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EXISTENCE OF SOLUTIONS OF THE DIRICHLET
PROBLEM FOR AN INFINITE SYSTEM OF NONLINEAR
DIFFERENTIAL-FUNCTIONAL EQUATIONS
OF ELLIPTIC TYPE

Abstract. The Dirichlet problem for an infinite weakly coupled system of semilinear
differential-functional equations of elliptic type is considered. It is shown the existence of
solutions to this problem. The result is based on Chaplygin’s method of lower and upper
functions.

Keywords: infinite systems, elliptic differential-functional equations, monotone iterative
technique, Chaplygin’s method, Dirichlet problem.

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1. INTRODUCTION

Let $S$ be an infinite set. Let $G \subset \mathbb{R}^m$ be an open bounded domain with $C^{2+\alpha}$
boundary ($\alpha \in (0, 1)$). Let $\mathcal{B}(S)$ be the Banach space of all bounded functions
$w: S \to \mathbb{R}$, $w(i) = w^i$ ($i \in S$) with the norm
$$
\|w\|_{\mathcal{B}(S)} := \sup_{i \in S} |w^i|.
$$

In $\mathcal{B}(S)$ there is a partly order $w \leq \tilde{w}$ defined as $w^i \leq \tilde{w}^i$ for every $i \in S$. Elements
of $\mathcal{B}(S)$ will be denoted by $(w^i)_{i \in S}$, too.

Let $\mathcal{C}(\bar{G})$ be the space of all continuous functions $v: \bar{G} \to \mathbb{R}$ with the norm
$$
|v|_{\mathcal{C}(\bar{G})} := \max_{x \in \bar{G}} |v(x)|.
$$

In this space $v \leq \tilde{v}$ means that $v(x) \leq \tilde{v}(x)$ for every $x \in \bar{G}$. By $\mathcal{C}^{l+\alpha}(\bar{G})$, where
$l = 0, 1, 2, \ldots$ and $\alpha \in (0, 1)$, we denote the space of all continuous functions in $\bar{G}$ whose
derivatives of order less or equal \( l \) exist and are Hölder continuous with exponent \( \alpha \) in \( G \) (see [5], pp. 52–53). By \( H^{1,p}(G) \) we denote the Sobolev space of all functions whose weak derivatives of order \( l \) are included in \( L^p(G) \) (see [1], pp. 44–46). A notation \( g \in C^{l+\alpha}(\partial G) \) (resp. \( g \in H^{l,p}(\partial G) \)) means that there exists a function \( f \in C^{l+\alpha}(\tilde{G}) \) (resp. \( f \in H^{l,p}(\tilde{G}) \cap C(\tilde{G}) \)) such that \( f(x) = g(x) \) for every \( x \in \partial G \). In these spaces norms are defined as

\[
|g|_{C^{l+\alpha}(\partial G)} := \inf \{|g|_{C^{l+\alpha}(\tilde{G})} : g \in C^{l+\alpha}(\tilde{G}) : \forall x \in \partial G : g(x) = g(x)\}
\]

and

\[
|g|_{H^{l,p}(\partial G)} := \inf \{|g|_{H^{l,p}(G)} : g \in H^{l,p}(G) \cap C(\tilde{G}) : \forall x \in \partial G : g(x) = g(x)\}.
\]

We denote \( z = (z^i)_{i \in S} \in C_S(\tilde{G}) \) if \( z : \tilde{G} \to \mathbb{B}(S) \) and \( z^i : \tilde{G} \to \mathbb{R} \) (\( i \in S \)) is a continuous function and \( \sup_{i \in S} |z^i|_{C(\tilde{G})} < \infty \). The space \( C_S(\tilde{G}) \) is a Banach space with the norm

\[
\|z\|_{C_S(\tilde{G})} := \sup_{i \in S} |z^i|_{C(\tilde{G})}
\]

and the partly order \( z \leq \tilde{z} \) defined as \( z^i(x) \leq \tilde{z}^i(x) \) for every \( x \in G, i \in S \). The space \( C_S^{l+\alpha}(\tilde{G}) \) is a space of all functions \( (z^i)_{i \in S} \) such that \( z^i \in C^{l+\alpha}(\tilde{G}) \) for every \( i \in S \) and \( \sup_{i \in S} |z^i|_{C^{l+\alpha}(\tilde{G})} < \infty \). In this space the norm is defined as

\[
\|z\|_{C_S^{l+\alpha}(\tilde{G})} = \sup_{i \in S} |z^i|_{C^{l+\alpha}(\tilde{G})}.
\]

We will write that \( z = (z^i)_{i \in S} \in L^p_S(G) \) if \( z^i \in L^p(G) \) for every \( i \in S \) and \( \sup_{i \in S} |z^i|_{L^p(G)} < \infty \). A notation \( z = (z^i)_{i \in S} \in H^{l,p}_S(G) \) means that \( z^i \in H^{l,p}(G) \) for every \( i \in S \) and \( \sup_{i \in S} |z^i|_{H^{l,p}(G)} < \infty \). In these spaces the norms are defined as

\[
\|z\|_{L^p_S(G)} = \sup_{i \in S} |z^i|_{L^p(G)}
\]

and

\[
\|z\|_{H^{l,p}_S(G)} = \sup_{i \in S} |z^i|_{H^{l,p}(G)}.
\]

We consider the Dirichlet problem for an infinite weakly coupled system of semilinear differential-functional equations of the following form

\[
-L^i[u^i](x) = f^i(x, u(x), u), \quad \text{for } x \in G, i \in S \tag{1}
\]

and

\[
u^i(x) = h^i(x), \quad \text{for } x \in \partial G, i \in S, \tag{2}
\]

where

\[
L^i[u^i](x) := \sum_{j,k=1}^{m} a^i_{jk}(x) u^j_{x_k}(x) + \sum_{j=1}^{m} b^i_j(x) u^j_{x_j}(x),
\]
which are strongly uniformly elliptic in \( \bar{G} \),
\[
f^i : \bar{G} \times B(S) \times C_S(\bar{G}) \ni (x,y,z) \mapsto f^i(x,y,z) \in \mathbb{R}
\]
for every \( i \in S \). The notation \( f(x,u(x),u) \) means that the dependence of \( f \) on the second variable is a function-type dependence and \( f(x,u(x),\cdot) \) is a functional-type dependence.

A function \( u \) is said to be regular in \( \bar{G} \) if \( u \in C_S(\bar{G}) \cap C^2(\bar{G}) \). A function \( u \) is said to be a classical (regular) solution of the problem (1), (2) in \( \bar{G} \) if \( u \) is regular in \( \bar{G} \) and fulfills the system of equations (1) in \( \bar{G} \) with the condition (2). A function \( u \) is said to be a weak solution of the problem (1), (2) in \( \bar{G} \) if \( u \in L^2(\bar{G}) \) such that
\[
-L^i[u^i] \in L^2(\bar{G})
\]
and
\[
-\int_{\bar{G}} L^i[u^i](x) \xi(x) \, dx = \int_{\bar{G}} f^i(x,u(x),u) \xi(x) \, dx
\]
for every \( i \in S \) and for any test function \( \xi \in C^\infty_0(\bar{G}) \).

We would like to find assumptions which guarantee existence of the classical solutions of the problem (1), (2)
\[
u : \bar{G} \to B(S).
\]

Regular functions \( u_0 = u_0(x) \) and \( v_0 = v_0(x) \) in \( \bar{G} \) satisfying the infinite systems of inequalities:
\[
\begin{cases}
-L^i[u_0^i](x) \leq f^i(x,u_0(x),u_0) & \text{for } x \in G, i \in S, \\
u_0^i(x) \leq h^i(x) & \text{for } x \in \partial G, i \in S,
\end{cases}
\]
\[
\begin{cases}
-L^i[v_0^i](x) \geq f^i(x,v_0(x),v_0) & \text{for } x \in G, i \in S, \\
v_0^i(x) \geq h^i(x) & \text{for } x \in \partial G, i \in S
\end{cases}
\]
are called a lower and an upper function for the problem (1), (2), respectively.

If \( u_0 \leq v_0 \), we define
\[
\mathcal{K} := \{(x,y,z) : x \in \bar{G}, y \in [m_0,M_0], z \in [u_0,v_0]\},
\]
where \( m_0 := (m_0^i)_{i \in S}, m_0^i := \min_{x \in G} v_0^i(x), M_0 := (M_0^i)_{i \in S}, M_0^i := \max_{x \in G} v_0^i(x) \)
and \( (u_0, v_0) := \{\xi \in C_S(\bar{G}) : u_0(x) \leq \varsigma(x) \leq v_0(x) \text{ for } x \in \bar{G}\} \).

Assumptions. We make the following assumptions.

(a) \( \mathcal{L} \) is a strongly uniformly elliptic operator in \( \bar{G} \), i.e., there exists a constant
\( \mu > 0 \) such that
\[
\sum_{j,k=1}^m a_{jk}^i(x)\xi_j\xi_k \geq \mu \sum_{j=1}^m \xi_j^2, \quad i \in S,
\]
for all \( \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m, x \in \bar{G} \).
(b) The functions $a^i_{jk}$, $b^i_j$ for $i \in S$, $j, k = 1, \ldots, m$ are functions of class $C^{0+\alpha}(\bar{G})$ and $a^i_{jk}(x) = a^i_{kj}(x)$ for every $x \in \bar{G}$.

(c) $h^i \in C^{2+\alpha}(\partial \bar{G})$ for every $i \in S$ and $\sup_{i \in S} |h^i|_{C^{2+\alpha}(\partial \bar{G})} < \infty$.

(d) There exists at least one ordered pair $u_0$, $v_0$ of a lower and an upper function for the problem (1), (2) in $\bar{G}$ such that

$$u_0(x) \leq v_0(x) \quad \text{for } x \in \bar{G}.$$ 

(e) $f(\cdot, y, z) \in C^{0+\alpha}_S(\bar{G})$ for $y \in B(S)$, $z \in C_S(\bar{G})$.

(f) For every $i \in S$, $x \in \bar{G}$, $y, \tilde{y} \in B(S)$, $z \in C_S(\bar{G})$

$$|f^i(x, y, z) - f^i(x, \tilde{y}, z)| \leq L_f \|y - \tilde{y}\|_{B(S)},$$

where $L_f > 0$ is a constant independent of $i$ and

$$|f^i(x, y^1, \ldots, y^j, \ldots, z) - f^i(x, \tilde{y}, \ldots, z)| \leq k^i |y^j - \tilde{y}^j|,$$

where $k^i > 0$ is a constant and there exists $k < \infty$ such that $k^i \leq k$ for every $i \in S$.

(g) $f(x, \cdot, \cdot, \cdot)$ is a continuous function for every $x \in \bar{G}$.

(h) $f^i(x, \cdot, z)$ is a quasi-increasing function for every $i \in S$, $x \in \bar{G}$, $z \in C_S(\bar{G})$ i.e., for every $i \in S$ for arbitrary $y, \tilde{y} \in B(S)$ if $y^j \leq \tilde{y}^j$ for all $j \in S$ such that $j \neq i$ and $y^i = \tilde{y}^i$, then $f^i(x, y, z) \leq f^i(x, \tilde{y}, z)$ for $x \in \bar{G}$, $z \in C_S(\bar{G})$.

(i) $f^i(x, y, \cdot)$ is an increasing function for every $i \in S$, $x \in \bar{G}$, $y \in B(S)$.

2. AUXILIARY RESULTS

From the assumption (f) we have $k := (k^i)_{i \in S} \in B(S)$. Let $\beta = (\beta^i)_{i \in S} \in C^{0+\alpha}_S(\bar{G})$. We define the operator

$$\mathcal{P} : C^{0+\alpha}_S(\bar{G}) \ni \beta \mapsto \gamma \in C^{2+\alpha}_S(\bar{G}),$$

where $\gamma = (\gamma^i)_{i \in S}$ is the solution (supposedly unique) of the following problem

$$\left\{
\begin{array}{ll}
-(L^i - k^i I)[\gamma^i](x) = f^i(x, \beta(x), \beta) + k^i \beta^i(x) & \text{for } x \in G, \ i \in S, \\
\gamma^i(x) = h^i(x) & \text{for } x \in \partial G, \ i \in .
\end{array}
\right.$$  \hspace{1cm} (5)

We remark that the problem (5) is a system of separate problems with only one equation, so $\mathcal{P}[\beta]$ is a collection of solutions of these problems.
Lemma 1. The operator $P : C^{0+\alpha}_S(\tilde{G}) \to C^{2+\alpha}_S(\tilde{G})$ is a continuous and bounded operator. If the operator $P$ maps $C^{0+\alpha}_S(\tilde{G})$ into $C^{0+\alpha}_S(\tilde{G})$, then it is a compact operator.

Proof. Let $\beta \in C^{0+\alpha}_S(\tilde{G})$, so
\[
|\beta^i(x) - \beta^i(\tilde{x})| \leq H_\beta \|x^j - \tilde{x}^j\|_{\mathbb{R}^m}^\alpha,
\]
where $H_\beta > 0$ is some constant independent of $i$ and $\|x\|_{\mathbb{R}^m} = (\sum_{j=1}^m x_j^2)^{1/2}$.

We define the operator $F = (F^i)_{i \in S} : C^{0+\alpha}_S(\tilde{G}) \ni \beta \mapsto \delta \in C^{0+\alpha}_S(\tilde{G})$ such that for every $i \in S$
\[
F^i[\beta](x) = \delta^i(x) := f^i(x, \beta(x), \beta) + k^i\beta^i(x).
\]

For arbitrary $i \in S$ we have:
\[
|\delta^i(x) - \delta^i(\tilde{x})| = |f^i(x, \beta(x), \beta) + k^i\beta^i(x) - f^i(\tilde{x}, \beta(\tilde{x}), \beta) - k^i\beta^i(\tilde{x})| \leq |f^i(x, \beta(x), \beta) - f^i(\tilde{x}, \beta(\tilde{x}), \beta)| + |f^i(\tilde{x}, \beta(x), \beta) - f^i(\tilde{x}, \beta(\tilde{x}), \beta)| + k^i |\beta^i(x) - \beta^i(\tilde{x})| \leq (H_f + L_f H_\beta + k H_\beta) \|x - \tilde{x}\|_{\mathbb{R}^m}^\alpha,
\]
where $H_f + L_f H_\beta + k H_\beta$ is some constant independent of $i$.

By the properties of $f$, we see that the operator $F$ is a continuous and bounded operator.

Now, we have our problem for arbitrary $i \in S$ in the following form
\[
\begin{align*}
-(L^i - k^i i^j)[\gamma^i(x)] &= \delta^i(x) & \text{for } x \in G, \\
\gamma^i(x) &= h^i(x) & \text{for } x \in \partial G,
\end{align*}
\]
which satisfies the assumptions of the Schauder theorem ([7], p. 115), so the problem (6) for every $i \in S$ has a unique solution $\gamma^i \in C^{2+\alpha}(\tilde{G})$ and the following estimate
\[
|\gamma^i|_{C^{2+\alpha}(G)} \leq C \left( |\delta^i|_{C^{0+\alpha}(G)} + |h^i|_{C^{0+\alpha}(\partial G)} \right)
\]
holds, where $C > 0$ is independent of $\delta$, $h$ and $i$.

Let us introduce the operator
\[
G = (G^i)_{i \in S} : C^{0+\alpha}_S(\tilde{G}) \ni \delta \mapsto \gamma \in C^{2+\alpha}_S(\tilde{G}).
\]

The function
\[
\gamma^i = G^i[\delta^i] = G^i_1[h^i] + G^i_2[\delta^i],
\]
where
\[
G^i_1 : C^{2+\alpha}(\tilde{G}) \ni h^i \mapsto G^i_1[h^i] \in C^{2+\alpha}(\tilde{G})
\]
and $G_1^i[h^i]$ is the unique solution of the problem (6) with $\delta^i(x) = 0$ in $\bar{G}$, and

$$G_1^i : C^{0+\alpha}(G) \ni \delta^i \mapsto G_1^i[\delta^i] \in C^{2+\alpha}(\bar{G})$$

and $G_2^i[\delta^i]$ is the unique solution of the problem (6) with $h^i(x) = 0$ on $\partial G$. The operator $G^i$ is a continuous operator because the operator $G_1^i$ is independent of $\delta^i$, and $G_2^i$ is a continuous operator (from (7)) with respect to $\delta^i$. By (7), we have

$$|\gamma^i|_{C^{2+\alpha}(\bar{G})} = |G^i \circ F^i[\beta^i]|_{C^{2+\alpha}(\bar{G})} \leq C \left( |\delta^i|_{C^{2+\alpha}(\bar{G})} + |h^i|_{C^{2+\alpha}(\partial G)} \right),$$

where $C > 0$ is independent of $\delta$, $h$ and $i$. Thus the operator $G \circ F$ is a continuous and bounded operator.

Since $\partial G \in C^{2+\alpha}$, the imbedding operator

$$I : C_S^{2+\alpha}(G) \rightarrow C_S^{0+\alpha}(G)$$

is a compact operator ([1], p. 11). So $P = I \circ G \circ F$ is a compact operator. \hfill $\square$

Next, let us consider the operator $P$ as a operator mapping $L^p_S(G)$.  

**Lemma 2.** The operator $P$ is a compact operator mapping $L^p_S(G)$ into $L^p_S(G)$.

**Proof.** We define $\delta$ and the operator $F$ such in the proof of Lemma 1 but on an element of $L^p_S(G)$.

The operator $F : L^p_S(G) \rightarrow L^p_S(G)$ and $F$ is a continuous and bounded operator by arguing as [6] (Th. 2.1, Th. 2.2 and Th. 2.3, pp. 31–37) and [9] (Th. 19.1, p. 204).

Now, we have our problem for arbitrary $i \in S$ in the following form

$$\begin{cases}
-(\mathcal{L}^i - k^i \mathcal{I})[\gamma^i](x) = \delta^i(x) & \text{for } x \in G, \\
\gamma^i(x) = h^i(x) & \text{for } x \in \partial G,
\end{cases} \quad \text{(8)}$$

which satisfies the assumptions of Agmon–Douglis–Nirenberg theorem for arbitrary $i \in S$, so the problem (8) has a unique weak solution $\gamma^i \in H^{2,p}(G)$ and the following estimate

$$|\gamma^i|_{H^{2,p}(G)} \leq C \left( |\delta^i|_{L^p(G)} + |h^i|_{H^{2,p}(\partial G)} \right) \quad \text{(9)}$$

holds, where $C > 0$ is independent of $\delta$, $h$ and $i$.

Let us introduce the operator

$$G = (G^i)_{i \in S} : L^p_S(G) \ni \delta \mapsto \gamma \in H^{2,p}_S(G).$$

The function

$$\gamma^i = G^i[\delta^i] = G_1^i[h^i] + G_2^i[\delta^i],$$

where

$$G_1^i : H^{2,p}(G) \ni h^i \mapsto G_1^i[h^i] \in H^{2,p}(G)$$
and $G^i_1[h^i]$ is the unique weak solution of the problem (8) with $\delta^i(x) = 0$ in $\bar{G}$, and

$$G^i_2: L^p(G) \ni \delta^i \mapsto G^i_2[\delta^i] \in H^{2,p}(G)$$

and $G^i_2[\delta^i]$ is the unique solution of the problem (8) with $h^i(x) = 0$ on $\partial G$. The operator $G^i$ is a continuous operator because the operator $G^i_1$ is independent of $\delta^i$, and $G^i_2$ is a continuous operator (from (9)) with respect to $\delta^i$. Also we know that

$$|\gamma^i|_{H^{2,p}(G)} = |G^i \circ F^i[\beta^i]|_{H^{2,p}(G)} \leq C \left( |\delta^i|_{L^p(G)} + |h^i|_{H^{2,p}(\partial G)} \right),$$

where $C > 0$ is independent of $\delta$, $h$ and $i$. Thus the operator $G \circ F$ is a continuous and bounded operator. Since $\partial G \in C^{2+\alpha}$, the imbedding operator

$$I: H^{2,p}_\delta(G) \to L^p(G)$$

is a compact operator ([1], p. 97), and $P = I \circ G \circ F$ is also a compact operator. $\square$

Now, we prove next some properties of the operator $P$.

**Lemma 3.** The operator $P$ is an increasing operator.

**Proof.** Let $\beta(x) \leq \tilde{\beta}(x)$ in $G$, so for all $i \in S$, $\beta^i(x) \leq \tilde{\beta}^i(x)$ in $\bar{G}$. Let $\gamma := P[\beta]$ and $\tilde{\gamma} := P[\tilde{\beta}]$. For arbitrary $i \in S$

$$\begin{cases}
- (L^i - k^iI)[\tilde{\gamma}^i - \gamma^i](x) = \\
\quad \quad f^i(x, \tilde{\beta}(x), \tilde{\beta}) - f^i(x, \beta(x), \beta) + k^i \left( \tilde{\beta}^i(x) - \beta^i(x) \right) & \text{for } x \in G, \\
(\tilde{\gamma}^i - \gamma^i)(x) = 0, & \text{for } x \in \partial G.
\end{cases}$$

By assumption (h), (i) and (f),

$$- (L^i - k^iI)[\tilde{\gamma}^i - \gamma^i](x) \geq f^i \left( x, \beta^1(x), \ldots, \beta^{i-1}(x), \tilde{\beta}^i(x), \beta^{i+1}(x), \ldots, \beta \right) - f^i(x, \beta(x), \beta) + k^i \left( \tilde{\beta}^i(x) - \beta^i(x) \right) \geq 0.$$

So for every $i \in S$

$$\begin{cases}
- (L^i - k^iI)[\tilde{\gamma}^i - \gamma^i](x) \geq 0 & \text{for } x \in G, \\
\tilde{\gamma}^i(x) - \gamma^i(x) = 0 & \text{for } x \in \partial G.
\end{cases}$$

By the maximum principle ([8], p. 64)

$$\tilde{\gamma}^i(x) - \gamma^i(x) \geq 0 \text{ in } \bar{G}.$$ 

We have for all $i \in S$ $\gamma^i(x) \leq \tilde{\gamma}^i(x)$, so

$$\gamma(x) \leq \tilde{\gamma}(x) \text{ in } \bar{G}. \square$$
Lemma 4. If $\beta$ is an upper (resp. a lower) function for the problem (1), (2) in $\bar{G}$, then $P[\beta] \leq \beta$ (resp. $P[\beta] \geq \beta$) in $\bar{G}$ and $P[\beta]$ is an upper (resp. a lower) function for problem (1), (2) in $\bar{G}$.

Proof. Let $\gamma = P[\beta]$. By (5) we have for every $i \in S$
$$-(L^i - k^i I)[\gamma^i - \beta^i](x) = -(L^i - k^i I)[\gamma]^i(x) + (L^i - k^i I)[\beta]^i(x) = f^i(x, \beta(x), \beta) + k^i \beta^i(x) + L^i[\beta^i](x) - k^i \beta^i(x) = f^i(x, \beta(x), \beta) + L^i[\beta^i](x)$$
and from (4)
$$f^i(x, \beta(x), \beta) + L^i[\beta^i](x) \leq 0$$
and
$$(\gamma^i - \beta^i)(x) = h^i(x) - \beta^i(x) \leq 0.$$
So for every $i \in S$
$$\begin{cases}-(L^i - k^i I)[\gamma^i - \beta^i](x) \leq 0 & \text{for } x \in G, \\(\gamma^i - \beta^i)(x) \leq 0 & \text{for } x \in \partial G.\end{cases}$$
Now, by using the maximum principle ([8], th. 6, p. 64) separately for every $i \in S$
$$\gamma^i(x) - \beta^i(x) \leq 0 \text{ in } \bar{G}.$$ 
So
$$\gamma(x) \leq \beta(x) \text{ in } \bar{G}.$$ 
From Lemma 1 it follows that $\gamma \in C^{2+\alpha}_S(\bar{G})$ and from (5) and the assumption (f), we get for every $i \in S$
$$-(L^i[\gamma^i](x) - f^i(x, \gamma(x), \gamma) = -(L^i - k^i I)[\gamma^i](x) - f^i(x, \gamma(x), \gamma) - k^i \gamma^i(x) = f^i(x, \beta(x), \beta) + k^i \beta^i(x) - f^i(x, \gamma(x), \gamma) - k^i \gamma^i(x) \geq (f^i(x, \gamma^i(x), \ldots, \gamma^{i-1}(x), \beta^i(x), \gamma^{i+1}(x), \ldots, \gamma) - f^i(x, \gamma(x), \gamma) + k^i(\beta^i(x) - \gamma^i(x)) \geq 0 \text{ in } \bar{G},$$
so it is a upper the function for problem (1), (2) in $\bar{G}$.

3. MAIN RESULT

Theorem. If the assumptions (a)–(i) hold, then the problem (1), (2) has at least one classical solution $u$ such that $u \in (u_0, v_0)$.

Proof. By induction, we define two sequences of functions $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ by setting:
$$u_1 = P[u_0], \quad u_n = P[u_{n-1}],$$
$$v_1 = P[v_0], \quad v_n = P[v_{n-1}].$$
Because $u_0$ and $v_0$ are regular functions, these sequences are well defined by Lemma 1. The sequence $\{u_n\}_{n=0}^{\infty}$ is increasing and $\{v_n\}_{n=0}^{\infty}$ is decreasing by Lemma 4:

$$u_1(x) = \mathcal{P}[u_0](x) \geq u_0(x) \quad \text{in } \bar{G},$$

$$v_1(x) = \mathcal{P}[v_0](x) \leq v_0(x) \quad \text{in } \bar{G},$$

and by induction:

$$u_n(x) = \mathcal{P}[u_{n-1}](x) \geq u_{n-1}(x) \quad \text{in } \bar{G}, \ n = 1, 2, \ldots$$

$$v_n(x) = \mathcal{P}[v_{n-1}](x) \leq v_{n-1}(x) \quad \text{in } \bar{G}, \ n = 1, 2, \ldots$$

Since the operator $\mathcal{P}$ is increasing and by the assumption (d) we have

$$u_1(x) = \mathcal{P}[u_0](x) \leq \mathcal{P}[v_0](x) = v_1(x) \quad \text{in } \bar{G}$$

and consequently by induction

$$u_n(x) \leq v_n(x) \quad \text{in } \bar{G}.$$ 

Therefore

$$u_0(x) \leq u_1(x) \leq \cdots \leq u_n(x) \leq \cdots \leq v_n(x) \leq \cdots \leq v_1(x) \leq v_0(x).$$

The sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are monotone and bounded, so they have pointwise limits and we can define:

$$\bar{u}(x) := \lim_{n \to \infty} u_n(x), \quad \bar{v}(x) := \lim_{n \to \infty} v_n(x)$$

for every $x \in \bar{G}$.

The functions $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are functions of class $L^p_G$. Let be $p \in (m, \infty)$ (we need this assumption to can use a imbedding theorem). Because $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are bounded functions in $L^p_G$ and $\mathcal{P}$ is an increasing compact operator in $L^p_G$ (from Lemma 2), $\{\mathcal{P}u_n\}$ and $\{\mathcal{P}v_n\}$ are converging sequences in $L^p_G$ and

$$\bar{u}(x) = \lim_{n \to \infty} \mathcal{P}[u_n](x) = \lim_{n \to \infty} \mathcal{P}[[u_{n-1}]](x) = \mathcal{P}[\bar{u}](x),$$

$$\bar{v}(x) = \lim_{n \to \infty} \mathcal{P}[v_n](x) = \lim_{n \to \infty} \mathcal{P}[[v_{n-1}]](x) = \mathcal{P}[\bar{v}](x).$$

Since $\bar{u}, \bar{v} \in L^p_G$ and:

$$\bar{u} = \mathcal{P}[\bar{u}], \quad \bar{v} = \mathcal{P}[\bar{v}]$$

by the Agmon–Douglis–Nirenberg theorem we have

$$\bar{u}, \bar{v} \in H^{2,p}_S(G).$$
Because the Sobolev space $H^{2,p}(\bar{G})$ for $p > m$ is continuously imbedding in $C^{0+\alpha}(\bar{G})$ and
\[ |u'|_{C^{0+\alpha}(\bar{G})} \leq C |u'|_{H^{2,p}(G)}, \]
where $C$ is independent of $i$ ([1], p. 97–98), we get
\[ u, \bar{v} \in C^{0+\alpha}(\bar{G}). \tag{11} \]
Applying the Schauder theorem to (10) separately for every $s \in S$ for (11) we get
\[ u, \bar{v} \in C^{2+\alpha}(S). \]

From the proof we know that
\[ u_0(x) \leq u(x) \leq \bar{v}(x) \leq v_0(x). \]

**Corollary.** The solutions $u, \bar{v}$ are minimal and maximal solution of the problem (1), (2) in $\langle u_0, v_0 \rangle$.

**Proof.** If $w$ is a solution of the problem (1), (2) then $w(x) = \mathcal{P}[w](x)$ and $u_0(x) \leq w(x) \leq v_0(x)$. Because $\mathcal{P}$ is an increasing operator, we have
\[ u_1(x) = \mathcal{P}[u_0](x) \leq \mathcal{P}[w](x) = w(x) = \mathcal{P}[w](x) \leq \mathcal{P}[v_0](x) = v_1(x) \]
and by induction we get
\[ u_n(x) \leq w(x) \leq v_n(x). \]
Thus
\[ u(x) = \lim_{n \to \infty} u_n(x) \leq w(x) \leq \lim_{n \to \infty} v_n(x) = \bar{v}(x). \]

**REFERENCES**


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