\|_{L^1([0,T],\mathbb{R}^+)}).\end{aligned}$$

The right hand side of the above bound is independent of t, so

$$\sup_{t\in[0,T]} \|\mathcal{T}(w)(t)\|_H < \infty.$$

Hence, $\mathcal{T}(D)$ is bounded in $C[0,T],H_n)$.

To prove equicontinuity, let $w \in D$, and $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then

$$\begin{aligned}\|\mathcal{T}(w)(t_1) - \mathcal{T}(w)(t_2)\|_H \leq\ & \|\mathcal{C}_n(t_1,0) - \mathcal{C}_n(t_2,0)\|_{\mathcal{L}(H)}\|P_n g(w)\|_H \\ &+ \|\mathcal{S}_n(t_1,0) - \mathcal{S}_n(t_2,0)\|_{\mathcal{L}(H)}\|P_n h(w)\|_H \\ &+ \left\| \int_0^{t_1} \mathcal{S}_n(t_1,s)P_n f(s,w(s))\, ds \right. \\ &\qquad \left. - \int_0^{t_2} \mathcal{S}_n(t_2,s)P_n f(s,w(s))\, ds \right\|_H\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{T}(w)(t_1) - \mathcal{T}(w)(t_2)\|_H \leq &\|\mathcal{C}_n(t_1,0) - \mathcal{C}_n(t_2,0)\|_{\mathcal{L}(H)}\|P_n g(w)\|_H \\ &+ \|\mathcal{S}_n(t_1,0) - \mathcal{S}_n(t_2,0)\|_{\mathcal{L}(H)}\|P_n h(w)\|_H \\ &+ \left\| \int_0^{t_1} [\mathcal{S}_n(t_1,s) - \mathcal{S}_n(t_2,s)] P_n f(s,w(s))\, ds \right\|_H \\ &+ \left\| \int_{t_1}^{t_2} \mathcal{S}_n(t_2,s) P_n f(s,w(s))\, ds \right\|_H .\end{aligned}$$

We know that

$$\mathcal{S}_n(t,s) = \mathcal{S}_n(t,r)\, \partial_r \mathcal{S}_n(r,s) - \partial_s \mathcal{S}_n(r,s)\, \mathcal{S}_n(r,s).$$

For $t = t_1$ and $t = t_2$, we have

$$\begin{aligned}&\mathcal{S}_n(t_1,s) - \mathcal{S}_n(t_2,s) \\ &= \big(\mathcal{S}_n(t_1,r) - \mathcal{S}_n(t_2,r)\big)\, \partial_r \mathcal{S}_n(r,s) - (\mathcal{C}_n(t_1,r) - \mathcal{C}_n(t_2,r))\, \mathcal{S}_n(r,s).\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\|\mathcal{T}(w)(t_1) - \mathcal{T}(w)(t_2)\|_H \\ &\leq \|\mathcal{C}_n(t_1,0) - \mathcal{C}_n(t_2,0)\|_{\mathcal{L}(H)}\|P_n g(w)\|_H \\ &\quad + \|\mathcal{S}_n(t_1,0) - \mathcal{S}_n(t_2,0)\|_{\mathcal{L}(H)}\|P_n h(w)\|_H \\ &\quad + \|\mathcal{S}_n(t_1,r) - \mathcal{S}_n(t_2,r)\|_{\mathcal{L}(H)} \int_0^{t_1} \|\partial_r \mathcal{S}_n(r,s)\|_{\mathcal{L}(H)} \|P_n f(s,w(s))\|_H ds \\ &\quad + \|\mathcal{C}_n(t_1,r) - \mathcal{C}_n(t_2,r)\|_{\mathcal{L}(H)} \int_0^{t_1} \|\mathcal{S}_n(r,s)\|_{\mathcal{L}(H)} \|P_n f(s,w(s))\|_H ds \\ &\quad + M_2 \int_{t_1}^{t_2} \|P_n f(s,w(s))\|_H ds.\end{aligned}$$

Observe that, since $\mathcal{S}_n(t,s)$ is the fundamental solution of the second order finite dimensional equation, we assume that $\|\mathcal{S}_n(t,s)\| \leq M_2$, $\|\partial_r \mathcal{S}_n(r,s)\| \leq M_1$ and, by hypothesis,

$$\|P_n f(s,w(s))\| \leq a\|u(s)\| + b(s) \leq a\ R + b(s).$$

Hence, we have

$$\begin{aligned}&\|\mathcal{T}(w)(t_1)-\mathcal{T}(w)(t_2)\|_H\\&\le\|\mathcal{C}_n(t_1,0)-\mathcal{C}_n(t_2,0)\|_{\mathcal{L}(H)}r_1+\|\mathcal{S}_n(t_1,0)-\mathcal{S}_n(t_2,0)\|_{\mathcal{L}(H)}r_2\\&\quad+\|\mathcal{S}_n(t_1,r)-\mathcal{S}_n(t_2,r)\|_{\mathcal{L}(H)}\,M_1(TaR+\|b\|_{L^1([0,T],\mathbb{R}^+)})\\&\quad+\Big\|\mathcal{C}_n(t_1,r)-\mathcal{C}_n(t_2,r)\Big\|_{\mathcal{L}(H)}\,M_2(TaR+\|b\|_{L^1([0,T],\mathbb{R}^+)})\\&\quad+M_2\int_{t_1}^{t_2}(aR+b(s))ds.\end{aligned}$$

Now, letting

$$C=\max\{r_1,r_2,M_1(TaR+\|b\|_{L^1([0,T],\mathbb{R}^+)}),M_2(2TaR+\|b\|_{L^1([0,T],\mathbb{R}^+)})\},$$

we have

$$\begin{aligned}\|\mathcal{T}(w)(t_1)-\mathcal{T}(w)(t_2)\|_H\le{}&C\|\mathcal{C}_n(t_1,0)-\mathcal{C}_n(t_2,0)\|_{\mathcal{L}(H)}\\&+C\|\mathcal{S}_n(t_1,0)-\mathcal{S}_n(t_2,0)\|_{\mathcal{L}(H)}\\&+C\|\mathcal{S}_n(t_1,r)-\mathcal{S}_n(t_2,r)\|_{\mathcal{L}(H)}\\&+C\|\mathcal{C}_n(t_1,r)-\mathcal{C}_n(t_2,r)\|_{\mathcal{L}(H)}\\&+M_2\left[\int_0^{t_1}b(s)ds-\int_0^{t_2}b(s)ds\right].\end{aligned}$$

Let $t_1=0$. Then we have

$$\begin{aligned}\|\mathcal{T}(w)(0)-\mathcal{T}(w)(t_2)\|_H\le{}&\|I-\mathcal{C}_n(t_2,0)\|_{\mathcal{L}(H)}\|P_ng(w)\|_H\\&+\|\mathcal{S}_n(t_2,0)\|_{\mathcal{L}(H)}\|P_nh(w)\|_H\\&+\Big\|\int_0^{t_2}\mathcal{S}_n(t_2,s)P_nf(s,w(s))\,ds\Big\|_H.\end{aligned}$$

By the continuity of norm $\|\mathcal{T}(w)(0)-\mathcal{T}(w)(t_2)\|_H\to0$ as $t_2\to0$. Hence, by the uniform continuity of $\mathcal{S}(\cdot,s)$ and $\mathcal{C}(\cdot,s)$, for $t_1,t_2\in[0,T]$ such that $|t_1-t_2|<\delta$, and for every $w\in D$, we have $\|\mathcal{T}(w)(t_1)-\mathcal{T}(w)(t_2)\|_H<\varepsilon$. Therefore, $\mathcal{T}$ is equi-continuous and by Arzelà–Ascoli Theorem $\mathcal{T}$ is compact.

Now, let

$$D_1=\{u\in C(0,T],H_n)\;:\;u=\lambda T(u)\text{ for }\lambda\in[0,1]\}.$$

To show that D_1 is bounded, let $u\in D_1$. Then

$$\begin{aligned}\|u(t)\|_H\le{}&\lambda\|\mathcal{C}_n(t,0)\,\mathcal{P}_ng(u)\|_H+\lambda\|\mathcal{S}_n(t,0)\,\mathcal{P}_nh(u)\|_H\\&+\lambda\int_0^t\|\mathcal{S}_n(t,s)\,\mathcal{P}_nf(s,u(s))\|_H\,ds,\end{aligned}$$

$$\|u(t)\|_H\le M_1r_1+M_2r_2+M_2\|b\|_{L^1([0,T],\mathbb{R}^+)}+M_2a\int_0^t\|u(s)\|_Hds.$$

By Gronwall's inequality, we have

$$\|u(t)\|_H \le Le^{M_2 a\ T}$$

where the right-hand side is independent of t. Hence,

$$\sup_{t\in[0,T]} \|u(t)\|_H \le Le^{M_2 a\ T},$$

where $L = M_1 r_1 + M_2 r_2 + M_2\|b\|_{L^1([0,T],\mathbb{R}^+)}$. This implies that D_1 is bounded in $C([0,T],H_n)$.

Thus, by Theorem 2.10, $\mathcal{T}$ has a fixed point in $C([0,T],H_n) \cap L^2([0,T],H_n)$. Consequently, the problem

$$\begin{cases} \ddot{u}(t) + A_n(t)u(t) = \mathcal{P}_n f(t,u(t)) & \text{a.e. } t\in[0,T], \\ u(0) = \mathcal{P}_n g(u), \quad \dot{u}(0) = \mathcal{P}_n h(u) \end{cases}$$

admits a solution u_n such that $u_n \in H^2([0,T],H_n) \cap H^1([0,T],V)$. Consider the sequence $\{u_n\}_{n\in\mathbb{N}} \subset L^2([0,T],H)$. Since for each $n\in\mathbb{N}$,

$$u_n \in H^2([0,T],H_n) \cap H^1([0,T],V),$$

then

$$\|u_n\|_{H^2([0,T],H)} + \|u_n\|_{H^1([0,T],V)} \le C < +\infty$$

for some positive constant C independent of $n\in\mathbb{N}$, which implies $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^2([0,T],H_n)\cap H^1([0,T],V)$. Since V is compactly embedded onto H, implies that $\{u_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2([0,T],H)$, by Theorem 2.11. Consequently, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ which converges to u_* in $\in L^2([0,T],H)$. We can also assume that for a.e. $t\in[0,T]$, $u_{n_k}(t)$ converges to $u_*(t)$ as $k\to\infty$. Moreover, up to a further subsequence, if needed, it can be seen that for any fixed $t\in[0,T]$,

$$\int_0^t \mathcal{S}_{n_k}(t,s)\mathcal{P}_{n_k} f(s,u_{n_k}(s)) - \mathcal{S}(t,s)f(s,u_*(s))ds \to 0. \tag{3.7}$$

Let $t\in[0,T]$ be fixed, we consider

$$y_{n_k}(t) := \int_0^t \mathcal{S}_{n_k}(t,s)\mathcal{P}_{n_k} f(s,u_{n_k}(s)) - \mathcal{S}(t,s)f(s,u_*(s))ds,$$

and $\{y_{n_k}\}_{k\in\mathbb{N}}$ is a bounded sequence in $H^2([0,T],H)$. By Lemma 3.3 and since $H^2([0,T],H_{n_k}) \hookrightarrow H^2([0,T],H)$, using the same reasoning as above, it is proved that

$y_{n_k}(t)$ converges to $y_*(t)$ as $k \to \infty$. On the other hand, for fixed $x \in H$ and $(t,s) \in \Delta$, we have

$$\begin{aligned}
&\langle \mathcal{S}_{n_k}(t,s)\mathcal{P}_{n_k} f(s,u_{n_k}(s)) - \mathcal{S}(t,s)f(s,u_*(s)), x\rangle_H \\
&= \langle \mathcal{S}_{n_k}(t,s)\mathcal{P}_{n_k}\big(f(s,u_{n_k}(s)) - f(s,u_*(s))\big), x\rangle_H \\
&\quad + \langle \big(\mathcal{S}_{n_k}(t,s)\mathcal{P}_{n_k} - \mathcal{S}(t,s)\big) f(s,u_*(s)), x\rangle_H \\
&= \langle f(s,u_{n_k}(s)) - f(s,u_*(s)), \mathcal{S}(t,s)^* x\rangle_H \\
&\quad + \langle f(s,u_{n_k}(s)) - f(s,u_*(s)), \big(\mathcal{S}_{n_k}(t,s)\mathcal{P}_{n_k}\big)^* x - \mathcal{S}(t,s)^* x\rangle_H \\
&\quad + \langle \big(\mathcal{S}_{n_k}(t,s)\mathcal{P}_{n_k} - \mathcal{S}(t,s)\big) f(s,u_*(s)), x\rangle_H .
\end{aligned}$$

By demicontinuity of $f(s,\cdot)$ and Remark 3.6, for $k \to \infty$ we have

$$\langle f(s,u_{n_k}(s)) - f(s,u_*(s)), \mathcal{S}(t,s)^* x\rangle_H \to 0,$$

and

$$\langle f(s,u_{n_k}(s)) - f(s,u_*(s)), (\mathcal{S}_{n_k}(t,s)\mathcal{P}_{n_k})^* x - \mathcal{S}(t,s)^* x\rangle_H \to 0$$

since $f(s,u_{n_k}(s)) - f(s,u_*(s))$ is bounded. Also, by Lemma 3.5,

$$\langle (\mathcal{S}_{n_k}(t,s)\mathcal{P}_{n_k} - \mathcal{S}(t,s)) f(s,u_*(s)), x\rangle_H \to 0 \text{ as } k \to \infty$$

Then

$$\lim_{k\to\infty} \langle \mathcal{S}_{n_k}(t,s)\mathcal{P}_{n_k} f(s,u_{n_k}(s)) - \mathcal{S}(t,s)f(s,u_*(s)), x\rangle_H = 0.$$

Also, it is easy to see that for any fixed $t \in [0,T]$ and any $s \in [0,t]$,

$$\begin{aligned}
&\langle \mathcal{S}_{n_k}(t,s)\mathcal{P}_{n_k} f(s,u_{n_k}(s)) - \mathcal{S}(t,s)f(s,u_*(s)), x\rangle_H \\
&\le \|x\| + (1+a^2)\|u(s)\|^2 + a^2\|u(s)\|_H^2 + |b(s)|^2
\end{aligned}$$

where $\|u_{n_k}(s)\|_H \le 1 + \|u_*(s)\|_H$ a.e. uniformly on $[0,t]$, and for k big enough. Since the last term of the inequality lies in $L^1([0,t],H)$, and the sequence is bounded (see [45, Proposition 23.9]), it implies that

$$\langle y_{n_k}(t), x\rangle_H = \int_0^t \langle \mathcal{S}_{n_k}(t,s)\mathcal{P}_{n_k} f(s,u_{n_k}(s)) - \mathcal{S}(t,s)f(s,u_*(s)), x\rangle_H ds \to 0,$$

as $k \to \infty$. To prove (3.7), observe that it is readily implied by the arbitrary choice of $x \in H$ and the uniqueness of the limit, which implies $y_* = 0$.

$$\begin{aligned}
\|\mathcal{S}_{n_k}(t,0)\mathcal{P}_{n_k} h(u_{n_k}) - \mathcal{S}(t,0)h(u_*)\|_H &\le \|\mathcal{S}_{n_k}(t,0)\mathcal{P}_{n_k}(h(u_{n_k}) - h(u_*))\|_H \\
&\quad + \|\big(\mathcal{S}_{n_k}(t,0)\mathcal{P}_{n_k} - \mathcal{S}(t,0)\big)h(u_*)\|_H
\end{aligned}$$

and by Lemma 3.5,

$$\|(\mathcal{S}_{n_k}(t,0)\mathcal{P}_{n_k} - \mathcal{S}(t,0))h(u_*)\|_H \to 0 \text{ as } k \to \infty.$$

Now, to prove $\|\mathcal{S}_{n_k}(t,0)\mathcal{P}_{n_k}(h(u_{n_k})-h(u_*))\|_H \to 0$, we introduce

$$z_{n_k}(t) := \mathcal{S}_{n_k}(t,0)\mathcal{P}_{n_k}(h(u_{n_k})-h(u_*)),$$

which is a bounded sequence in $H^2([0,T],H)$. If needed, by passing to further subsequences, it is proved that $z_{n_k}(t)$ converges to $z_*(t)$ as $k\to\infty$. For fixed $x\in H$ and $(t,s)\in\Delta$, we have

$$\begin{aligned}\langle z_{n_k},x\rangle_H &= \langle h(u_{n_k})-h(u_*),(\mathcal{S}_{n_k}(t,0)\mathcal{P}_{n_k})^*x-\mathcal{S}(t,0)^*x\rangle_H\\ &\quad+\langle h(u_{n_k})-h(u_*),\mathcal{S}(t,0)^*x\rangle_H.\end{aligned}$$

We observe that $\langle h(u_{n_k})-h(u_*),\mathcal{S}(t,0)^*x\rangle_H\to 0$ as $k\to\infty$. By the same reasoning as above, we have

$$\langle h(u_{n_k})-h(u_*),(\mathcal{S}_{n_k}(t,0)\mathcal{P}_{n_k})^*x-\mathcal{S}(t,0)^*x\rangle_H\to 0.$$

This proves that $\langle z(t),x\rangle_H\to 0$ as $k\to\infty$ and so $z_*=0$. The fact that $h(u_{n_k})\to h(u_*)$ as $k\to\infty$ follows by arguing as before. Similarly,

$$z'_{n_k}(t) := \mathcal{C}_{n_k}(t,0)\mathcal{P}_{n_k}(g(u_{n_k})-g(u_*)).$$

Hence, for any $t\in[0,T]$, we have the following

$$\begin{aligned}u_*(t)=\lim_{k\to\infty}u_{n_k}(t)&=\lim_{k\to\infty}\Bigg(\mathcal{C}_{n_k}(t,0)\,\mathcal{P}_{n_k}g(u_{n_k})+\mathcal{S}_{n_k}(t,0)\,\mathcal{P}_{n_k}h(u_{n_k})\\ &\qquad+\int_0^t\mathcal{S}_{n_k}(t,s)\,\mathcal{P}_{n_k}f(s,u_{n_k}(s))\,ds\Bigg)\\ &=\mathcal{C}(t,0)\,g(u_*)+\mathcal{S}(t,0)\,h(u_*)+\int_0^t\mathcal{S}(t,s)\,f(s,u_*(s))\,ds.\end{aligned}$$

Hence, by Lemmas 3.2 and 3.3, $u_*\in H^2([0,T],H)\cap H^1([0,T],V)$ solves problem (3.5). □

4. DAMPED WAVE EQUATION

In this section, we discuss the regularity and well-posedness of the complete equation, i.e., the abstract wave equation with a damping term. Here, V, H are Hilbert spaces such that V is densely and compactly embedded in H. $\mathcal{A}(t)$, $\mathcal{B}(t)\in\mathcal{L}(V,V')$ are associated with non-autonomous forms

$$a:[0,T]\times V\times V\to\mathbb{C}\quad\text{and}\quad b:[0,T]\times V\times V\to\mathbb{C}.$$

$A(t)$, $B(t)$ are such that $A(t)u=\mathcal{A}(t)u$ and $B(t)u=\mathcal{B}(t)u$ with

$$D(B(t))\subset D(A(t))\text{ for all }t\in[0,T],$$

on the non-empty sets

$$D(A(t)) = \{u \in V \mid \mathcal{A}(t)u \in H\} \quad \text{and} \quad D(B(t)) = \{u \in V \mid \mathcal{B}(t)u \in H\}.$$

Theorem 4.1. *Assume that $B, A : [0,T] \to \mathcal{L}(V,H)$ are strongly measurable, $f \in L^2([0,T],H)$ and there exists $h \in L^2(0,T)$ such that*

$$\|A(t)\|_{\mathcal{L}(V,H)} \le h(t)$$

for almost every t. Then the following holds. If the first-order Cauchy problem

$$\dot{u} + B(t)u = f \quad (t \in [0,T]), \quad u(0) = u_0 \in V$$

has L^2-maximal regularity, then the second-order problem

$$\ddot{u} + B(t)\dot{u} + A(t)u = f \quad (t \in [0,T]), \quad u(0) = u_0, \quad \dot{u}(0) = u_1 \in V, \tag{4.1}$$

admits at least one solution u in $H^2([0,T],H) \cap H^1([0,T],V)$.

Proof. We present a detailed argument based on the first-order formulation and maximal regularity for the associated block operator system. Define the phase variable $U(t) := (u(t), \dot{u}(t))^T$ and the non-autonomous operator matrix

$$\mathcal{A}(t) := \begin{pmatrix} 0 & -I \\ A(t) & B(t) \end{pmatrix}, \quad D(\mathcal{A}(t)) := D(A(t)) \times V.$$

Then the second-order problem (4.1) is equivalent to the first-order Cauchy problem on $\mathcal{H} := H \times H$ (with the natural pivot $V \hookrightarrow H \cong H' \hookrightarrow V'$):

$$\dot{U}(t) + \mathcal{A}(t)U(t) = F(t) := (0, f(t))^T, \quad U(0) = (u_0, u_1)^T.$$

Under our standing assumptions on $A(\cdot)$ and $B(\cdot)$ (strong measurability, coercivity, V-boundedness, and the square-root property for $A(t)$), the first-order non-autonomous system governed by $\mathcal{A}(t)$ admits L^2-maximal regularity on $\mathcal{H}$; see, e.g., [9, 26, 29]. In particular, for every $F \in L^2(0,T;\mathcal{H})$ and $(u_0, u_1) \in D(A(0)) \times V$, there exists at least one solution $U \in H^1(0,T;\mathcal{H}) \cap L^2\big(0,T; D(\mathcal{A}(\cdot))\big)$ with $\mathcal{A}(\cdot)U(\cdot) \in L^2(0,T;\mathcal{H})$.

Writing the components of U, this yields $u \in H^1(0,T;H)$, $\dot{u} \in H^1(0,T;H)$ and, a.e. in t, $(u(t), \dot{u}(t)) \in D(A(t)) \times V$, $u \in H^2(0,T;H)$ and $u \in H^1(0,T;V)$, while the equation $\ddot{u} + B(t)\dot{u} + A(t)u = f$ holds in H for a.e. $t \in (0,T)$. This is exactly the L^2-maximal regularity conclusion for (4.1).

For completeness, we record the variation-of-constants formula. Let $E(t,s)$ denote the evolution family associated with $\mathcal{A}(t)$ and write its block decomposition as

$$E(t,s) = \begin{pmatrix} v_1(t,s) & v_2(t,s) \\ v_3(t,s) & v_4(t,s) \end{pmatrix}.$$

Then

$$U(t) = E(t,0)\,(u_0, u_1)^T + \int_0^t E(t,s)\,(0, f(s))^T\,ds,$$

so the first component satisfies the explicit representation

$$u(t) = v_1(t,0)u_0 + v_2(t,0)u_1 + \int_0^t v_2(t,s)f(s)\,ds.$$

The regularity of u stated above follows from the maximal regularity of the first-order system and the fact that $D(\mathcal{A}(t)) = D(A(t)) \times V$. □

Theorem 4.2. *Suppose V is densely and compactly embedded in H, the components of $E(t,s)$ are uniformly bounded in H, and the following conditions hold:*

- (i) *$A(t)$ and $B(t)$ are strongly measurable, coercive, V-bounded, and $A(t)$ satisfies the square-root property,*
- (ii) *$f : [0,T] \times H \to H$ is measurable in t and uniformly Lipschitz in the second variable, i.e., $\|f(t,u) - f(t,v)\|_H \le L\,\|u-v\|_H$ for a.e. t and all $u, v \in H$, with $f(\cdot,0) \in L^2(0,T;H)$,*
- (iii) *$g, h : L^2([0,T],H) \to V$ are Lipschitz continuous.*

Then the following nonlocal semilinear evolution problem admits at least one solution u in $H^2([0,T],H) \cap H^1([0,T],V)$:

$$\begin{cases} \ddot{u}(t) + B(t)\dot{u}(t) + A(t)u(t) = f(t,u(t)), \\ u(0) = g(u), \quad \dot{u}(0) = h(u). \end{cases} \tag{4.2}$$

Proof. Define the solution map $P : C([0,T];H) \to C([0,T];H)$ by

$$(Pu)(t) := v_1(t,0)\,g(u) + v_2(t,0)\,h(u) + \int_0^t v_2(t,s)\,f(s,u(s))\,ds,$$

where

$$E(t,s) = \begin{pmatrix} v_1(t,s) & v_2(t,s) \\ v_3(t,s) & v_4(t,s) \end{pmatrix}$$

is the evolution family of the linear first-order system associated with $\mathcal{A}(t)$ in the proof of the previous theorem. By the uniform boundedness of the components of $E(t,s)$ in H there exist constants $M_1, M_2 > 0$ with

$$\sup_{t\in[0,T]} \|v_1(t,0)\|_{\mathcal{L}(H)} \le M_1, \quad \sup_{t\in[0,T]} \|v_2(t,0)\|_{\mathcal{L}(H)} \le M_2,$$

and, for each fixed t, $\int_0^t \|v_2(t,s)\|_{\mathcal{L}(H)}\,ds \le M_{2,T}$ for some $M_{2,T}$ depending on T. The Lipschitz continuity of g, h from $L^2(0,T;H)$ to $V \hookrightarrow H$ implies the existence of $L_g, L_h > 0$ such that

$$\|g(u) - g(v)\|_H \le L_g\,\|u-v\|_{L^2(0,T;H)}, \quad \|h(u) - h(v)\|_H \le L_h\,\|u-v\|_{L^2(0,T;H)}.$$

Moreover, the Lipschitz continuity of f in u yields

$$\|f(s,u(s)) - f(s,v(s))\|_H \le L\|u(s) - v(s)\|_H$$

for a.e. s. Combining these estimates, for any $u, v \in C([0,T];H)$ we obtain

$$\begin{aligned}
&\|(Pu)-(Pv)\|_{C([0,T];H)} \\
&\leq M_1L_g\,\|u-v\|_{L^2(0,T;H)} + M_2L_h\,\|u-v\|_{L^2(0,T;H)} \\
&\quad + \sup_{t\in[0,T]} \int\limits_0^t \|v_2(t,s)\|\,L\,\|u(s)-v(s)\|_H\,ds \\
&\leq (M_1L_g + M_2L_h)\,T^{1/2}\|u-v\|_{C([0,T];H)} + LM_{2,T}\|u-v\|_{C([0,T];H)}.
\end{aligned}$$

Thus, there exists $T_* > 0$ such that the right-hand coefficient is strictly less than 1 on $[0,T_*]$, so P is a contraction on $C([0,T_*];H)$. By Banach's fixed point theorem there exists a mild solution on $[0,T_*]$. Iterating this argument on a finite partition $0 = T_0 < T_1 < \ldots < T_N = T$ (with subintervals of length at most T_*) yields a mild solution $u \in C([0,T];H)$.

To lift the mild solution to a strong one with maximal regularity, observe that $f(\cdot,u(\cdot)) \in L^2(0,T;H)$ by the Lipschitz assumption and $u \in L^2(0,T;H)$ (since $u \in C([0,T];H)$ on a finite interval). Therefore, the right-hand side of (4.2) belongs to $L^2(0,T;H)$. Applying the linear maximal regularity result from the previous theorem to the equation with inhomogeneity $f(\cdot,u(\cdot))$ shows that $u \in H^2(0,T;H) \cap H^1(0,T;V)$ and that (4.2) holds in H for a.e. $t \in (0,T)$. Finally, the nonlocal initial conditions are satisfied by construction of the fixed point (through v_1 and v_2 terms). This proves maximal regularity for the nonlocal semilinear problem. □

5. APPLICATIONS

In this section, we illustrate our abstract results through two model problems. We present the PDEs and verify the hypotheses in a concise, self-contained manner, focusing on how the structural assumptions translate into the abstract framework.

5.1. UNDAMPED WAVE EQUATION WITH NEUMANN BOUNDARY CONDITIONS AND NONLOCAL INITIAL DATA

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and fixed $T > 0$. Consider the following problem:

$$\begin{cases}
\partial_{tt}u(t,x) - \nabla\cdot\big(a(t,x)\nabla u(t,x)\big) + c(t,x)u(t,x) = f\big(t,u(t,x)\big), & (t,x)\in(0,T)\times\Omega, \\
\partial_\nu u(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega, \\
u(0,\cdot) = \displaystyle\int\limits_0^T \kappa_1(s,\cdot)\,u(s,\cdot)\,ds, \quad \partial_t u(0,\cdot) = \displaystyle\int\limits_0^T \kappa_2(s,\cdot)\,u(s,\cdot)\,ds.
\end{cases} \tag{5.1}$$

We set $H := L^2(\Omega)$, $V := H^1(\Omega)$, and for each $t \in [0, T]$ define the sesquilinear form

$$a(t; u, v) := \int_\Omega a(t, x)\, \nabla u \cdot \nabla v\, dx + \int_\Omega c(t, x)\, u\, v\, dx, \quad u, v \in V.$$

We assume that

$$a \in L^\infty\big((0, T) \times \Omega\big), \quad a(t, x) \geq a_0 > 0, \quad c \in L^\infty\big((0, T) \times \Omega\big), \quad c \geq 0,$$

and that f satisfies $(F1)$—$(F3)$. We also define the nonlocal operators

$$g(u) := \int_0^T \kappa_1(s, \cdot)\, u(s, \cdot)\, ds, \quad h(u) := \int_0^T \kappa_2(s, \cdot)\, u(s, \cdot)\, ds.$$

Since Ω is bounded with Lipschitz boundary, we have the continuous and compact embedding

$$V = H^1(\Omega) \hookrightarrow H = L^2(\Omega),$$

and V is densely embedded in H. Thus, assumption (i) of Theorem 3.7 holds.

The map $t \mapsto a(t; u, v)$ is measurable for all $u, v \in V$, since $a(\cdot, \cdot)$ and $c(\cdot, \cdot)$ are measurable and essentially bounded. To verify the boundedness assumption, note that for every $u, v \in V$,

$$|a(t; u, v)| \leq \|a\|_{L^\infty}\, \|\nabla u\|_{L^2}\, \|\nabla v\|_{L^2} + \|c\|_{L^\infty}\, \|u\|_{L^2}\, \|v\|_{L^2} \;\leq\; C\|u\|_V\, \|v\|_V.$$

Moreover, for every $u \in V$,

$$\operatorname{Re} a(t; u, u) = \int_\Omega a(t, x) |\nabla u|^2\, dx + \int_\Omega c(t, x) |u|^2\, dx \geq a_0 \|\nabla u\|_{L^2}^2.$$

Therefore, setting $\omega > 0$ arbitrarily,

$$\operatorname{Re} a(t; u, u) + \omega \|u\|_H^2 \geq \min\{a_0, \omega\}\, \|u\|_V^2.$$

For uniformly elliptic divergence-form operators with Neumann boundary conditions on Lipschitz domains, it is well-known (see [5]) that the square-root property holds:

$$D(A(t)^{1/2}) = V,$$

with equivalence of norms. Thus, $\{A(t)\}$ satisfies (A_1)–(A_5), so assumption (ii) is satisfied.

Passing to the nonlinearity $f : [0, T] \times H \to H$, we note that, by assumption, it satisfies $(F1)$–$(F3)$.

We now check that the operators

$$(g(u))(x) = \int_0^T \kappa_1(s, x) u(s, x)\, ds, \quad (h(u))(x) = \int_0^T \kappa_2(s, x) u(s, x)\, ds$$

map $L^2(0,T;H)$ into V and satisfy the required Lipschitz and boundedness conditions. Assume

$$\kappa_i \in L^2\big(0,T;W^{1,\infty}(\Omega)\big) \quad (i=1,2).$$

Then, by Hölder's inequality and Sobolev embeddings, for any $u \in L^2(0,T;H)$, we have

$$\|g(u)\|_H \le \|\kappa_1\|_{L^2(0,T;L^\infty)} \|u\|_{L^2(0,T;H)},$$
$$\|\nabla g(u)\|_H \le \|\nabla \kappa_1\|_{L^2(0,T;L^\infty)} \|u\|_{L^2(0,T;H)},$$

and similarly for $h(u)$. Hence, $g,h : L^2(0,T;H) \to V$ are well-defined and continuous, thus demicontinuous. Since $V \hookrightarrow H$ compactly and g,h are integral operators with essentially bounded kernels, they map bounded subsets of $L^2(0,T;H)$ into relatively compact subsets of V.

Thus, assumption (iv) is satisfied.

In conclusion, all hypotheses of Theorem 3.7 are verified. Therefore, problem (5.1) admits at least one solution

$$u \in H^2(0,T;H) \cap H^1(0,T;V).$$

5.2. POPULATION DYNAMICS WITH MEMORY EFFECTS

Our last application concerns a class of nonlocal semilinear evolution problems arising in population dynamics with spatial diffusion and hereditary effects. Such memory dependence is common in ecological models, where adaptation mechanisms or genetic persistence influence long-term behavior. Let $u(t,x)$ represent the population density at time t and location x in a bounded habitat $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, where homogeneous Neumann boundary conditions describe a closed ecosystem. The dynamics are governed by the system

$$\begin{cases} \partial_{tt}u(t,x) + \sigma(t,x)\partial_t u(t,x) - d(t,x)\Delta u(t,x) + \mu(t,x)u(t,x) = f(t,u(t,x)), \\ \partial_\nu u(t,x) = 0, \quad (t,x) \in (0,T)\times\partial\Omega, \\ u(0,x) = \int\limits_0^T \kappa_1(s,x)u(s,x)\,ds, \quad \partial_t u(0,x) = \int\limits_0^T \kappa_2(s,x)u(s,x)\,ds. \end{cases} \tag{5.2}$$

The coefficients satisfy

$$d \in L^\infty((0,T)\times\Omega) d(t,x) \ge d_0 > 0,$$
$$\sigma \in L^\infty((0,T)\times\Omega)\sigma(t,x) \ge 0,$$
$$\mu \in L^\infty((0,T)\times\Omega)\mu(t,x) \ge 0.$$

The nonlinear term $f(t,u)$ represents population growth and interaction effects, and κ_1, κ_2 are memory kernels encoding hereditary dependence.

To apply Theorem 4.2, we set $V := H^1(\Omega)$ and $H := L^2(\Omega)$, so that V embeds continuously and compactly into H. For $u \in V$, define

$$A(t)u := -d(t,\cdot)\Delta u + \mu(t,\cdot)u, \quad B(t)u := \sigma(t,\cdot)u.$$

The operators $A(t), B(t) : V \to H$ are well defined since $d, \mu, \sigma \in L^\infty$, and their strong measurability follows from the measurability of the coefficients.

The form associated with $A(t)$ is given by

$$a(t; u, v) = \int_\Omega d(t,x)\,\nabla u \cdot \nabla v\,dx + \int_\Omega \mu(t,x)\,u\,v\,dx, \quad u, v \in V,$$

and satisfies, for almost every $t \in [0,T]$,

$$|a(t; u, v)| \leq \|d\|_{L^\infty}\,\|\nabla u\|_{L^2}\,\|\nabla v\|_{L^2} + \|\mu\|_{L^\infty}\,\|u\|_{L^2}\,\|v\|_{L^2} \leq C\|u\|_V\|v\|_V.$$

Moreover, for all $u \in V$,

$$a(t; u, u) = \int_\Omega d(t,x)|\nabla u|^2\,dx + \int_\Omega \mu(t,x)|u|^2\,dx \geq d_0\|\nabla u\|_{L^2}^2 \geq \alpha\|u\|_V^2,$$

so $A(t)$ is uniformly coercive. By the solution of the Kato square-root problem [5], for uniformly elliptic divergence-form operators with Neumann boundary conditions on Lipschitz domains, we have

$$D(A(t)^{1/2}) = H^1(\Omega) = V,$$

with equivalence of norms, thus verifying the square-root property required by Theorem 4.2.

The damping operator $B(t)$ acts by pointwise multiplication with $\sigma(t,x) \in L^\infty$, so $B(t) \in \mathcal{L}(V, H)$ and $t \mapsto B(t)$ is strongly measurable. Since $\sigma \geq 0$, $B(t)$ is accretive, and the linear evolution family $E(t,s)$ generated by the first-order problem $\dot{u} + B(t)u = 0$ is uniformly bounded in H.

Regarding the nonlinearity, we assume that $f : [0,T] \times H \to H$ is measurable in t and uniformly Lipschitz in u, i.e.

$$\|f(t,u) - f(t,v)\|_H \leq L\|u - v\|_H,$$

for almost every $t \in [0,T]$ and all $u, v \in H$, with $f(\cdot, 0) \in L^2(0,T;H)$. This verifies condition (ii) of Theorem 4.2.

The nonlocal operators induced by the kernels κ_1 and κ_2 are defined by

$$(g(u))(x) = \int_0^T \kappa_1(s,x)\,u(s,x)\,ds, \quad (h(u))(x) = \int_0^T \kappa_2(s,x)\,u(s,x)\,ds.$$

Assuming $\kappa_i \in L^2(0,T;W^{1,\infty}(\Omega))$ for $i = 1, 2$, we have

$$\|g(u) - g(v)\|_V \leq \|\kappa_1\|_{L^2(0,T;W^{1,\infty})}\,\|u - v\|_{L^2(0,T;H)},$$

and similarly for h, proving that both $g, h : L^2(0,T;H) \to V$ are Lipschitz continuous. Thus, condition (iii) is satisfied.

Since V is compactly embedded in H, $A(t)$ and $B(t)$ are strongly measurable, bounded and coercive, $A(t)$ satisfies the square-root property, the nonlinearity f is uniformly Lipschitz, and the nonlocal operators g, h are Lipschitz from $L^2(0,T;H)$ into V. Therefore, all assumptions of Theorem 4.2 are verified. Consequently, the nonlocal semilinear evolution problem (5.2) admits at least one solution

$$u \in H^2(0,T;H) \cap H^1(0,T;V),$$

which satisfies the equation, the Neumann boundary conditions, and the nonlocal initial conditions.

6. CONCLUSIONS

This work has been devoted to the analysis of second-order non-autonomous evolution equations with nonlocal initial data. We have established existence and regularity results for both damped and undamped problems by combining fundamental solution techniques, regularity theory for non-autonomous forms, and fixed-point arguments. The applicability of the developed framework has been illustrated through model problems inspired by realistic physical scenarios, including wave propagation, population dynamics with memory effects, and nonlocal semilinear phenomena. Overall, the results demonstrate how abstract form methods provide solutions with optimal regularity and offer a systematic approach for treating concrete classes of partial differential equations.

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