A NECESSARY AND SUFFICIENT CONDITION
FOR $\sigma$-HURWITZ STABILITY
OF THE CONVEX COMBINATION
OF THE POLYNOMIALS

Abstract. In the paper are given a necessary and sufficient condition for $\sigma$-Hurwitz stability of the convex combination of the polynomials.

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1. INTRODUCTION

We will consider the set of real polynomials
\[ F(x, Q) = \left\{ a_n(q)x^n + a_{n-1}(q)x^{n-1} + \cdots + a_1(q)x + a_0(q) \right\}, \]
where \( q = (q_1, q_2, \ldots, q_k) \in Q \subset R^k \), \( Q \) is a compact set, \( a_i(q): Q \to R \) \( i = 0, 1, \ldots, n \), \( a_n(q) \neq 0 \) for each \( q \in Q \).

Let \( \sigma \in R \) and \( \sigma > 0 \).

Definition 1. We shall say that the real polynomial
\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n (x - x_1)(x - x_2) \cdots (x - x_n) \]  (1)
where \( a_n \neq 0 \), is Hurwitz stable if \( \text{Re}(x_i) < 0 \) \( i = 1, 2, \ldots, n \). The polynomial (1) is called $\sigma$-Hurwitz stable if \( \text{Re}(x_i) < -\sigma \) \( i = 1, 2, \ldots, n \).

Definition 2. The set of the polynomials \( F(x, Q) \) is called $\sigma$-Hurwitz stable if each polynomial \( g(x) \in F(x, Q) \) is $\sigma$-Hurwitz stable.
Consider the interval polynomial
\[ G(x) = [a_n, \overline{a}_n]x^n + [a_{n-1}, \overline{a}_{n-1}]x^{n-1} + \cdots + [a_1, \overline{a}_1]x + [a_0, \overline{a}_0], \]
and the set of the polynomials
\[ W(x) = \{ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 : a_i \in [a_i, \overline{a}_i], i = 0, 1, \ldots, n \}. \]

The following theorem is true

**Theorem 1 (Bhattacharyya, Chapellat, Keel [2]).** The interval real polynomial
\[ G(x) = [a_n, \overline{a}_n]x^n + [a_{n-1}, \overline{a}_{n-1}]x^{n-1} + \cdots + [a_1, \overline{a}_1]x + [a_0, \overline{a}_0], \]
where \( 0 \not\in [a_n, \overline{a}_n] \), is \( \sigma \)-Hurwitz stable if and only if the set of the polynomials \( W(x) \) is \( \sigma \)-Hurwitz stable.

Let
\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n (x - x_1)(x - x_2) \cdots (x - x_n), \]
where \( a_n \neq 0 \).

Denote by \( H(f) \) the Hurwitz matrix for the polynomial \( f(x) \), i.e.
\[
H(f) = \begin{bmatrix}
 a_{n-1} & a_n & 0 & 0 & 0 & \cdots & 0 \\
 a_{n-2} & a_{n-1} & a_n & 0 & 0 & \cdots & 0 \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
 & & & & & & \\
 0 & 0 & 0 & \cdots & \cdots & \cdots & a_0 \\
 \end{bmatrix}.
\]

It is easy to see that \( H(f) \in \mathbb{R}^{n \times n} \).

Consider the real polynomials
\[ f_j(x) = a_n^{(j)} x^n + a_{n-1}^{(j)} x^{n-1} + \cdots + a_1^{(j)} x + a_0^{(j)} \quad (2) \]
for \( j = 1, 2, \ldots, m \), where \( a_n^{(j)} \neq 0 \) \( (j = 1, 2, \ldots, m) \), and the convex combinations of these polynomials
\[ C(f_1, f_2, \ldots, f_m) = \{ \alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x) : \alpha_j \geq 0 \quad (j = 1, 2, \ldots, m), \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1 \}. \]

In this paper we give the necessary and sufficient condition for \( \sigma \)-Hurwitz stability of the convex combination \( C(f_1, f_2, \ldots, f_m) \).

We assume that the polynomials (2) are Hurwitz stable. Hence, follows that there exists the inverse matrix \( H^{-1}(f_j) \) \( (j = 1, 2, \ldots, m) \).

Let
\[ \lambda_k \left( H^{-1}(f_j) H(f_i) \right) \quad (k = 1, 2, \ldots, n; i, j = 1, 2, \ldots, m; j < i) \]
denote the eigenvalues of the matrix \( H^{-1}(f_j) H(f_i) \).
The following theorems are true:

**Theorem 2 (Białas [3])**. If the real polynomials

\[
  f_1(x) = a_n^{(1)} x^n + a_{n-1}^{(1)} x^{n-1} + \cdots + a_1^{(1)} x + a_0^{(1)},
\]

\[
  f_2(x) = a_n^{(2)} x^n + a_{n-1}^{(2)} x^{n-1} + \cdots + a_1^{(2)} x + a_0^{(2)},
\]

where \( a_n^{(1)} \neq 0, a_n^{(2)} \neq 0 \), are Hurwitz stable, then the convex combination

\[
  C(f_1, f_2) = \{ \alpha_1 f_1(x) + \alpha_2 f_2(x) : \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1 \}
\]

is Hurwitz stable if and only if

\[
  \lambda_k(H^{-1}(f_1)H(f_2)) \notin (-\infty, 0) \quad (k = 1, 2, \ldots, n).
\]

**Theorem 3 (Bartlett, Hollot, Huang [1])**. If the polynomials (2) are Hurwitz stable, then the convex combination

\[
  C(f_1, f_2, \ldots, f_m) = \{ \alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x) : \alpha_j \geq 0 \quad (j = 1, 2, \ldots, m), \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1 \}
\]

(3)

is Hurwitz stable if and only if the convex combinations \( C(f_i, f_j) \) are Hurwitz stable for each \( i, j = 1, 2, \ldots, m; i < j \).

2. **MAIN RESULT**

Let

\[
  f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,
\]

where \( a_n \neq 0 \).

It is easy to note that for \( \alpha \in R \) we have

\[
  g(s) = f(s + \alpha) = b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0,
\]

where

\[
  b_0 = f(\alpha),
\]

\[
  b_i = \frac{1}{i!} \left. \frac{d^i f(x)}{dx^i} \right|_{x=\alpha} \quad (i = 1, 2, \ldots, n).
\]

As it is easy to see, we have the following result.
Lemma 1. The real polynomial
\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \]
where \( a_n \neq 0 \), is \( \sigma \)-Hurwitz stable if and only if the polynomial
\[ g(s) = f(s - \sigma) \]
is Hurwitz stable.

Now, we will prove

**Theorem 4.** If the polynomials (2) are \( \sigma \)-Hurwitz stable, then the convex combination
\[ C(f_1, f_2, \ldots, f_m) = \{ \alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x) : \alpha_j \geq 0 \ (j = 1, 2, \ldots, m), \ \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1 \} \]
is \( \sigma \)-Hurwitz stable if and only if
\[ \lambda_k(H^{-1}(g_i)H(g_j)) \not\in (-\infty, 0) \ (k = 1, 2, \ldots, n) \] (4)
for \( i, j = 1, 2, \ldots, m; \ i < j \), where \( g_i(s) = f_i(s - \delta), g_j(s) = f_j(s - \delta) \).

**Proof.** From Lemma 1, it follows that the convex combination \( C(f_i, f_j) \) is \( \sigma \)-Hurwitz stable if and only if the convex combination \( C(g_i, g_j) \) is Hurwitz stable.

However, from Theorem 2 and 3 follows that the set \( C(g_i, g_j) \) is Hurwitz stable if and only if the conditions (4) holds. This completes the proof of Theorem 4.

**REFERENCES**


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