

Dedicated to Professor Giuseppe Marino  
on the occasion of his retirement

## FIXED POINT THEORY FOR JAGGI L-TYPE MAPPINGS

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**Abstract.** We present a class of nonlinear mappings, properly containing the nonexpansive ones, enjoying the fixed point property in orthogonally convex Banach spaces.

**Keywords:** fixed point, generalized nonexpansive mapping, normal structure, orthogonal convexity.

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### 1. INTRODUCTION

A Banach space  $(X, \|\cdot\|)$  is said to have the fixed point property (FPP, for short) for nonexpansive mappings if every nonexpansive map  $T : C \rightarrow C$  – that is, a map satisfying  $\|T(x) - T(y)\| \leq \|x - y\|$  for all  $x, y \in C$  – has a fixed point in  $C$ , whenever  $C \subset X$  is nonempty, closed, bounded, and convex. If, in this definition, the condition “closed and bounded” is replaced by “weakly compact”, the resulting property is called the weak fixed point property, abbreviated as wFPP. In reflexive Banach spaces, the fixed point property and the weak fixed point property are equivalent.

Since the mid-1960s, it has been understood that the fixed point properties (FPP and wFPP) of a Banach space  $X$  are closely related to certain geometric characteristics of its norm. For example, it was established early on that Hilbert spaces and uniformly convex Banach spaces possess the FPP.

In a seminal work by Kirk [13], it was shown that normal structure – a geometric condition slightly more general than uniform convexity – also guarantees the weak fixed point property (wFPP) (see the definitions below). However, the question of whether all reflexive Banach spaces enjoy the FPP remains open.

Since 1965, a substantial body of research has been aimed at identifying sufficient geometric conditions to ensure the wFPP. One such condition is the *orthogonal convexity* (OC), an isometric property of norms introduced in 1988 [5, 7].

The class of OC spaces is neither contained in nor contains the class of Banach spaces with normal structure. Orthogonal convexity is a more general property than uniform convexity and is therefore satisfied by every  $\ell_p$  space with  $1 < p < \infty$ . Moreover, several nonreflexive Banach spaces, such as  $\ell_1$ ,  $c_0$ , and the James semireflexive space (see [9]), also satisfy the OC condition.

It is also known that OC implies the weak Banach–Saks property (see [16]).

On the other hand, since the mid-1970s, numerous authors have introduced various classes of nonlinear mappings that generalize the class of nonexpansive mappings, while still preserving key fixed-point results. In this work, we focus on two such classes: the generalized nonexpansive mappings introduced by Jaggi in 1983, and the so-called L-type mappings, introduced in 2011 [19].

It is worth noting that reflexive Banach spaces with normal structure possess the fixed point property (FPP) for both Jaggi-type mappings and L-type mappings. However, standard techniques in fixed point theory, developed for nonexpansive mappings, often do not directly extend to these generalized mappings, particularly in spaces that lack normal structure.

In a recent paper [17], it was observed that mappings satisfying both the Jaggi condition and condition (L) also fulfill two key results in fixed point theory: those due to Goebel–Karlovitc and Maurey. Building on these insights, we will show that Banach spaces with orthogonal convexity (OC) possess the weak fixed point property (wFPP) for mappings in this class.

## 2. PRELIMINAIRES

We suppose that  $(X, \|\cdot\|)$  is a Banach space and  $0_X$  its zero vector. From now on,  $C$  stands for a given nonempty, closed, convex, and bounded subset of  $X$ . To say that a sequence  $(x_n)$  in  $X$  converges weakly to a point  $x \in X$  we will use the standard symbol  $x_n \rightharpoonup x$ .

A mapping  $T : C \rightarrow X$  is *nonexpansive* if for all  $x, y \in C$ ,  $\|T(x) - T(y)\| \leq \|x - y\|$ . Let  $\mathcal{A}$  be a class of self mappings of the set  $C$ . To simplify the statements of the fixed point results, we will say that a class of Banach spaces has the “FPP for  $\mathcal{A}$ -mappings” if for every nonempty, closed, convex, bounded subset  $C$  of any space  $X$  belonging to this class, every mapping  $T \in \mathcal{A}$  has a fixed point in  $C$ . If the same holds for nonempty, weakly compact, convex subsets  $C$  of  $X$ , we will use “wFPP” instead of “FPP”. For instance, the original Kirk contribution in this area [13] was to show that reflexive Banach spaces with normal structure enjoy the FPP for nonexpansive mappings.

A sequence  $(x_n)$  in  $C$  is called an *almost fixed point sequence* (a.f.p.s. for short) for  $T$  in  $C$  whenever  $x_n - T(x_n) \rightarrow 0_X$ . It is well known that if  $T : C \rightarrow C$  is nonexpansive and  $D \subset C$  is nonempty and  $T$ -invariant (i.e.  $T(D) \subset D$ ), then  $T$  has a.f.p. sequences in  $D$  provided that  $D$  is closed and convex. This fact is often called property (A).

Recall that the Banach space  $(X, \|\cdot\|)$  has weak normal structure (WNS, in short) if every nonempty convex and weakly compact subset  $C$  of  $X$ , with  $\text{diam}(C) > 0$ ,

contains a nondiametral point, that is, there exists  $x_0 \in C$  such that

$$\sup\{\|x - x_0\| : x \in C\} < \text{diam}(C).$$

If we replace “weakly compact” by “bounded and closed”, we obtain the definition of normal structure (NS).

For  $x, y \in X$  and  $\lambda \in (0, \infty)$ , denote

$$M_\lambda(x, y) := \left\{ z \in X : \max\{\|z - x\|, \|z - y\|\} \leq \frac{1}{2}(1 + \lambda)\|x - y\| \right\}.$$

If  $A$  is a nonempty bounded subset of  $X$ , we will write

$$|A| := \sup\{\|x\| : x \in A\}.$$

If  $(x_n)$  is a bounded sequence in  $X$  and  $\lambda > 0$ , we will use the following notations.

$$D[(x_n)] := \limsup_{n \rightarrow \infty} \left( \limsup_{m \rightarrow \infty} \|x_n - x_m\| \right),$$

$$A_\lambda[(x_n)] := \limsup_{n \rightarrow \infty} \left( \limsup_{m \rightarrow \infty} |M_\lambda(x_n, x_m)| \right).$$

**Definition 2.1.** The Banach space  $(X, \|\cdot\|)$  is called *orthogonally convex* (OC) if for each sequence  $(x_n)$  in  $X$  weakly convergent to 0, with  $D[(x_n)] > 0$ , there exists  $\lambda > 0$ , such that  $A_\lambda[(x_n)] < D[(x_n)]$ .

This property can be regarded as a generalization of the well known uniform convexity (see [12, p. 213]). It is satisfied by spaces with a very different kind of geometry, such as, for instance,  $c_0$  or the James sequence space. For more information and examples, see [5, 7–10, 14, 27], among others. In particular, the two following examples are relevant to separate the (OC) and the normal structure.

**Example 2.2.** For  $\beta \geq 1$ , let  $E_\beta$  be the Banach space  $\ell_2$  endowed with the norm

$$\|x\|_\beta := \max\{\|x\|_2, \beta\|x\|_\infty\},$$

where, for  $x = (x_n) \in \ell_2$ ,  $\|x\|_2^2 = \sum_{n=1}^{\infty} |x_n|^2$ , and  $\|x\|_\infty = \sup\{|x_n| : n = 1, 2, \dots\}$ .

According to [5], the spaces  $E_\beta$  are OC. From [1], for  $\beta > \sqrt{2}$  the space  $E_\beta$  fails to have normal structure.

**Example 2.3.** Let  $Y$  be the Banach space  $\ell_2$  endowed with the norm defined for  $x = (x_n) \in \ell_2$  as

$$\|x\| = \max\{\|x\|_2, \sup\{|x_k| + |x_r| : 1 \leq k < r\}\} + \sum_{n=1}^{\infty} \frac{|x_n|}{2^n},$$

According [18, Example 3.7] (see also [5]), the space  $Y$  has normal structure, but it fails to be OC.

Next we present the definitions of the classes of generalized nonexpansive mapping that we will be concerned with.

In 1983, D.S. Jaggi [4] defined a class of mappings properly wider than the class of the nonexpansive mappings.

**Definition 2.4** ([4]). A mapping  $T : C \rightarrow C$  is said to be Jaggi nonexpansive (JNE, in short) if for every  $T$ -invariant, closed, convex subset  $E$  of  $C$  with at least two points, and for every  $x \in E$ ,

$$\sup\{\|T(x) - T(y)\| : y \in E\} \leq \sup\{\|x - y\| : y \in E\}. \quad (2.1)$$

Of course, every nonexpansive mapping is JNE, but the converse fails.

The original Jaggi's fixed point theorem extended Kirk's result, showing that the reflexive Banach spaces with normal structure have the FPP property for JNE mappings (see [4, Theorem 2]). However, the following properties of the JNE mappings are not shared with the nonexpansive mappings:

1. In general Banach spaces, a JNE mapping  $T : C \rightarrow C$  where  $C$  is closed convex and bounded need not have a.f.p. sequences in  $C$ . (See Example 2.8 below). In other words, for JNE mappings, property (A) may fail. However, in reflexive Banach spaces, JNE mappings satisfy property (A) [17, Theorem 6].
2. Recall that a mapping  $T : C \rightarrow C$  is said to be *quasi-nonexpansive*, (QNE in short), provided that  $T$  has at least one fixed point  $p \in C$ , and for every fixed point  $p$  of  $T$  and for all  $x \in C$ ,  $\|p - T(x)\| \leq \|p - x\|$ . Nonexpansive mappings with fixed points are, of course QNE. However, JNE mappings with fixed points may fail to be QNE (see [2, Example 3.4]).
3. Every Banach space lacking normal structure fails the FPP for JNE mappings ([11, Theorem 3.1]). A deeper recent result about this matter can be seen in [24, Theorem 3.3].
4. The average of the identity mapping with a JNE mapping is not necessarily a JNE mapping ([2, Proposition 3.5]).

For more on JNE mappings, one can see, for instance, [2, 4, 11, 17, 24].

**Definition 2.5** (see [3]). For  $\mu \geq 1$  we say that a mapping  $T : C \rightarrow X$  satisfies condition  $E_\mu$  on  $C$  if there exists  $\mu \geq 1$  such that for all  $x, y \in C$ ,

$$\|x - T(y)\| \leq \mu\|x - T(x)\| + \|x - y\|. \quad (2.2)$$

We say that  $T$  satisfies condition (E) on  $C$  whenever it satisfies condition  $E_\mu$  for some  $\mu \geq 1$ .

It is obvious that if  $T : C \rightarrow X$  is nonexpansive, then it satisfies the condition  $E_1$ . The converse is not true (see Example 2.7 below). Let  $T : C \rightarrow X$  be a mapping that satisfies condition (E) on  $C$ . If  $T(p) = p$  for some  $p \in C$ , then by inequality (2.2) it follows that for every  $y \in C$

$$\|p - T(y)\| \leq \|p - y\|,$$

that is,  $T$  is QNE on  $C$ .

We introduce the following definition.

**Definition 2.6.** We say that the mapping  $T : C \rightarrow C$  is an JE mapping, provided that it is JNE and it satisfies condition (E).

Condition (E) alone does not guarantee fixed points for a mapping, even in the case of compact convex domains. To see this, consider the following easy example.

**Example 2.7.** Let  $T : [0, 1] \rightarrow [0, 1]$  be the mapping defined as

$$T(x) = \begin{cases} x + \frac{1}{2} & x \in [0, \frac{1}{2}], \\ x - \frac{1}{2} & x \in (\frac{1}{2}, 1]. \end{cases}$$

It is obvious that, for every  $x \in [0, 1]$ , we have  $|x - T(x)| = \frac{1}{2}$ . Therefore, for every  $y \in [0, 1]$ ,

$$|x - T(y)| = \left| x - \left( y \pm \frac{1}{2} \right) \right| \leq |x - y| + \frac{1}{2} = |x - T(x)| + |x - y|.$$

Thus,  $T$  satisfies condition  $E_1$  in  $[0, 1]$ . Of course  $T$ , is fixed point free, and by [4, Theorem 2] it fails to be JNE.

There are fixed-point-free JE self-mappings of closed convex bounded subsets of some Banach spaces.

**Example 2.8** (see [2, Example 3.2]). In the standard Banach space  $(\mathcal{C}([0, 1]), \|\cdot\|_\infty)$ , consider the subset

$$K = \{x \in \mathcal{C}([0, 1]) : x(0) = 0, \text{ and } \forall t \in [0, 1], t \leq x(t) \leq 1 = x(1)\},$$

and the mapping  $T : K \rightarrow K$  given by

$$T(x) = \sqrt{x}.$$

It is clear that  $T(K) \subset K$ ,  $T$  is fixed point free, and

$$(T \circ T)(x) = x^{1/4}, \dots, (T)^n(x) = x^{1/2^n}.$$

If  $D$  is a non trivial  $T$ -invariant subset of  $K$ , in [2, Example 3.2] it is seen that for every  $y \in D$ ,

$$\sup\{\|y - x\|_\infty : x \in D\} = 1 = \text{diam}(K).$$

Hence, for every nonempty  $T$ -invariant, closed, convex subset  $D$  of  $K$  with at least two points, for every  $x \in D$ ,

$$\sup\{\|y - x\|_\infty : x \in D\} = 1 = \text{diam}(K) \geq \sup\{\|T(x) - T(y)\|_\infty : y \in D\}.$$

This implies that  $T$  is a Jaggi-nonexpansive self-mapping of  $K$ .

Moreover,  $T$  has positive minimal displacement, that is, for every  $x \in K$ ,

$$\|x - T(x)\|_\infty = \frac{1}{4} > 0.$$

Consequently, this mapping fails to satisfy property (A). In addition, for  $x, y \in K$ , we have

$$\|x - T(y)\|_\infty \leq 1 \leq 4\|x - T(x)\|_\infty \leq 4\|x - T(x)\|_\infty + \|x - y\|_\infty.$$

This means that  $T$  satisfies condition  $E_4$ . Thus, the mapping  $T$  is a JE fixed point free selfmapping of the closed convex bounded set  $K$ .

The asymptotic behaviour of mappings satisfying Condition (E) has been recently studied. See, for instance, [22] and the references therein.

**Definition 2.9** ([19]). We say that the mapping  $T : C \rightarrow C$  is an L-type mapping, (or that  $T$  satisfies condition (L)) on  $C$ , provided that

1. (L.a) If  $D \subset C$  is nonempty, closed, convex and  $T$  invariant, then there exists an a.f.p.s.  $(x_n)$  for  $T$  in  $D$ .
2. (L.b) For every a.f.p.s.  $(x_n)$  for  $T$  in  $C$ , and for each  $x \in C$ ,

$$\limsup_{n \rightarrow \infty} \|x_n - T(x)\| \leq \limsup_{n \rightarrow \infty} \|x_n - x\|.$$

Condition (L.a) is none other than Property (A). It is automatically satisfied by several classes of mappings. For instance, if a mapping  $T : C \rightarrow C$  is nonexpansive with respect to some equivalent renorming of  $X$ , then  $T$  satisfies (L.a). The (C) mappings introduced by Suzuki in 2008 (see [28]) satisfy condition (L.a) too.

From Condition (L.b) it follows that L-type mappings with a fixed point are QNE.

The class of those mappings  $T : C \rightarrow C$  which satisfy condition (L) has been largely studied (see [15, 19, 23, 24, 26, 29] among others). In particular, a useful tool to check that a mapping satisfies condition (L) is the following.

**Proposition 2.10** (see [19, Proposition 3.5]). *If a mapping  $T : C \rightarrow C$  satisfies conditions (A) and (E), then it satisfies condition (L).*

Reflexive Banach spaces with normal structure have the FPP for L-type mappings ([19, Theorem 4.4]). Non-reflexive Banach spaces that enjoy this property have recently been studied in [20, 25]. However, the following properties of the nonexpansive mappings are not shared with L-type mappings:

- (L.a) If  $T : C \rightarrow C$  is an L-type mapping,  $x_0 \in C$  and  $\lambda \in (0, 1)$ , then the mappings  $T_\lambda : C \rightarrow C$  defined as  $T_\lambda(x) := \lambda x_0 + (1-\lambda)x$  may not satisfy the condition (L).
- (L.b) From [24, Theorem 4.3] in every Banach space lacking normal structure, there exist weakly compact convex subsets admitting L-type fixed-point free selfmappings. In other words, the fixed point property for L-type mappings characterizes the reflexive Banach spaces with normal structure.

We have seen (Example 2.7) an (E) mapping which fails to be JNE. In [2, Example 3.4], one can read an example of a JNE failing to be QNE, and hence failing to satisfy condition (E). Consequently, the class of the JNE mappings neither contains nor is contained in the class of the E-type mappings. Notice that, in the setting of the reflexive Banach spaces, every JNE mapping satisfies condition (A). Thus, in these spaces, the JE mappings also satisfy condition (L).

### 3. MAIN RESULT

We are now in a position to state and prove our main result, which establishes the existence of fixed points for JE mappings under the assumption of orthogonal convexity of the space. This existence result, in turn, allows us to assert that the mapping under consideration is quasi-nonexpansive.

According to [19, Theorem 4.6], in a Banach space satisfying the so-called Opial condition, in particular any Hilbert space, an L-type mapping  $T : C \rightarrow C$  also satisfies that  $Id - T$  is demiclosed at  $0_X$ . Consequently, if  $T$  is a JE mapping in such a setting, then it is QNE and  $Id - T$  is demiclosed at  $0_X$ .

These are the key conditions under which [21, Theorem 3.2] can be applied to guarantee convergence results for the general viscosity midpoint rule algorithm proposed by Marino *et al.* in 2017.

Let  $T : C \rightarrow C$  be a nonexpansive mapping. Put  $d := \text{diam}(C)$ . Take  $x_0 \in C$  and define  $\tilde{C} := \frac{1}{d}(C - x_0)$ . It is easy to see that the mapping  $\tilde{T} : \tilde{C} \rightarrow \tilde{C}$  defined as  $\tilde{T}(\frac{1}{d}(x - x_0)) = \frac{1}{d}(T(x) - x_0)$  is nonexpansive too. However, if the mapping  $T$  is either JNE or L or  $E_\mu$ , it is not straightforward that the same property holds for  $\tilde{T}$ . We will need this feature in the proof of the main result of this section.

**Lemma 3.1.** *Let  $T : C \rightarrow C$  be a mapping. Put  $d := \text{diam}(C)$ . Take  $x_0 \in C$  and define  $\tilde{C} := \frac{1}{d}(C - x_0)$  and  $\tilde{x} = \frac{1}{d}(x - x_0)$ . Consider the mapping  $\tilde{T} : \tilde{C} \rightarrow \tilde{C}$  defined as*

$$\tilde{T}\left(\frac{1}{d}(x - x_0)\right) = \frac{1}{d}(T(x) - x_0)$$

- (1) *If  $(x_n)$  is an a.f.p.s. for  $T$  in  $C$ , then  $(\tilde{x}_n)$  is an a.f.p.s. for  $\tilde{T}$  in  $\tilde{C}$ . Conversely, if  $(\frac{1}{d}(y_n - x_0))$  is an a.f.p.s. for  $\tilde{T}$  in  $\tilde{C}$ , then  $(y_n)$  is an a.f.p.s. for  $T$  in  $C$ .*
- (2) *If  $T$  is JNE on  $C$ , then  $\tilde{T}$  is JNE on  $\tilde{C}$ .*
- (3) *If  $T$  satisfies the condition  $E_\mu$  on  $C$ , then  $\tilde{T}$  satisfies condition  $E_\mu$  on  $\tilde{C}$ .*
- (4) *If  $T$  is an L-type self mapping of  $C$ , then  $\tilde{T}$  is an L-type self mapping of  $\tilde{C}$ .*

*Proof.* For  $x, y \in C$ , put  $\tilde{x} := \frac{1}{d}(x - x_0)$ ,  $\tilde{y} := \frac{1}{d}(y - x_0)$ . Then

$$\tilde{T}(\tilde{x}) - \tilde{T}(\tilde{y}) = \frac{1}{d}(T(x) - x_0) - \frac{1}{d}(T(y) - x_0) = \frac{1}{d}(T(x) - T(y)).$$

In the same way, if  $x \in C$ , we have

$$\tilde{x} - \tilde{T}(\tilde{x}) = \frac{1}{d}(x - x_0) - \frac{1}{d}(T(x) - x_0) = \frac{1}{d}(x - T(x)). \quad (3.1)$$

Therefore, if  $(x_n)$  is a sequence in  $C$ . Then

$$\tilde{x}_n - \tilde{T}(\tilde{x}_n) = \frac{1}{d}(x_n - T(x_n))$$

Consequently,  $(x_n)$  is an a.f.p.s. for  $T$  in  $C$  if and only if  $(\tilde{x}_n)$  is an a.f.p.s. for  $\tilde{T}$  in  $\tilde{C}$ .

Let us suppose that  $T : C \rightarrow C$  satisfies the condition  $E_\mu$ . Then, for  $\tilde{x} = \frac{1}{d}(x - x_0)$ ,  $\tilde{y} = \frac{1}{d}(y - x_0) \in \tilde{C}$ ,

$$\begin{aligned} \|\tilde{x} - \tilde{T}(\tilde{y})\| &= \left\| \frac{1}{d}(x - x_0) - \frac{1}{d}(T(y) - x_0) \right\| \\ &= \frac{1}{d}\|x - T(y)\| \\ &\stackrel{(E)}{\leq} \frac{1}{d}(\mu\|x - T(x)\| + \|x - y\|) \\ &\stackrel{(3.1)}{=} \mu\|\tilde{x} - \tilde{T}(\tilde{x})\| + \|\tilde{x} - \tilde{y}\|. \end{aligned}$$

Therefore, the mapping  $\tilde{T}$  satisfies the condition  $E_\mu$  on  $\tilde{C}$ .

Notice that, if  $E$  is a  $T$  invariant subset of  $C$ , then  $\tilde{E} := \{\frac{1}{d}(x - x_0) : x \in E\}$  is a  $\tilde{T}$ -invariant subset of  $\tilde{C}$ . Indeed, if  $\frac{1}{d}(x - x_0) \in \tilde{E}$ , then

$$\tilde{T}\left(\frac{1}{d}(x - x_0)\right) = \frac{1}{d}(T(x) - x_0) \in \tilde{E},$$

since  $T(x) \in E$ . Conversely, if  $\tilde{E} := \{\frac{1}{d}(x - x_0) : x \in E\}$  is a  $\tilde{T}$ -invariant subset of  $\tilde{C}$ , then for every  $x \in E$ ,

$$\tilde{T}\left(\frac{1}{d}(x - x_0)\right) \in \tilde{E},$$

that is,

$$\frac{1}{d}(T(x) - x_0) \in \left\{ \frac{1}{d}(x - x_0) : x \in E \right\}.$$

There exists  $y \in E$  such that

$$\frac{1}{d}(T(x) - x_0) = \frac{1}{d}(y - x_0)$$

which implies that  $y = T(x) \in E$ .

In other words, the mapping  $E \rightarrow \tilde{E}$  is a bijection between the  $T$ -invariant subsets of  $C$  and the  $\tilde{T}$ -invariant subsets of  $\tilde{C}$ .

Let  $\tilde{E}$  be a nontrivial closed convex  $\tilde{T}$ -invariant subset of  $\tilde{C}$ , and let  $\tilde{x} = \frac{1}{d}(x - x_0) \in \tilde{E}$ . Then, we have

$$\begin{aligned} \sup\{\|\tilde{T}(\tilde{x}) - \tilde{T}(\tilde{y})\| : \tilde{y} \in \tilde{E}\} &= \frac{1}{d} \sup\{\|T(x) - T(y)\| : y \in E\} \\ &\stackrel{(J)}{\leq} \sup\left\{ \frac{1}{d}\|x - y\| : y \in E \right\} \\ &= \sup\left\{ \left\| \frac{1}{d}(x - x_0) - \frac{1}{d}(y - x_0) \right\| : y \in E \right\} \\ &= \sup\{\|\tilde{x} - \tilde{y}\| : \tilde{y} \in \tilde{E}\}. \end{aligned}$$

This means that  $\tilde{T}$  is a JNE selfmapping of  $\tilde{C}$ .

Finally, suppose that  $T$  is an L-type mapping on  $C$ . Let  $(\tilde{x}_n) := (\frac{1}{d}(x_n - x_0))$  be an a.f.p.s. for  $\tilde{T}$  in  $\tilde{C}$ . Then we have seen that  $(x_n)$  is an a.f.p.s. for  $T$  in  $C$ . Consequently, for  $\tilde{x} \in \tilde{C}$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|\tilde{x}_n - \tilde{T}(\tilde{x})\| &= \limsup_{n \rightarrow \infty} \left\| \frac{1}{d}(x_n - x_0) - \frac{1}{d}(T(x) - x_0) \right\| \\ &= \frac{1}{d} \limsup_{n \rightarrow \infty} \|x_n - T(x)\| \\ &\leq \frac{1}{d} \limsup_{n \rightarrow \infty} \|x_n - x\| \\ &= \limsup_{n \rightarrow \infty} \left\| \frac{1}{d}(x_n - x_0) - \frac{1}{d}(x - x_0) \right\| \\ &= \limsup_{n \rightarrow \infty} \|\tilde{x}_n - \tilde{x}\|. \end{aligned}$$

Regarding condition (L.a) if  $\tilde{E}$  is a closed convex  $\tilde{T}$ -invariant subset of  $\tilde{C}$  then there exists a closed convex subset  $E$  of  $C$  such that

$$\tilde{E} = \left\{ \frac{1}{d}(x - x_0) : x \in E \right\}.$$

It is obvious that the set  $E$  is closed and convex. We have seen that  $E$  is  $T$ -invariant also. If  $T$  satisfies condition (L.a) then there exists an  $T$ -a.f.p.s.  $(x_n)$  in  $E$ . Therefore,

$$\left( \frac{1}{d}(x_n - x_0) \right)$$

is an a.f.p.s for  $\tilde{T}$  in  $\tilde{E}$ . Thus, the mapping  $\tilde{T}$  satisfies both conditions (L.a) and (L.b).  $\square$

It is clear that  $x^* \in C$  is a fixed point of  $T$  if and only if  $\tilde{x}^* = \frac{1}{d}(x^* - x_0)$  is a fixed point of  $\tilde{T}$  in  $\tilde{C}$ .

**Theorem 3.2.** *Every reflexive OC Banach space has the w-FPP for JE mappings.*

*Proof.* If  $(X, \|\cdot\|)$  fails to have the w-FPP for JE mappings then there exists a nonempty weakly compact convex subset  $K$  of  $X$  and a fixed point free JE mapping  $T : K \rightarrow K$ . We may suppose that  $K$  is minimal for  $T$  (i.e. it does not have any proper convex weakly compact  $T$ -invariant subset). As  $T$  is fixed point free, the set  $K$  is not a singleton. Then,  $d := \text{diam}(K) > 0$ .

Since  $X$  is a reflexive space and  $T$  is JNE, according to [17, Theorem 6], there exists an a.f.p.s.  $(x_n)$  for  $T$  in  $K$ . Passing to a subsequence, if necessary, we may suppose that  $x_n \rightharpoonup x_0 \in K$ .

Consider the weakly compact convex set

$$\tilde{K} := \left\{ \frac{1}{d}(x - x_0) : x \in K \right\},$$

and, for each  $x \in K$  put  $\tilde{x} = \frac{1}{d}(x - x_0) \in \tilde{K}$ . Define  $\tilde{T} : \tilde{K} \rightarrow \tilde{K}$  by

$$\tilde{T}(\tilde{x}) = \frac{1}{d}(T(x) - x_0).$$

1.  $0_X = \frac{1}{d}(x_0 - x_0) \in \tilde{K}$ , and  $\text{diam}(\tilde{K}) = 1$ .
2. The set  $\tilde{K}$  is minimal for  $\tilde{T}$ . Indeed, if  $\tilde{K}_0 \subset \tilde{K}$  is closed convex and  $\tilde{T}$ -invariant, then there exists  $K_0 \subset K$ , such that  $\tilde{K}_0 = \{\frac{1}{d}(x - x_0) : x \in K_0\}$ . If  $x \in K_0$ , then

$$\tilde{T}(\tilde{x}) = \frac{1}{d}(T(x) - x_0) \in \tilde{T}(\tilde{K}_0) \subset \tilde{K}_0.$$

Then there exists  $\tilde{y} \in \tilde{K}_0$  such that  $\tilde{y} = \tilde{T}(\tilde{x})$ , that is,  $\frac{1}{d}(y - x_0) = \frac{1}{d}(T(x) - x_0)$ . Therefore,  $T(x) = y \in K_0$ . Consequently, the set  $K_0$  is closed convex and  $T$  invariant. For minimality,  $K_0 = K$  and hence  $\tilde{K}_0 = \tilde{K}$ , which means that  $\tilde{K}$  is minimal for  $\tilde{T}$ .

According to Lemma (3.1), the mapping  $\tilde{T} : \tilde{K} \rightarrow \tilde{K}$  is JE and fixed-point free. The sequence  $(\tilde{x}_n)$  is an a.f.p. for  $\tilde{T}$  and, of course,  $\tilde{x}_n \rightarrow 0_X \in \tilde{K}$ , and  $\text{diam}(\tilde{K}) = 1$ .

Since  $X$  is a reflexive Banach space, every JNE mapping satisfies condition (A). Consequently,  $T$  satisfies conditions (E) and (A), which from Proposition 2.10 implies that  $T$  is an L-type mapping. Then, from [17, Theorem 5], for every  $\tilde{x} \in \tilde{K}$ ,

$$\|\tilde{x} - \tilde{x}_n\| \rightarrow \text{diam}(K) = 1.$$

One follows easily that  $D[(\tilde{x}_n)] = 1$ .

By hypothesis, there exists  $\lambda > 0$  such that

$$A_\lambda[(\tilde{x}_n)] < 1.$$

Therefore, we can choose  $b < 1$  such that

$$\limsup_{n \rightarrow \infty} \left( \limsup_{m \rightarrow \infty} |M_\lambda(\tilde{x}_n, \tilde{x}_m)| \right) < b$$

and there exists a positive integer  $n_0$  such that, for  $n \geq n_0$ ,

$$\limsup_{m \rightarrow \infty} |M_\lambda(\tilde{x}_n, \tilde{x}_m)| < b.$$

For any fixed  $n \geq n_0$  we can get a positive integer  $m = m(n)$  such that  $m(n) < m(n+1)$  and

$$|M_\lambda(\tilde{x}_n, \tilde{x}_m)| < b$$

for  $m \geq m(n)$ .

We have now the bounded sequence of non-negative real numbers

$$(\|\tilde{x}_n - \tilde{x}_{m(n)}\|)$$

and (by passing to a subsequence if necessary) we can suppose that there exists

$$\lim_{n \rightarrow \infty} \|\widetilde{x}_n - \widetilde{x_{m(n)}}\| =: L$$

Applying the Maurey-type theorem [17, Theorem 7] we obtain a sequence  $(z_n)$  in  $\widetilde{K}$  such that

$$z_n - T(z_n) \rightarrow 0_X, \quad \|z_n - \widetilde{x}_n\| \rightarrow \frac{1}{2}L, \quad \|z_n - \widetilde{x_{m(n)}}\| \rightarrow \frac{1}{2}L.$$

Then, there exists  $n_1 \geq n_0$  such that, for  $n \geq n_1 \geq n_0$ ,

$$\max \{ \|z_n - \widetilde{x}_n\|, \|z_n - \widetilde{x_{m(n)}}\| \} \leq \frac{1 + \lambda}{2} \|\widetilde{x}_n - \widetilde{x_{m(n)}}\|.$$

Therefore,

$$z_n \in M_\lambda(\widetilde{x}_n, \widetilde{x_{m(n)}}).$$

It follows that

$$\|z_n\| \leq |M_\lambda(\widetilde{x}_n, \widetilde{x_{m(n)}})| < b < 1.$$

However, since  $\widetilde{0}_X \in \widetilde{K}$  and  $(z_n)$  is an a.f.p.s., from [17, Theorem 5],

$$\|z_n\| = \|0_K - z_n\| \rightarrow 1,$$

which yields a contradiction, completing the proof.  $\square$

**Corollary 3.3.** *The following statements hold.*

- (1) *The reflexive spaces  $E_\beta$  (see Example 2.2) have the FPP for JE mappings.*
- (2) *If  $F$  is a Banach space such that  $d(F, \ell_p) < 2^{1-1/p}$  for some  $p \in (1, \infty)$  then  $F$  has the FPP for JE mappings.*
- (3) *If  $(X, \|\cdot\|)$  is a reflexive Banach space with a  $\lambda$ -unconditional basis and with characteristic of convexity  $\varepsilon_0$ , then it has the FPP for JE mappings whenever  $\lambda/2 + \varepsilon_0/4 < 1$ .*

*Proof.* All of these spaces are OC. See [7, Corollary 1] for Statement (2) and [6, Corollary 1] for Statement (3).  $\square$

**Remark 3.4.** Regarding Example 2.2, we have seen that the space  $E_\beta$  is OC, but it lacks NS for  $\beta > \sqrt{2}$ . According to Kassay [11], there exists a weakly compact convex subset  $C$  of  $\ell_2$  and a JNE mapping  $T : C \rightarrow C$  that is fixed-point-free. Consequently, the OC is not enough to guarantee the fixed point property for JNE mappings.

Similarly, according to Piatek [24, Theorem 4.3], in this space  $E_\beta$  for  $\beta > \sqrt{2}$ , that is, OC, there exists a weakly compact convex set, say  $K$ , and a continuous L-type self-mapping of  $K$  that is fixed-point free. Thus, the property OC is not enough to guarantee the fixed point property for L-type mappings.

## 4. CONCLUSIONS

In Banach spaces lacking normal structure, there exists weakly compact convex subsets admitting fixed point free self-mappings satisfying either Jaggi's condition or the condition (L). However, for mappings satisfying simultaneously both conditions, we have presented several fixed point results, mainly within the framework of reflexive Banach spaces. We have shown that such mappings possess the weak fixed point property in orthogonally convex Banach spaces. Orthogonal convexity is an isometric property of the norm known to ensure the fixed point property for nonexpansive mappings. It is satisfied by a wide class of Banach spaces, but it neither implies nor is implied by normal structure

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