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# RADIAL SOLUTIONS FOR A NEUMANN ELLIPTIC SYSTEM WITH QUADRATIC GROWTH IN THE GRADIENT

Filomena Cianciaruso and Paolamaria Pietramala

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**Abstract.** We prove the existence of multiple positive solutions for elliptic systems with linear boundary conditions of Neumann type. We suppose that the nonlinearities grow quadratically with respect to gradient. A key step is to obtain a priori bound on the derivatives by using a Gronwall-type inequality. Our approach is topological and relies on the fixed point index.

**Keywords:** elliptic system, annular domain, radial solution, fixed point index, linear Neumann boundary conditions, Gronwall inequality.

**Mathematics Subject Classification:** 35B07, 34B18, 35J57, 47H10.

## 1. INTRODUCTION

In this paper we focus on the ellictic problem in $\mathbb{R}^n$, $n \geq 3$,

$$\begin{cases} -\Delta u = \tilde{f}_1(|x|, u, v, |\nabla u|, |\nabla v|) \quad \text{in } E, \\ -\Delta v = \tilde{f}_2(|x|, u, v, |\nabla u|, |\nabla v|) \quad \text{in } E, \\ \dfrac{\partial u}{\partial r} = 0 \text{ for } |x| = R_0, \quad \dfrac{\partial u}{\partial r} + \beta_1[u] = 0 \text{ for } |x| = R_1, \\ \dfrac{\partial v}{\partial r} = 0 \text{ for } |x| = R_0, \quad \dfrac{\partial v}{\partial r} + \beta_2[v] = 0 \text{ for } |x| = R_1, \end{cases} \tag{1.1}$$

where

$$E = \{x \in \mathbb{R}^n : R_0 < |x| < R_1\}, \quad 0 < R_0 < R_1 < +\infty,$$

is an annulus, the functions $\tilde{f}_i$ are continuous, $\beta_i$ are linear functionals and $\frac{\partial}{\partial r}$ denotes (as in [11]) differentiation in the radial direction $r = |x|$. The problem with Neumann boundary conditions has been an object of interest by a number of authors, see for example [1–7, 20–22, 26] and the existence of solutions for this problem is investigated via different tools. The approach that we want to use is topological and relies on fixed point index theory. In fact, under suitable transformations, radially symmetric solutions of a PDE can correspond to the solutions of an associated ordinary differential problem and therefore of a Hammerstein integral equation, see for example [7, 9, 10, 12–14, 16–19]. Nevertheless, when dealing with the existence of radial solutions of the Neumann problem, the linear part of a local associated ordinary differential problem is not invertible and the corresponding Green's function does not exist, but it is possible for the strictly nonlocal problem, i.e. $\beta_i[w] \neq 0$, to have a Green's function.

The boundary conditions in our problem are nonlocal and can be read as some kind of feedback mechanisms. For example, the single associate ordinary differential equation under Neumann boundary conditions corresponds to a thermostat model for a heated bar occupying the interval $[0, 1]$, with the end at 0 insulated and a controller at 1 adding or removing heat according to feedback received from measurements of the temperature along some part of the interval (see [23]).

For problems with derivative dependence it is often supposed that the nonlinearities $\tilde{f}_i$ satisfy a Nagumo condition or the method of upper and lower solutions is used. The Gronwall inequality is a well-known tool in the study of differential equations and of Volterra integral equations in establishing a priori bounds. We make use a Gronwall-type inequality due to Webb ([24, 25]), which, when applied to second order ODEs, allows quadratic growth in the derivative and can be regarded as a possible replacement of a Nagumo condition in suitable circumstances. Moreover, even in situations where other methods are available one can give explicit bounds and one covers cases where no systematic methods are known.

In our problem we don't need the assumption that $\tilde{f}_i$ are sublinear or superlinear, and we don't require any monitonicity hypotheses, but we assume conditions, relate to some constants depending on the kernel and on the nonlocal BCs, on the upper and lower bounds of the nonlinearity on boxes. Under these oscillation assumptions the multiplicity of positive solutions is obtained.

As far as we know, our results seems to be the first one proving multiplicity of positive solutions for this kind of problems that combine some topics in the literature, in particular a system, the derivative dependence and the nonlocal Neumann boundary conditions.

The outline of our paper is as follows. In Section 2 we construct the associated integral equation, state the properties of the kernel and related to the BCs, and we obtain the priori bound on the derivative of the solution. In Section 3 we give existence and multiplicity results for the problem (1.1), and we briefly illustrate our theory by an example.

## 2. PRELIMINARIES AND SETTING

### 2.1. THE ASSOCIATE ORDINARY DIFFERENTIAL PROBLEM

We consider in $\mathbb{R}^n$, $n \geq 3$, the equation

$$-\triangle w = \tilde{f}(|x|, w, |\nabla w|) \quad \text{in } E. \tag{2.1}$$

Since we are looking for the existence of positive radial solutions $w = w(r)$, $r = |x|$ of the problem (1.1), we rewrite (2.1) for $w$ in polar coordinates as

$$-w''(r) - \frac{n-1}{r} w'(r) = \tilde{f}(r, w(r), |w'(r)|) \quad \text{in } [R_0, R_1]. \tag{2.2}$$

Set $w(t) := w(r(t))$, where, for $t \in [0,1]$,

$$r(t) := \left(\frac{A}{B-t}\right)^{\frac{1}{n-2}},$$

$$A = \frac{(R_0R_1)^{n-2}}{R_1^{n-2} - R_0^{n-2}} \quad \text{and} \quad B = \frac{R_1^{n-2}}{R_1^{n-2} - R_0^{n-2}}$$

(see [8, 9]). Take, for $t \in [0,1]$, the increasing function

$$j(t) := \left(\frac{R_0R_1\left(R_1^{n-2} - R_0^{n-2}\right)}{n-2}\right)^2 \frac{1}{\left(R_1^{n-2} - (R_1^{n-2} - R_0^{n-2})t\right)^{\frac{2(n-1)}{n-2}}}.$$

Thus, the equation (2.2) becomes

$$-w''(t) = j(t)\tilde{f}\left(r(t), w(t), \left|\frac{w'(t)}{r'(t)}\right|\right) := j(t) f\left(t, w(t), \left|\frac{w'(t)}{r'(t)}\right|\right).$$

Hence, to the elliptic problem

$$\begin{cases} -\Delta u = \tilde{f}_1(|x|, u, v, |\nabla u|, |\nabla v|) \text{ in } E, \\ -\Delta v = \tilde{f}_2(|x|, u, v, |\nabla u|, |\nabla v|) \text{ in } E, \\ \dfrac{\partial u}{\partial r} = 0 \text{ for } |x| = R_0, \quad \dfrac{\partial u}{\partial r} + \beta_1[u] = 0 \text{ for } |x| = R_1, \\ \dfrac{\partial v}{\partial r} = 0 \text{ for } |x| = R_0, \quad \dfrac{\partial v}{\partial r} + \beta_2[v] = 0 \text{ for } |x| = R_1 \end{cases} \tag{2.3}$$

we can associate the ordinary boundary value problem

$$\begin{cases} -u''(t) = j(t) f_1\left(t, u(t), v(t), \left|\frac{u'(t)}{r'(t)}\right|, \left|\frac{v'(t)}{r'(t)}\right|\right), \quad t \in (0,1), \\ -v''(t) = j(t) f_2\left(t, u(t), v(t), \left|\frac{u'(t)}{r'(t)}\right|, \left|\frac{v'(t)}{r'(t)}\right|\right), \quad t \in (0,1), \\ u'(0) = 0, \quad u'(1) + \frac{R_1^{n-1}}{A(n-2)}\beta_1[u] = 0, \\ v'(0) = 0, \quad v'(1) + \frac{R_1^{n-1}}{A(n-2)}\beta_2[v] = 0. \end{cases} \tag{2.4}$$

By a *radial solution* of the problem (2.3) we mean a solution $(u, v)$ of the boundary value problem (2.4). We remark that the functions $u, v$ are concave, i.e. the derivatives $u', v'$ are decreasing; moreover, since $u'(0) = v'(0) = 0$, also $u, v$ are decreasing.

2.2. THE ASSOCIATED INTEGRAL SYSTEM

We study the existence of positive solutions to the problem (2.4) by means of an associate suitable Hammerstein integral system. When dealing with the existence of radial solutions of the Neumann problem, the homogenous problem is at resonance and the Green's function does not exist; nevertheless in our case it would have to be found by another method. We denote by $h$ the function

$$h(t,s) := \begin{cases} t-s, & \text{if } t \geq s, \\ 0, & \text{if } t < s, \end{cases}$$

and by $N$ the positive constant

$$N := \frac{R_1^{n-1}}{A(n-2)}.$$

**Proposition 2.1.** *Suppose that $\beta[1] \neq 0$, where $\beta$ is a linear functional. Let $y \in L^1[0,1]$. Then $w$ is a solution of the boundary value problem*

$$-w''(t) = y(t), \text{ for a.e. } t \in (0,1), \quad w'(0) = 0, \quad w'(1) + N\beta[w] = 0 \tag{2.5}$$

*if and only if $w \in \mathcal{C}^1[0,1]$ is a solution of*

$$w(t) = \int_0^1 G(t,s)y(s)ds,$$

*with*

$$G(t,s) := \frac{1}{\beta[1]}\left(\frac{1}{N} + \beta[(h(\cdot,s)]\right) - h(t,s).$$

*Proof.* Suppose that $w$ is a solution of the boundary value problem (2.5). By integration, we have

$$w'(t) = w'(0) - \int_0^t y(s)ds = -\int_0^t y(s)ds.$$

Set $t = 1$, we have

$$N\beta[w] = \int_0^1 y(s)ds. \tag{2.6}$$

Integrating again gives

$$\begin{aligned} w(t) &= w(0) - \int_0^t \left(\int_0^s y(\tau)d\tau\right) ds \\ &= w(0) - \int_0^t (t-s)y(s)ds = w(0) - \int_0^1 h(t,s)y(s)ds. \end{aligned} \tag{2.7}$$

Applying the functional $\beta$ to (2.7), we have

$$\beta[w] = w(0)\beta[1] - \int_0^1 \beta[h(\cdot, s)]y(s)ds.$$

Using (2.6) we obtain

$$w(0)\beta[1] = \frac{1}{N}\int_0^1 y(s)ds + \int_0^1 \beta[h(\cdot, s)]y(s)ds.$$

Substituting $w(0)$ in (2.7)

$$\begin{aligned} w(t) &= \frac{1}{N\beta[1]}\int_0^1 y(s)ds + \frac{1}{\beta[1]}\int_0^1 \beta[h(\cdot, s)]y(s)ds - \int_0^1 h(t,s)y(s)ds \\ &= \int_0^1 \left(\frac{1}{N\beta[1]} + \frac{1}{\beta[1]}\beta[h(\cdot, s)] - h(t,s)\right) y(s)ds. \end{aligned}$$

□

Thus, we associate to the problem (2.4) the Hammerstein integral system

$$\begin{cases} u(t) = \int_0^1 G_1(t,s)j(s)f_1\left(s, u(s), v(s), \left|\frac{u'(s)}{r'(s)}\right|, \left|\frac{v'(s)}{r'(s)}\right|\right) ds, & t \in [0,1], \\ v(t) = \int_0^1 G_2(t,s)j(s)f_2\left(s, u(s), v(s), \left|\frac{u'(s)}{r'(s)}\right|, \left|\frac{v'(s)}{r'(s)}\right|\right) ds, & t \in [0,1], \end{cases} \tag{2.8}$$

where the Green's functions $G_i$ are given by

$$G_i(t,s) := \frac{1}{\beta_i[1]}\left(\frac{A(n-2)}{R_1^{n-1}} + \beta_i[h(\cdot, s)]\right) - h(t,s).$$

## 2.3. THE INTEGRAL OPERATOR

From now on, we assume that

(a) the functions $\tilde{f}_i : [R_0, R_1] \times [0, +\infty)^4 \to [0, +\infty[$ are continuous,
(b) the linear bounded functionals $\beta_i : C^1[0,1] \to \mathbb{R}$ are positive, i.e. $\beta_i[w] \geq 0$ for $w \geq 0$,
(c) $\frac{A(n-2)}{R_1^{n-1}} + \beta_i[h(\cdot, s)] \geq \beta_i[1](1-s)$ for $s \in [0,1]$.

The condition (c) is necessary to have $G_i(t,s) \geq 0$. Moreover, one has, for $t,s \in [0,1]$,

$$c(t)\phi_i(s) \leq G_i(t,s) \leq \phi_i(s),$$

where

$$c(t) = 1 - t \quad \text{and} \quad \phi_i(s) = \frac{\frac{A(n-2)}{R_1^{n-1}} + \beta_i[h(\cdot,s)]}{\beta_i[1]}.$$

We study the existence of solutions of the system (2.8) by means of the fixed points of a suitable operator on the space $C^1[0,1] \times C^1[0,1]$ equipped with the norm

$$\|(u,v)\| := \max\{\|u\|_{C^1}, \|v\|_{C^1}\},$$

where

$$\|w\|_{C^1} := \max\{\|w\|_\infty, \|w'\|_\infty\} \quad \text{and} \quad \|y\|_\infty := \max_{t\in[0,1]} |y(t)|.$$

If $(u,v)$ is a positive solution of our problem, the functions $u,v$ are concave and decreasing, then it follows that $u(t) \geq (1-t)\|u\|_\infty$ and $v(t) \geq (1-t)\|v\|_\infty$ for $t \in [0,1]$. Motivated by this fact, for $i = 1,2$ we consider the cone in $C^1[0,1]$

$$\mathcal{K}_i = \{w \in C^1[0,1] : w(t) \geq c_{a_i,b_i}\|w\|_\infty \text{ for } t \in [a_i,b_i]\},$$

where $[a_i,b_i]$ is a subinterval of $[0,1)$ and

$$c_{a_i,b_i} := \min_{t\in[a_i,b_i]} c(t) = 1 - b_i > 0,$$

and the cone in $C^1[0,1] \times C^1[0,1]$

$$\mathcal{K} := \{(u,v) \in \mathcal{K}_1 \times \mathcal{K}_2\}.$$

The well-defined integral operator $T : C^1[0,1] \times C^1[0,1] \to C^1[0,1] \times C^1[0,1]$ given by

$$\begin{aligned} T(u,v)(t) &:= \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix} \\ &= \begin{pmatrix} \int_0^1 G_1(t,s)j(s)f_1\left(s,u(s),v(s),\left|\frac{u'(s)}{r'(s)}\right|,\left|\frac{v'(s)}{r'(s)}\right|\right)\,ds \\ \int_0^1 G_2(t,s)j(s)f_2\left(s,u(s),v(s),\left|\frac{u'(s)}{r'(s)}\right|,\left|\frac{v'(s)}{r'(s)}\right|\right)\,ds \end{pmatrix} \end{aligned}$$

satisfies the following result:

**Theorem 2.2.** *The operator $T$ leaves the cone $\mathcal{K}$ invariant and is completely continuous.*

*Proof.* In order to prove that $T$ leaves the cone $\mathcal{K}$ invariant, take $r > 0$ and $(u, v) \in \mathcal{K}$ such that $\|(u, v)\| \leq r$. Then we have, for $t \in [0, 1]$,

$$0 \leq T_i(u,v)(t) \leq \int_0^1 \phi_i(s)j(s)f_i\left(s, u(s), v(s), \left|\frac{u'(s)}{r'(s)}\right|, \left|\frac{v'(s)}{r'(s)}\right|\right) ds, \tag{2.9}$$

so that

$$\|T_i(u,v)\|_\infty \leq \int_0^1 \phi_i(s)j(s)f_i\left(s, u(s), v(s), \left|\frac{u'(s)}{r'(s)}\right|, \left|\frac{v'(s)}{r'(s)}\right|\right) ds.$$

On the other hand, we have, for $t \in [a_i, b_i]$,

$$\begin{aligned} T_i(u,v)(t) &\geq c(t)\int_0^1 \phi_i(s)j(s)f_i\left(s, u(s), v(s), \left|\frac{u'(s)}{r'(s)}\right|, \left|\frac{v'(s)}{r'(s)}\right|\right) ds \\ &\geq c(t)\|T_i(u,v)\|_\infty \geq c_{a_i,b_i}\|T_i(u,v)\|_\infty. \end{aligned}$$

In order to prove the complete continuity of $T$, let us firstly note that the continuity of $f_i$, $j$, $G_i$ and $\beta_i$ give the continuity of $T$. Let $U$ be a bounded subset of $\mathcal{K}$; it follows from (2.9) that $T(U)$ is bounded in $\mathcal{K}$. It is a standard argument based on the uniform continuity of the kernels $G_i$ and on Ascoli–Arzelá Theorem that $T(U)$ is relatively compact in $C^1[0,1] \times C^1[0,1]$. □

## 2.4. A PRIORI BOUNDS ON THE DERIVATIVE

We will use the following Gronwall-type inequality due to Webb [24]:

**Lemma 2.3** (Gronwall-type inequality)**.** *Let $w \in L^\infty_+[0, t_0]$. Suppose that there are a constant $\alpha_0 > 0$ and two positive functions $\alpha_1, \alpha_2 \in L^1[0, t_0]$ such that*

$$w(t) \leq \alpha_0 + \int_0^t \alpha_1(s)w(s)ds + \int_0^t \alpha_2(s)w^2(s)ds \text{ for a.e. } t \in [0, t_0]$$

*and suppose it is known that there is a constant $R > 0$ such that*

$$\int_0^{t_0} \alpha_2(s)w(s)ds \leq R.$$

*Then we have*

$$w(t) \leq \alpha_0 e^R e^{\int_0^t \alpha_1(s)ds} \text{ for a.e. } t \in [0, t_0].$$

We can now give a priori bounds on the derivative of the variables when the nonlinearities have quadratic growth in the derivative term and arbitrary growth with respect one of the variables.

**Theorem 2.4.** *For $i = 1, 2$, suppose that there exist two constants $c_i$, $p_i \geq 0$ and two functions $h_i, l_i \in L^1_+[0,1]$ and a non-negative continuous increasing function $g_i$ such that*

$$f_i(t, w_1, w_2, z_1, z_2) \leq c_i + h_i(t)g_i(w_i) + l_i(t)z_i + p_i z_i^2 \quad \text{in } [0,1] \times [0,+\infty)^4. \tag{2.10}$$

*Let $\lambda \geq 1$, $\sigma_i \geq 0$, $\rho_i > 0$ and two non-negative functions $u_i \in C^1[0,1]$ such that $\|u_i\|_\infty = \rho_i$ and $\lambda(u_1, u_2)(t) = T(u_1, u_2)(t) + (\sigma_1, \sigma_2)$. Then, for $t \in [0,1]$,*

$$|u_i'(t)| \leq j(1)\left(c_i + g_i(\rho_i)\int_0^1 h_i(s)ds\right) e^{\frac{p_i j(1)}{(r'(0))^2}\rho_i} e^{\int_0^1 \frac{j(s)l_i(s)}{|r'(s)|}ds} := Q_i(\rho_i).$$

*Proof.* For $i = 1, 2$, from the hypotheses we have

$$\lambda u_i''(t) = (T_i(u_1, u_2))''(t) = -j(t)\, f_i\left(t, u_1(t), u_2(t), \frac{|u_1'(t)|}{|r'(t)|}, \frac{|u_2'(t)|}{|r'(t)|}\right).$$

Thus,

$$\lambda u_i'(t) = -\int_0^t j(s)\, f_i\left(s, u_1(s), u_2(s), \frac{|u_1'(s)|}{|r'(s)|}, \frac{|u_2'(s)|}{|r'(s)|}\right) ds,$$

so $u_i'(t) \leq 0$ for $t \in [0,1]$. Then, since $\lambda \geq 1$, we have

$$\begin{aligned}
|u_i'(t)| \leq \lambda|u_i'(t)| &\leq \int_0^t j(s)\, f_i\left(s, u_1(s), u_2(s), \frac{|u_1'(s)|}{|r'(s)|}, \frac{|u_2'(s)|}{|r'(s)|}\right) ds \\
&\leq \int_0^t j(s)\left(c_i + h_i(s)g_i(u_i(s)) + l_i(s)\frac{|u_i'(s)|}{|r'(s)|} + p_i\frac{|u_i'(s)|^2}{|r'(s)|^2}\right) ds \\
&\leq j(1)\left(c_i + g_i(\rho_i)\int_0^1 h_i(s)ds\right) + \int_0^t \left(\frac{j(s)l_i(s)}{|r'(s)|}\,|u_i'(s)| + \frac{p_i j(s)}{|r'(s)|^2}\,|u_i'(s)|^2\right) ds.
\end{aligned}$$

Because we have

$$\begin{aligned}
\int_0^1 \frac{p_i j(s)}{|r'(s)|^2}\,|u_i'(s)|ds = \int_0^1 \frac{p_i j(s)}{|r'(s)|^2}\,(-u_i'(s))ds &\leq \frac{p_i j(1)}{(r'(0))^2}\int_0^1 (-u_i'(s))ds \\
&= \frac{p_i j(1)}{(r'(0))^2}\,(u_i(0) - u_i(1)) \\
&\leq \frac{p_i j(1)}{(r'(0))^2}u_i(0) \leq \frac{p_i j(1)}{(r'(0))^2}\|u_i\|_\infty = \frac{p_i j(1)}{(r'(0))^2}\rho_i,
\end{aligned}$$

we can apply Lemma 2.3 to the function $|u_i'|$, and we obtain that

$$|u_i'(t)| \leq j(1)\left(c_i + g_i(\rho_i)\int_0^1 h_i(s)ds\right) e^{\frac{p_i j(1)}{(r'(0))^2}\rho_i} e^{\int_0^1 \frac{j(s)l_i(s)}{|r'(s)|}ds} := Q_i(\rho_i). \qquad \square$$

## 3. INDEX CALCULATIONS

In this section we study the existence of multiple solutions of the integral equation

$$(u,v)(t) = T(u,v)(t), \quad t \in [0,1].$$

The following theorem follows from classical results about fixed point index (see, for example, [15]).

**Theorem 3.1.** *Let $K$ be a cone in an ordered Banach space $X$. Let $\Omega$ be an open bounded subset with $0 \in \Omega \cap K$ and $\overline{\Omega \cap K} \neq K$. Let $\Omega^1$ be open in $X$ with $\overline{\Omega^1} \subset \Omega \cap K$. Let $F : \overline{\Omega \cap K} \to K$ be a compact map. Suppose that:*

(1) *$Fx \neq \mu x$ for all $x \in \partial(\Omega \cap K)$ and for all $\mu \geq 1$,*
(2) *there exists $h \in K \setminus \{0\}$ such that $x \neq Fx + \lambda h$ for all $x \in \partial(\Omega^1 \cap K)$ and all $\lambda > 0$.*

*Then $F$ has at least one fixed point $x \in (\Omega \cap K) \setminus \overline{(\Omega^1 \cap K)}$. Denoting by $i_K(F,U)$ the fixed point index of $F$ in some $U \subset X$, we have*

$$i_K(F, \Omega \cap K) = 1 \quad \text{and} \quad i_K(F, \Omega^1 \cap K) = 0\,.$$

*The same result holds if*

$$i_K(F, \Omega \cap K) = 0 \quad \text{and} \quad i_K(F, \Omega^1 \cap K) = 1.$$

For the index calculations, as in [25] we shall use the open bounded sets (relative to $\mathcal{K}$), namely, for $\rho_1, \rho_2 > 0$,

$$K_{\rho_1,\rho_2} := \{(u_1,u_2) \in \mathcal{K} : \|u_i\|_\infty < \rho_i,\ \|u_i'\|_\infty < Q_i(\rho_i) + 1,\ \text{for } i = 1,2\}$$

and

$$V_{\rho_1,\rho_2} := \left\{(u_1,u_2) \in \mathcal{K} : \min_{t \in [a_i,b_i]} u_i(t) < \rho_i,\ \|u_i'\|_\infty < Q_i\left(\frac{\rho_i}{c_{a_i,b_i}}\right) + 1,\ \text{for } i = 1,2\right\}.$$

Since the functions $Q_i$ are increasing, the sets defined above have the following properties:

($P_1$) $K_{\rho_1,\rho_2} \subset V_{\rho_1,\rho_2} \subset K_{\frac{\rho_1}{c_{a_1,b_1}},\frac{\rho_2}{c_{a_2,b_2}}}$,
($P_2$) $(u_1,u_2) \in \partial K_{\rho_1,\rho_2}$ if and only if $(u_1,u_2) \in \mathcal{K}$ and belongs to

$$\bigcup_{i=1,2} \{\|u_i\|_\infty = \rho_i,\ c_{a_i,b_i}\rho_i \leq u_i(t) \leq \rho_i \text{ for } t \in [a_i,b_i], \|u_i'\|_\infty \leq Q_i(\rho_i) + 1\}$$
$$\cup \bigcup_{i=1,2} \{\|u_i\|_\infty \leq \rho_i \text{ and } \|u_i'\|_\infty = Q_i(\rho_i) + 1\},$$

$(P_3)$ $(u_1,u_2)\in\partial V_{\rho_1,\rho_2}$ if and only if $(u_1,u_2)\in\mathcal{K}$ and belongs to

$$\bigcup_{i=1,2}\left\{\min_{t\in[a_i,b_i]}u_i(t)=\rho_i,\ \|u_i'\|_\infty\le Q_i\left(\frac{\rho_i}{c_{a_i,b_i}}\right)+1\right\}$$
$$\cup\bigcup_{i=1,2}\left\{\min_{t\in[a_i,b_i]}u_i(t)\le\rho_i,\ \|u_i'\|_\infty= Q_i\left(\frac{\rho_i}{c_{a_i,b_i}}\right)+1\right\}.$$

Set

$$\frac{1}{m_i}=\max_{t\in[0,1]}\int_0^1 G_i(t,s)j(s)ds \quad\text{and}\quad \frac{1}{M_i}=\min_{t\in[a_i,b_i]}\int_{a_i}^{b_i}G_i(t,s)j(s)ds$$

and define the following sets:

$$\Omega_1^{s_1,s_2}=[0,1]\times[c_{a_1,b_1}s_1,s_1]\times[0,s_2]\times\left[0,\frac{Q_1(s_1)}{r'(0)}\right]\times\left[0,\frac{Q_2(s_2)}{r'(0)}\right],$$
$$\Omega_2^{s_1,s_2}=[0,1]\times[0,s_1]\times[c_{a_2,b_2}s_2,s_2]\times\left[0,\frac{Q_1(s_1)}{r'(0)}\right]\times\left[0,\frac{Q_2(s_2)}{r'(0)}\right],$$
$$A_1^{s_1,s_2}=[a_1,b_1]\times\left[s_1,\frac{s_1}{c_{a_1,b_1}}\right]\times\left[0,\frac{s_2}{c_{a_2,b_2}}\right]\times\left[0,\frac{Q_1\left(\frac{s_1}{c_{a_1,b_1}}\right)}{r'(0)}\right]\times\left[0,\frac{Q_2\left(\frac{s_2}{c_{a_2,b_2}}\right)}{r'(0)}\right],$$
$$A_2^{s_1,s_2}=[a_2,b_2]\times\left[0,\frac{s_1}{c_{a_1,b_1}}\right]\times\left[s_2,\frac{s_2}{c_{a_2,b_2}}\right]\times\left[0,\frac{Q_1\left(\frac{s_1}{c_{a_1,b_1}}\right)}{r'(0)}\right]\times\left[0,\frac{Q_2\left(\frac{s_2}{c_{a_2,b_2}}\right)}{r'(0)}\right].$$

**Theorem 3.2.** *For $i=1,2$, suppose that the function $f_i$ satisfies the condition* (2.10)*. Moreover, assume that for some $\rho_i,\sigma_i,\tau_i>0$ with $\rho_i<\sigma_i$ and $\frac{\sigma_i}{c_{a_i,b_i}}<\tau_i$ we have*

$$\begin{aligned} f_i(t,w_1,w_2,z_1,z_2)&<m_i\rho_i &&\text{in } \Omega_i^{\rho_1,\rho_2},\\ f_i(t,w_1,w_2,z_1,z_2)&>M_i\sigma_i &&\text{in } A_i^{\sigma_1,\sigma_2},\\ f_i(t,w_1,w_2,z_1,z_2))&<m_i\tau_i &&\text{in } \Omega_i^{\tau_1,\tau_2}.\end{aligned}$$

*Then the system*

$$\begin{cases} -u''(t)=j(t)f_1\left(t,u(t),v(t),\left|\frac{u'(t)}{r'(t)}\right|,\left|\frac{v'(t)}{r'(t)}\right|\right), & t\in(0,1),\\ -v''(t)=j(t)f_2\left(t,u(t),v(t),\left|\frac{u'(t)}{r'(t)}\right|,\left|\frac{v'(t)}{r'(t)}\right|\right), & t\in(0,1),\\ u'(0)=0,\quad u'(1)+\frac{R_1^{n-1}}{A(n-2)}\beta_1[u]=0,\\ v'(0)=0,\quad v'(1)+\frac{R_1^{n-1}}{A(n-2)}\beta_2[v]=0\end{cases}\tag{3.1}$$

*admits at least two positive solutions in $\mathcal{K}$.*

*Proof.* The choice of the numbers $\rho_i$, $\sigma_i$ and $\tau_i$ assures the compatibility of conditions on the nonlinearities $f_i$ and the inclusions

$$K_{\rho_1,\rho_2}\subset V_{\sigma_1,\sigma_2}\subset K_{\frac{\sigma_1}{c_{a_1,b_1}},\frac{\sigma_2}{c_{a_2,b_2}}}\subset K_{\tau_1,\tau_2}.$$

We want to show that

$$i_{\mathcal{K}}(T, K_{\rho_1,\rho_2}) = 1, \quad i_{\mathcal{K}}(T, V_{\sigma_1,\sigma_2}) = 0 \quad \text{and} \quad i_{\mathcal{K}}(T, K_{\tau_1,\tau_2}) = 1.$$

From Theorem 3.1 it follows that the operator $T$ has a fixed point in $V_{\sigma_1,\sigma_2} \setminus \overline{K_{\rho_1,\rho_2}}$ and a fixed point in $K_{\tau_1,\tau_2} \setminus \overline{V_{\sigma_1,\sigma_2}}$. Then the system (3.1) admits at least two positive solutions in $\mathcal{K}$.

Firstly we claim that

$$\lambda(u, v) \neq T(u, v) \text{ for } \lambda \geq 1 \quad \text{and} \quad (u, v) \in \partial K_{\rho_1,\rho_2}$$

that ensures, by Theorem 3.1, that the index is 1 on $K_{\rho_1,\rho_2}$. Assume this is not true, then there exist $(u, v) \in \partial K_{\rho_1,\rho_2}$ and $\lambda \geq 1$ such that $\lambda(u, v) = T(u, v)$. From Property $(P_2)$ derive that there are four cases on the estimates of the involved norms. Without loss generality we can consider only the cases that involve the variable $u$. If $\|u\|_\infty \leq \rho_1$, by Theorem 2.4 we have $\|u'\|_\infty \leq Q_1(\rho_1) < Q_1(\rho_1) + 1$, and so we must have only the case $\|u\|_\infty = \rho_1$ and $|u'(t)| \leq Q_1(\rho_1)$ for $t \in [0, 1]$. Now we have

$$\begin{aligned} \lambda\, u(t) &= \int_0^1 G_1(t, s) j(s) f_1 \left( s, u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|} \right) ds \\ &< m_1 \rho_1 \int_0^1 G_1(t, s) j(s) ds \leq \rho_1 \end{aligned}$$

and taking the maximum over $[0, 1]$ we can conclude that $\lambda\rho_1 < \rho_1$. This contradicts the fact that $\lambda \geq 1$ and proves the result. Note that the constant function $e(t) \equiv 1$ for $t \in [0, 1]$ belongs to $\mathcal{K}_i$. Now we claim that

$$(u, v) \neq T(u, v) + \lambda(e, e) \text{ for } (u, v) \in \partial V_{\sigma_1,\sigma_2} \quad \text{and} \quad \lambda \geq 0,$$

which ensures, by Theorem 3.1, that the index is 0 on the set $V_{\sigma_1,\sigma_2}$. If this is not true, then there exist $(u, v) \in \partial V_{\sigma_1,\sigma_2}$ and $\lambda \geq 0$ such that $(u, v) = T(u, v) - \lambda(e, e)$. From Property $(P_3)$ derive that there are four cases that involve the variables. Without loss generality we can consider only the cases that involve the variable $u$. Since $(u, v) \in \partial V_{\sigma_1,\sigma_2}$, we have $\min_{t\in[a_1,b_1]} u(t) \leq \sigma_1$ and $\|u\|_\infty \leq \frac{\sigma_1}{c_{a_1,b_1}}$. By Proposition 2.4, this implies

$$|u'(t)| \leq Q_1 \left( \frac{\sigma_1}{c_{a_1,b_1}} \right) < Q_1 \left( \frac{\sigma_1}{c_{a_1,b_1}} \right) + 1.$$

Thus, we must have only the case $\min_{t\in[a_1,b_1]} u(t) = \sigma_1$, hence $\sigma_1 \le u(t) \le \frac{\sigma_1}{c_{a_1,b_1}}$ for $t \in [a_1, b_1]$, and $\|u'\|_\infty \le Q_1\left(\frac{\sigma_1}{c_{a_1,b_1}}\right)$. So we have

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,s)j(s)f_1\left(s, u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds + \lambda \\ &\ge \int_{a_1}^{b_1} G_1(t,s)j(s)f_1\left(s, u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds + \lambda \\ &> M_1\sigma_1 \int_{a_1}^{b_1} G_1(t,s)j(s)ds + \lambda \ge \sigma_1 + \lambda. \end{aligned}$$

Taking the minimum on $[a_1, b_1]$ gives the contradiction $\sigma_1 > \sigma_1 + \lambda$. The proof that the index of $T$ is 1 on $K_{\tau_1,\tau_2}$ is as in the first step. □

Define the following sets:

$$\begin{aligned} \tilde{\Omega}_1^{s_1,s_2} &= [R_0, R_1] \times [c_{a_1,b_1}s_1, s_1] \times [0, s_2] \times [0, Q_1(s_1)] \times [0, Q_2(s_2)], \\ \tilde{\Omega}_2^{s_1,s_2} &= [R_0, R_1] \times [0, s_1] \times [c_{a_2,b_2}s_2, s_2] \times [0, Q_1(s_1)] \times [0, Q_2(s_2)], \\ \tilde{A}_1^{s_1,s_2} &= [r(a_1), r(b_1)] \times \left[s_1, \frac{s_1}{c_{a_1,b_1}}\right] \times \left[0, \frac{s_2}{c_{a_2,b_2}}\right] \\ &\qquad \times \left[0, Q_1\left(\frac{s_1}{c_{a_1,b_1}}\right)\right] \times \left[0, Q_2\left(\frac{s_2}{c_{a_2,b_2}}\right)\right], \\ \tilde{A}_2^{s_1,s_2} &= [r(a_2), r(b_2)] \times \left[0, \frac{s_1}{c_{a_1,b_1}}\right] \times \left[s_2, \frac{s_2}{c_{a_2,b_2}}\right] \\ &\qquad \times \left[0, Q_1\left(\frac{s_1}{c_{a_1,b_1}}\right)\right] \times \left[0, Q_2\left(\frac{s_2}{c_{a_2,b_2}}\right)\right]. \end{aligned}$$

**Theorem 3.3.** *For $i = 1, 2$, suppose that there exist two constants $c_i$, $p_i \ge 0$ and two functions $h_i, l_i \in L^1_+[0,1]$ and a non-negative continuous increasing function $g_i$ such that*

$$\tilde{f}_i(r, w_1, w_2, z_1, z_2) \le c_i + h_i(r)g_i(w_i) + l_i(r)z_i + p_i z_i^2 \quad \text{in } [R_0, R_1] \times [0, +\infty[^4.$$

*Moreover, assume that for some $\rho_i, \sigma_i, \tau_i > 0$ with $\rho_i < \sigma_i$ and $\frac{\sigma_i}{c_{a_i,b_i}} < \tau_i$ we have*

$$\begin{aligned} \tilde{f}_i(r, w_1, w_2, z_1, z_2) &< m_i\rho_i \quad \text{in } \tilde{\Omega}_i^{\rho_1,\rho_2}, \\ \tilde{f}_i(r, w_1, w_2, z_1, z_2) &> M_i\sigma_i \quad \text{in } \tilde{A}_i^{\sigma_1,\sigma_2}, \\ \tilde{f}_i(r, w_1, w_2, z_1, z_2)) &< m_i\tau_i \quad \text{in } \tilde{\Omega}_i^{\tau_1,\tau_2}. \end{aligned}$$

*Then the elliptic system*

$$\begin{cases} -\Delta u = \tilde{f}_1(|x|, u, v, |\nabla u|, |\nabla v|) & \textit{in } E, \\ -\Delta v = \tilde{f}_2(|x|, u, v, |\nabla u|, |\nabla v|) & \textit{in } E, \\ \dfrac{\partial u}{\partial r} = 0 \textit{ for } |x| = R_0, \quad \dfrac{\partial u}{\partial r} + \beta_1[u] = 0 \textit{ for } |x| = R_1, \\ \dfrac{\partial v}{\partial r} = 0 \textit{ for } |x| = R_0, \quad \dfrac{\partial v}{\partial r} + \beta_2[v] = 0 \textit{ for } |x| = R_1, \end{cases} \tag{3.2}$$

*admits at least two positive radial solutions.*

*Proof.* We note that when $\tilde{f}_i$ acts on $\tilde{\Omega}_i^{\rho_1,\rho_2}$, $\tilde{A}_i^{\sigma_1,\sigma_2}$ and $\tilde{\Omega}_i^{\tau_1,\tau_2}$, $f_i$ acts respectively on $\Omega_i^{\rho_1,\rho_2}$, $A_i^{\sigma_1,\sigma_2}$ and $\Omega_i^{\tau_1,\tau_2}$. Since a radial solution of the problem (3.2) is a solution of the problem (3.1), the thesis follows from Theorem 3.2. □

**Remark 3.4.** The hypotheses of Theorem 3.3 can be directily verify for example for the function

$$\tilde{f}(r, u, v, |\nabla u|, |\nabla v|) = 0.0019 \arctan(u^2(|x|) + |\nabla v|^2)u^5 + 0.19|\nabla u|^2$$

and the boundary conditions

$$\frac{\partial u}{\partial r}|_{|x|=1} = 0, \quad \frac{\partial u}{\partial r}|_{|x|=2} + \frac{1}{2}u(|x|)|_{|x|=\frac{8}{7}} = 0$$

with the choices

$$E := \{x \in \mathbb{R}^3 : 1 \le |x| \le 2\} \quad \text{and} \quad [a, b] = \left[0, \frac{3}{4}\right].$$

In this case the positive Green's function in the associate integral equation is

$$G(t, s) = 1 + \begin{cases} \frac{1}{4} - s, & \text{if } \frac{1}{4} \ge s \\ 0, & \text{if } \frac{1}{4} < s \end{cases} - \begin{cases} t - s, & \text{if } t \ge s, \\ 0, & \text{if } t < s. \end{cases}$$

We note that the function $G(t, s)j(s)$ is a positive nonincreasing function of $t$. Thus, we have

$$\frac{1}{m} = \max_{t \in [0,1]} \int_0^1 G(t, s)j(s)ds = \int_0^1 G(0, s)j(s)ds$$

and

$$\frac{1}{M} = \min_{t \in \left[0, \frac{3}{4}\right]} \int_0^{\frac{3}{4}} G(t, s)j(s)ds = \int_0^{\frac{3}{4}} G\left(\frac{3}{4}, s\right) j(s)ds.$$

So by direct computations $m = 0.85$ and $M = 2.56$. Moreover, since

$$\tilde{f}(r, w_1, w_2, z_1, z_2) \le 0.0019 \frac{\pi}{2} w_1^5 + 0.19 z_1^2 \quad \text{in } [R_0, R_1] \times [0, +\infty[^4,$$

we have $Q(\rho) = 0.012\rho^5 e^{3\rho}$. With the choice of $\rho_1 = \rho_2 = 1$ and $\sigma_1 = \sigma_2 = 7$, the function $f$ satisfies the conditions of Theorem 3.3, i.e.

$$\tilde{f}(r, w_1, w_2, z_1, z_2) \leq 0.003 < m\rho_1 \text{ in } [1,2] \times \left[\frac{1}{4}, 1\right] \times [0,1] \times [0, 0.24] \times [0, 0.24],$$

$$\tilde{f}(r, w_1, w_2, z_1, z_2) \geq 49.5 > M\sigma_1 \text{ in } \left[1, \frac{8}{5}\right] \times [7, 28] \times [0, 28] \times [0, Q(28)] \times [0, Q(28)].$$

**Remark 3.5.** Based on the choice of the radii that assures the compatibility of growth conditions of the nonlinearities and the properties of the fixed point index, we can combine the results above in order to prove the existence of one, two or three positive solutions. For example, we have: The elliptic problem (3.2) has at least three radial solutions if there exist $\rho_i, \sigma_i, \tau_i, \theta_i \in (0, \infty)$ with $\rho_i < c_{a_i,b_i}\sigma_i$, $\sigma_i < \tau_i$ and $\tau_i < c_{a_i,b_i}\theta_i$ such that

$$\begin{aligned} \tilde{f}_i(r, w_1, w_2, z_1, z_2) &> M_i\rho_i \quad \text{in } \tilde{A}_i^{\rho_1,\rho_2}, \\ \tilde{f}_i(r, w_1, w_2, z_1, z_2) &< m_i\sigma_i \quad \text{in } \tilde{\Omega}_i^{\sigma_1,\sigma_2}, \\ \tilde{f}_i(r, w_1, w_2, z_1, z_2) &> M_i\tau_i \quad \text{in } \tilde{A}_i^{\tau_1,\tau_2}, \\ \tilde{f}_i(r, w_1, w_2, z_1, z_2) &< m_i\rho_4 \quad \text{in } \tilde{\Omega}_i^{\theta_1,\theta_2}. \end{aligned}$$

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Filomena Cianciaruso
filomena.cianciaruso@unical.it
https://orcid.org/0000-0002-3522-1765

University of Calabria
Department of Mathematics and Computer Sciences
87036 Rende, Cosenza, Italy

Paolamaria Pietramala (corresponding author)
pietramala@unical.it
https://orcid.org/0000-0002-7435-9767

University of Calabria
Department of Mathematics and Computer Sciences
87036 Rende, Cosenza, Italy

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