AN UPPER BOUND ON THE TOTAL OUTER-INDEPENDENT DOMINATION NUMBER OF A TREE

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Abstract. A total outer-independent dominating set of a graph $G = (V(G), E(G))$ is a set $D$ of vertices of $G$ such that every vertex of $G$ has a neighbor in $D$, and the set $V(G) \setminus D$ is independent. The total outer-independent domination number of a graph $G$, denoted by $\gamma_{oi}(G)$, is the minimum cardinality of a total outer-independent dominating set of $G$. We prove that for every tree $T$ of order $n \geq 4$, with $l$ leaves and $s$ support vertices we have $\gamma_{oi}(T) \leq (2n + s - l)/3$, and we characterize the trees attaining this upper bound.

Keywords: total outer-independent domination, total domination, tree.

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1. INTRODUCTION

Let $G = (V(G), E(G))$ be a graph. By the neighborhood of a vertex $v$ of $G$ we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex $v$, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The path on $n$ vertices we denote by $P_n$. Let $T$ be a tree, and let $v$ be a vertex of $T$. We say that $v$ is adjacent to a path $P_n$ if there is a neighbor of $v$, say $x$, such that the subtree resulting from $T$ by removing the edge $vx$ and which contains the vertex $x$ as a leaf, is a path $P_n$. By a star we mean a connected graph in which exactly one vertex has degree greater than one. By a double star we mean a graph obtained from a star by joining a positive number of vertices to one of its leaves.

We say that a subset of $V(G)$ is independent if there is no edge between every two of its vertices. A subset $D \subseteq V(G)$ is a dominating set of $G$ if every vertex of $V(G) \setminus D$ has a neighbor in $D$, while it is a total dominating set if every vertex of $G$ has a neighbor in $D$. The domination (total domination, respectively) number of $G$, denoted by $\gamma(G)$ ($\gamma_t(G)$, respectively), is the minimum cardinality of a dominating set of $G$.
A subset $D \subseteq V(G)$ is a total outer-independent dominating set, abbreviated TOIDS, of $G$ if every vertex of $G$ has a neighbor in $D$, and the set $V(G) \setminus D$ is independent. The total outer-independent domination number of $G$, denoted by $\gamma_{oi}(G)$, is the minimum cardinality of a total outer-independent dominating set of $G$. A total outer-independent dominating set of $G$ of minimum cardinality is called a $\gamma_{oi}(G)$-set.

The study of total outer-independent domination in graphs was initiated in [5]. Chellali and Haynes [1] established the following upper bound on the total domination number of a tree. For every nontrivial tree $T$ of order $n$ with $s$ support vertices we have $\gamma(T) \leq (n + s)/2$.

We prove the following upper bound on the total outer-independent domination number of a tree. For every tree $T$ of order $n \geq 4$, with $l$ leaves and $s$ support vertices we have $\gamma_{oi}(T) \leq (2n + s - l)/3$. Moreover, we characterize the trees attaining this upper bound.

2. RESULTS

Since the one-vertex graph does not have a total outer-independent dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following two straightforward observations.

**Observation 2.1.** Every support vertex of a graph $G$ is in every $\gamma_{oi}(G)$-set.

**Observation 2.2.** For every connected graph $G$ of diameter at least three there exists a $\gamma_{oi}(G)$-set that contains no leaf.

We show that if $T$ is a tree of order $n \geq 4$, with $l$ leaves and $s$ support vertices, then $\gamma_{oi}(T)$ is bounded above by $(2n + s - l)/3$. For the purpose of characterizing the trees attaining this bound we introduce a family $T$ of trees $T = T_k$ that can be obtained as follows. Let $T_1$ be a path $P_6$, and let $A(T_1)$ be a set containing all vertices of $P_6$ which are not leaves. Let $T_2$ be a path $P_3$, and let $A(T_2)$ be a set containing all vertices of $P_3$ which are not leaves and is adjacent to a support vertex. Let $A(T) = A(T') \cup \{u, v\}$.

Now we prove that for every tree $T$ of the family $T$, the set $A(T)$ defined above is a TOIDS of minimum cardinality equal to $(2n + s - l)/3$. 

– Operation $O_1$: Attach a copy of $H$ by joining the vertex $u$ to a vertex of $T_k$ adjacent to a path $P_3$. Let $A(T) = A(T') \cup \{u, v\}$.

– Operation $O_2$: Attach a copy of $H$ by joining the vertex $u$ to a vertex of $T_k$ which is not a leaf and is adjacent to a support vertex. Let $A(T) = A(T') \cup \{u, v\}$.

– Operation $O_3$: Attach a copy of $H$ by joining the vertex $u$ to a weak support vertex of $T_k$ adjacent to a weak support vertex. Let $A(T) = A(T') \cup \{u, v\}$.
Lemma 2.3. If $T \in \mathcal{T}$, then the set $A(T)$ defined above is a $\gamma_{oi}^{\alpha}(T)$-set of size $(2n + s - l)/3$.

Proof. We use the terminology of the construction of the trees $T = T_k$, the set $A(T)$, and the graph $H$ defined above. To show that $A(T)$ is a $\gamma_{oi}^{\alpha}(T)$-set of cardinality $(2n + s - l)/3$ we use induction on the number $k$ of operations performed to construct the tree $T$. If $T = T_1 = P_6$, then $(2n + s - l)/3 = (12 + 2 - 2)/3 = 4 = |A(T)| = \gamma_{oi}^{\alpha}(T)$.

Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T'' = T_k$ of the family $T$ constructed by $k - 1$ operations. Let $n'$ mean the order of the tree $T'$, $l'$ the number of its leaves, and $s'$ the number of support vertices. Let $T = T_{k+1}$ be a tree of the family $T$ constructed by $k$ operations.

First assume that $T$ is obtained from $T'$ by operation $O_1$. We have $n = n' + 3$, $s = s' + 1$, and $l = l' + 1$. The vertex of $T''$ to which is attached $P_3$ we denote by $x$. Let $abc$ mean a path $P_3$ adjacent to $x$, and such that $a \neq u$. It is easy to see that $A(T) = A(T') \cup \{u, v\}$ is a TOIDS of the tree $T$. Thus $\gamma_{oi}^{\alpha}(T) \leq \gamma_{oi}^{\alpha}(T') + 2$. Now let $D$ be a $\gamma_{oi}^{\alpha}(T)$-set that contains no leaf. By Observation 2.1, we have $v \in D$. Each one of the vertices $v$ and $b$ has to have a neighbor in $D$, thus $u, a \in D$. Let us observe that $D \setminus \{u, v\}$ is a TOIDS of the tree $T'$ as the vertex $x$ has a neighbor in $D \setminus \{u, v\}$. Therefore $\gamma_{oi}^{\alpha}(T') \leq \gamma_{oi}^{\alpha}(T) - 2$. Now we conclude that $\gamma_{oi}^{\alpha}(T) = \gamma_{oi}^{\alpha}(T') + 2$. We get $\gamma_{oi}^{\alpha}(T) = |A(T)| = |A(T')| + 2 = (2n' + s' - l')/3 + 2 = (2n - 6 + s - 1 - l + 1)/3 + 2 = (2n + s - l)/3$.

Now assume that $T$ is obtained from $T'$ by operation $O_2$. We have $n = n' + 3$, $s = s' + 1$, and $l = l' + 1$. The vertex of $T''$ to which is attached $P_3$ we denote by $y$. Let $y$ mean a support vertex adjacent to $x$. It is easy to see that $A(T) = A(T') \cup \{u, v\}$ is a TOIDS of the tree $T$. Thus $\gamma_{oi}^{\alpha}(T) \leq \gamma_{oi}^{\alpha}(T') + 2$. Now let $D$ be a $\gamma_{oi}^{\alpha}(T)$-set that contains no leaf. By Observation 2.1 we have $v, y \in D$. The vertex $v$ has to have a neighbor in $D$, thus $u \in D$. Let us observe that $D \setminus \{u, v\}$ is a TOIDS of the tree $T'$ as the vertex $x$ has a neighbor in $D \setminus \{u, v\}$. Therefore $\gamma_{oi}^{\alpha}(T') \leq \gamma_{oi}^{\alpha}(T) + 2$. Now we conclude that $\gamma_{oi}^{\alpha}(T) = \gamma_{oi}^{\alpha}(T') + 2$. In the same way as in the previous possibility we get $\gamma_{oi}^{\alpha}(T) = (2n + s - l)/3$.

Now assume that $T$ is obtained from $T'$ by operation $O_3$. We have $n = n' + 3$, $s = s' + 1$, and $l = l'$. The leaf to which is attached $P_3$ we denote by $y$. Let $y$ mean a neighbor of $x$ other than $u$. It is easy to see that $A(T) = A(T') \cup \{u, v\}$ is a TOIDS of the tree $T$. Thus $\gamma_{oi}^{\alpha}(T) \leq \gamma_{oi}^{\alpha}(T') + 2$. Now let us observe that there exists a $\gamma_{oi}^{\alpha}(T)$-set that does not contain the vertex $x$, and does not contain any leaf. Let $D$ be such a set. By Observation 2.1 we have $v \in D$. The vertex $v$ has to have a neighbor in $D$, thus $u \in D$. The set $V(T) \setminus D$ is independent, thus $y \in D$. Let us observe that $D \setminus \{u, v\}$ is a TOIDS of the tree $T'$ as the vertex $x$ has a neighbor in $D \setminus \{u, v\}$. Therefore $\gamma_{oi}^{\alpha}(T') \leq \gamma_{oi}^{\alpha}(T) - 2$. Now we conclude $\gamma_{oi}^{\alpha}(T) = \gamma_{oi}^{\alpha}(T') + 2$. We get $\gamma_{oi}^{\alpha}(T) = |A(T)| = |A(T')| + 2 = (2n' + s' - l')/3 + 2 = (2n - 6 + s - l)/3 + 2 = (2n + s - l)/3$.

Now we establish the main result, an upper bound on the total outer-independent domination number of a tree together with the characterization of the extremal trees.
Theorem 2.4. If $T$ is a tree of order $n \geq 4$, with $l$ leaves and $s$ support vertices, then $\gamma^a_i(T) \leq (2n + s - l)/3$ with equality if and only if $T = K_{1,3}$ or $T \in T$.

Proof. First assume that $\text{diam}(T) = 2$. Thus $T$ is a star $K_{1,m}$ with $m \geq 3$. If $m = 3$, then $T = K_{1,3}$. We have $\gamma^a_i(T) = 2 = (8 + 1 - 3)/3 = (2n + s - l)/3$. If $m \geq 4$, then $(2n + s - l)/3 = (2n + 2 + 1 - m)/3 = (m + 3)/3 \geq (4 + 3)/3 > 2 = \gamma^a_i(T)$. Now let us assume that $\text{diam}(T) = 3$. Thus $T$ is a double star. We have $(2n + s - l)/3 = (2n + 2 - n + 2)/3 = (n + 4)/3 \geq (4 + 4)/3 > 2 = \gamma^a_i(T)$. Now assume that $\text{diam}(T) = 4$. Let $v_1v_2v_3v_4$ be a longest path in $T$. If $v_3$ is adjacent to a leaf, then all support vertices of $T$ form a TOIDS of the tree $T$. Thus $\gamma^a_i(T) \leq s$. Now we get $\gamma^a_i(T) \leq s = s/3 + 2s/3 = s/3 + 2(n - l)/3 < (2n + s - l)/3$. Now assume that $T$ is not adjacent to any leaf. It is easy to observe that all support vertices of $T$ together with the vertex $v_3$ form a TOIDS of the tree $T$. Thus $\gamma^a_i(T) \leq s + 1$. We have $n = l + s + 1$. Now we get $\gamma^a_i(T) \leq s + 1 = s/3 + 2s/3 + 1 = s/3 + 2(n - l - 1)/3 + 1 = (2n + s - 2l - 2)/3 + 1 = (2n + s - l)/3 + (l - 1)/3 < (2n + s - l)/3$. Now assume that $\text{diam}(T) = 5$. Let $v_1v_2v_3v_4v_5$ be a longest path in $T$. If both vertices $v_3$ and $v_4$ are adjacent to a leaf, then all support vertices of $T$ form a TOIDS of the tree $T$. Thus $\gamma^a_i(T) \leq s$. Now we get $\gamma^a_i(T) \leq s = s/3 + 2s/3 = (2n - 2l)/3 < (2n + s - l)/3$. Now assume that exactly one of the vertices $v_3$ and $v_4$ is adjacent to a leaf. Without loss of generality we assume that $v_3$ is adjacent to a leaf. It is easy to observe that all support vertices of $T$ together with the vertex $v_4$ form a TOIDS of the tree $T$. Thus $\gamma^a_i(T) \leq s + 1$. We have $n = l + s + 2$. Now we get $\gamma^a_i(T) \leq s + 1 = s/3 + 2s/3 + 1 = s/3 + 2(n - l - 2)/3 + 2 = (2n + s - l)/3 + 2 = (2n + s - l)/3 + (l - 2)/3$. If $T$ has exactly two leaves, then $T = P_2 = T_1 \in T$. By Lemma 2.3 we have $\gamma^a_i(T) = (2n + s - l)/3$. Now assume that $T$ has at least three leaves. We have $\gamma^a_i(T) \leq (2n + s - l)/3 + (l - 2)/3 < (2n + s - l)/3$.

Now assume that $\text{diam}(T) \geq 6$. Thus the order of the tree $T$ is an integer $n \geq 7$. The result we obtain by the induction on the number $n$. Assume that the theorem is true for every tree $T'$ of order $n' < n$, with $l'$ leaves and $s'$ support vertices.

First assume that some support vertex of $T$, say $x$, is strong. Let $y$ mean a leaf adjacent to $x$. Let $T' = T - y$. We have $n' = n - 1$, $s' = s$, and $l' = l - 1$. Let $D'$ be any $\gamma^a_i(T')$-set. By Observation 2.1 we have $x \in D'$. Of course, $D'$ is a TOIDS of the tree $T$. Thus $\gamma^a_i(T) \leq \gamma^a_i(T')$. Now we get $\gamma^a_i(T) \leq \gamma^a_i(T') = (2n' + s' - l')/3 = (2n' - 2 + s - l + 1)/3 = (2n' + s - l)/3 - 1/3 < (2n + s - l)/3$. Therefore every support vertex of $T$ is weak.

We now root $T$ at a vertex $r$ of maximum eccentricity $\text{diam}(T)$. Let $t$ be a leaf at maximum distance from $r$, $u$ be the parent of $t$, $v$ be the parent of $u$, $w$ be the parent of $u$, and $d$ be the parent of $w$ in the rooted tree. By $T_z$ let us denote the subtree induced by a vertex $x$ and its descendants in the rooted tree $T$.

First assume that $d_T(v) \geq 3$. Assume that among the descendants of $u$ there is a support vertex, say $x$, different than $v$. Let $T' = T - T_z$. We have $n' = n - 2$, $s' = s - 1,$
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The vertex \( v \) has to have a neighbor in \( D', \) thus \( u \in D' \). It is easy to see that \( D' \cup \{ v \} \) is a TOIDS of the tree \( T \).

Thus \( \gamma_i^{\alpha}(T) \leq \gamma_i^{\alpha}(T') + 1 \). Now we get \( \gamma_i^{\alpha}(T) \leq \gamma_i^{\alpha}(T') + 1 \leq (2n' + s' - l')/3 + 1 = (2n - 4 + s - 1 - l + 1)/3 + 1 = (2n + s - l)/3 - 1/3 < (2n + s - l)/3 \).

Now assume that some descendant of \( u \), say \( x \), is a leaf. Let \( T' = T - x \). We have \( n' = n - 1, s' = s - 1, \) and \( l' = l - 1 \). Let \( D' \) be a \( \gamma_i^{\alpha}(T') \)-set that contains no leaf. The vertex \( v \) has to have a neighbor in \( D' \), thus \( u \in D' \). It is easy to see that \( D' \cup \{ v \} \) is a TOIDS of the tree \( T \).

Thus \( \gamma_i^{\alpha}(T) \leq \gamma_i^{\alpha}(T') \). Now we get \( \gamma_i^{\alpha}(T) \leq \gamma_i^{\alpha}(T') \leq (2n' + s' - l')/3 = (2n - 2 + s - 1 - l + 1)/3 = (2n + s - l)/3 - 2/3 < (2n + s - l)/3 \).

Now assume that \( d_T(u) = 2 \). First assume that there is a descendant of \( u \), say \( k \), such that the distance of \( w \) to the most distant vertex of \( T_k \) is three. It suffices to consider only the possibility when \( T_k \) is a path \( P_3 \), say \( klm \). Let \( T' = T - T_u \). We have \( n' = n - 3, s' = s - 1, \) and \( l' = l - 1 \). Let \( D' \) be any \( \gamma_i^{\alpha}(T') \)-set. It is easy to see that \( D' \cup \{ u \} \) is a TOIDS of the tree \( T \). Thus \( \gamma_i^{\alpha}(T) \leq \gamma_i^{\alpha}(T') + 2 \). Now we get \( \gamma_i^{\alpha}(T) \leq \gamma_i^{\alpha}(T') + 2 \leq (2n' + s' - l')/3 + 2 = (2n - 6 + s - 1 - l + 1)/3 + 2 = (2n + s - l)/3 \).

If \( \gamma_i^{\alpha}(T) = (2n + s - l)/3 \), then obviously \( \gamma_i^{\alpha}(T') = (2n' + s' - l')/3 \). The tree \( T' \) has at least seven vertices. By the inductive hypothesis we have \( T' \in T \). The tree \( T \) can be obtained from \( T' \) by operation \( O_1 \). Thus \( T \in T \).

Now assume that there is a descendant of \( w \), say \( k \), such that the distance of \( w \) to the most distant vertex of \( T_k \) is two. Thus \( k \) is a support vertex. Let \( T' = T - T_u \). In the same way as in the previous possibility we get \( \gamma_i^{\alpha}(T) \leq (2n + s - l)/3 \). If \( \gamma_i^{\alpha}(T) \leq (2n + s - l)/3 \), then \( \gamma_i^{\alpha}(T') = (2n' + s' - l')/3 \). The tree \( T' \) has at least six vertices. By the inductive hypothesis we have \( T' \in T \). The tree \( T \) can be obtained from \( T' \) by operation \( O_2 \). Thus \( T \in T \).

Now assume that some descendant of \( w \), say \( k \), is a leaf. Let \( T' = T - t - k \). We have \( n' = n - 2, s' = s - 1, \) and \( l' = l - 1 \). Let \( D' \) be a \( \gamma_i^{\alpha}(T') \)-set that contains no leaf. By Observation 2.1 we have \( u \in D' \). The vertex \( u \) has to have a neighbor in \( D' \), thus \( w \in D' \). It is easy to observe that \( D' \cup \{ v \} \) is a TOIDS of the tree \( T \). Thus \( \gamma_i^{\alpha}(T) \leq \gamma_i^{\alpha}(T') + 1 \). Now we get \( \gamma_i^{\alpha}(T) \leq \gamma_i^{\alpha}(T') + 1 \leq (2n' + s' - l')/3 + 1 = (2n - 4 + s - 1 - l + 1)/3 + 1 = (2n + s - l)/3 - 1/3 < (2n + s - l)/3 \).

Now assume that \( d_T(u) = 2 \). First assume that \( d \) is adjacent to a leaf. Let \( T' = T - T_u \). We have \( n' = n - 3, s' = s - 1, \) and \( l' = l \). Let \( D' \) be any \( \gamma_i^{\alpha}(T') \)-set. It is easy to see that \( D' \cup \{ u, v \} \) is a TOIDS of the tree \( T \). Thus \( \gamma_i^{\alpha}(T) \leq \gamma_i^{\alpha}(T') + 2 \). Now we get \( \gamma_i^{\alpha}(T) \leq \gamma_i^{\alpha}(T') + 2 \leq (2n + s - l)/3 + 2 = (2n - 6 + s - 1 - l + 1)/3 + 2 = (2n + s - l)/3 - 1/3 < (2n + s - l)/3 \).

Now assume that \( d \) is not adjacent to any leaf. Let \( T' = T - T_u \). We have \( n' = n - 3, s' = s, \) and \( l' = l \). Let \( D' \) be any \( \gamma_i^{\alpha}(T') \)-set. It is easy to see that \( D' \cup \{ u, v \} \) is a TOIDS of the tree \( T \). Thus \( \gamma_i^{\alpha}(T) \leq \gamma_i^{\alpha}(T') + 2 \). Now we get \( \gamma_i^{\alpha}(T) \leq \gamma_i^{\alpha}(T') + 2 \leq (2n + s' - l')/3 + 2 = (2n - 6 + s - 1 - l + 1)/3 + 2 = (2n + s - l)/3 \). If \( \gamma_i^{\alpha}(T) = (2n + s - l)/3 \), then \( \gamma_i^{\alpha}(T') = (2n' + s' - l')/3 \). The tree \( T' \) has at least four vertices and is different from \( K_{1,3} \) as \( T' \) has no strong support vertex. By the inductive hypothesis we have \( T' \in T \). The tree \( T \) can be obtained from \( T' \) by operation \( O_3 \). Thus \( T \in T \).
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