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WEAKLY CONVEX AND CONVEX DOMINATION NUMBERS

Abstract. Two new domination parameters for a connected graph $G$: the weakly convex domination number of $G$ and the convex domination number of $G$ are introduced. Relations between these parameters and the other domination parameters are derived. In particular, we study for which cubic graphs the convex domination number equals the connected domination number.

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1. INTRODUCTION

Let $G = (V, E)$ be a connected undirected graph. The neighbourhood of a vertex $v \in V$ in $G$ is the set $N_G(v)$ of all vertices adjacent to $v$ in $G$. For a set $X \subseteq V$, the open neighbourhood $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and the closed neighbourhood $N_G[X] = N_G(X) \cup X$. Let $X$ be a set of vertices and let $u \in X$. We say that a vertex $v$ is a private neighbour of $u$, with respect to $X$, if $N_G[v] \cap X = \{u\}$.

The degree $\deg_G(v)$ of a vertex $v$ is the number of edges incident to $v$, $\deg_G(v) = |N_G(v)|$. If $\deg_G(v) = r$ for every vertex $v \in V$ in $G$, then $G$ is said to be regular of degree $r$, or simply $r$-regular. A 3-regular graph is also called a cubic graph. A 2-regular graph of order $n$ is a cycle and is denoted by $C_n$. If $\deg_G(v) = 1$, then $v$ is called an end-vertex or a leaf of $G$. A vertex which is a neighbour of an end-vertex let us call a support. A vertex $x \in V$ is called a universal vertex (or a dominating vertex) if $\deg(x) = |V(G)| - 1$. A set $D \subset V$ is a dominating set of $G$ if $N_G[D] = V$. A dominating set $D$ is a perfect dominating set if $|N_G(v) \cap D| = 1$ for each $v \in V - D$. Further, $D$ is a connected dominating set if $D$ is dominating and the subgraph $\langle D \rangle$ induced by $D$, is connected. A set $D \subset V$ is independent
in $G$ if no two vertices of $D$ are adjacent. A set is independent dominating if it is independent and dominating. The domination number of $G$, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in $G$, while the connected domination number $\gamma_c(G)$ is the minimum cardinality of a connected dominating set in $G$. The distance $d_G(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of the shortest $(u - v)$ path in $G$. A $(u - v)$ path of length $d_G(u, v)$ is called $(u - v)$-geodesic. The diameter diam $G$ of a connected graph $G$ is $\max_{u, v \in V(G)} d_G(u, v)$. A set $X \subseteq V$ is called weakly convex (or isometric [4]) in $G$ if for every two vertices $a, b \in X$ exists $(a - b)$-geodesic whose vertices belong to $X$. A set $X$ is convex in $G$ if vertices from all $(a - b)$-geodesic belong to $X$ for every two vertices $a, b \in X$. A set $X \subseteq V$ is a weakly convex dominating set if $X$ is weakly convex and dominating. Further, $X$ is a convex dominating set if it is convex and dominating. The weakly convex domination number $\gamma_{wcon}(G)$ of a graph $G$ is the minimum cardinality of a weakly convex dominating set, while the convex domination number $\gamma_{con}(G)$ of a graph $G$ is the minimum cardinality of a convex dominating set. Convex and weakly convex domination numbers were first introduced by Jerzy Topp, Gdańsk University of Technology, 2002. The union $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$ and the join $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv: u \in V(G_1), v \in V(G_2)\}$. The corona $G = H \circ K_1$ is the graph $G$ constructed from a copy of $H$, where for each vertex $v \in V(H)$, a new vertex $v'$ and a pendant edge $vv'$ are added.

2. RELATIONS BETWEEN $\gamma_{wcon}$, $\gamma_{con}$ AND THE OTHER DOMINATION PARAMETERS

Since every convex dominating set is weakly convex dominating set and every weakly convex dominating set is connected dominating set, we have following inequality chain.

**Lemma 1.** For any connected graph $G$ is

$$\gamma(G) \leq \gamma_c(G) \leq \gamma_{wcon}(G) \leq \gamma_{con}(G).$$

We show that the differences $\gamma_{con} - \gamma_c$ and $\gamma_{con} - \gamma_{wcon}$ can be arbitrarily large.

**Theorem 1.** For any $k, r \in N$ where $r \geq 3$, there exists a graph $G$ such that $\gamma_c(G) = \gamma_{wcon}(G) = r$ and $\gamma_{con}(G) - \gamma_c(G) = \gamma_{con}(G) - \gamma_{wcon}(G) = k$.

**Proof.** Assume first that $k \in N, r = 3$.

Let $H$ be a complete bipartite graph isomorphic to $K_{k+1,2}$ and let $x$ be a vertex of minimum degree in $H$. Let $y$ and $z$ be two neighbours of $x$. Let $G$ be a graph
which results if we identify \( y \) with one support of \( P_6 \) (a path with six vertices) and \( z \) with another support of \( P_6 \). The set \( \{x, y, z\} \) is a minimum connected dominating set and a weakly convex set in \( G \), so we have \( \gamma_c(G) = \gamma_{wcon}(G) = 3 \). It is easy to observe that \( V(H) \) is a minimum convex dominating set of \( G \) and therefore \( \gamma_{con}(G) = k + 3 \). Thus \( \gamma_{con}(G) - \gamma_c(G) = \gamma_{con}(G) - \gamma_{wcon}(G) = k \).

If \( r \geq 4 \), then the graph \( G \) is constructed as follows. Let \( x \) and \( y \) be two non-end-vertices at distance two in the corona \( C_r \circ K_1 \). Let \( u \) and \( v \) be two vertices of a complete bipartite graph \( K_{k,2} \) which form its partite set. Let \( G \) be a graph which results if we identify \( x \) with \( u \) and \( y \) with \( v \) (see Fig. 1 for \( r = 8, k = 2 \)).

![Fig. 1](image)

In such a graph \( G \), the minimum connected dominating set consists of all supports. This set is also weakly convex, so \( \gamma_c(G) = \gamma_{wcon}(G) = r \). It is obvious that the minimum convex dominating set consists of all non-end-vertices, so \( \gamma_{con}(G) = n - r = k + r \). Thus \( \gamma_{con}(G) - \gamma_c(G) = \gamma_{con}(G) - \gamma_{wcon}(G) = k \).

**Theorem 2.** For any positive integers \( k \) and \( l, k \geq 3 \), there is a graph \( G \), for which \( \gamma_c(G) = k \) and \( \gamma_{con}(G) = k + l \).

**Proof.** We begin with the corona \( C_k \circ K_1 \) and then, to receive a graph \( G \), we replace any non-end edge \( uv \) by \( l \) inner disjoint \( (u - v) \) paths of length two (see Fig. 2 for \( k = 7, l = 3 \)).

It is easy to observe that in such a graph \( G \), the minimum connected dominating set consists of all supports of \( G \) and certainly \( \gamma_c(G) = k \). It is also easy to observe that the minimum convex dominating set consists of all non-end vertices of \( G \) and \( \gamma_{con}(G) = k + l \).

Duchet and Meyniel [1] have shown that for any connected graph \( G \) is \( \gamma_c(G) \leq 2/\beta_0(G) - 1 \) and \( \gamma_c(G) \leq 2\Gamma(G) - 1 \), where \( \Gamma(G) \) is the maximum cardinality of a minimal dominating set of \( G \) and \( \beta_0 \) is the maximum cardinality of an independent dominating set of \( G \).
The next theorem shows that there is no case for the convex domination number of a graph.

**Theorem 3.** Every of differences $\gamma_{\text{con}} - \beta_0$ and $\gamma_{\text{con}} - \Gamma$ can be arbitrarily large.

*Proof.* The graph $G$ is constructed as follows. We begin with a cycle $C_{2k+2}$ with $2k + 2$ vertices and label consecutive vertices $v_1, \ldots, v_k, z, w_k, w_{k-1}, \ldots, w_1, u$. Then we add the edges $v_iw_i, v_iw_{i+1}$ and $w_iw_{i+1}$, for $i = 1, \ldots, k-1$ and to each of vertices $u$ and $z$ we add an end edge (see Fig. 3 for $k = 3$). This completes the construction of $G$. If we want to find a minimum convex dominating set of $G$, we must put there supports $u$ and $z$. Vertices $v_i$ and $w_i$, where $i = 1, \ldots, k$ belong to the shortest paths between $u$ and $z$, so we must also put them to a minimum convex dominating set.
Thus we have $\gamma_{\text{con}}(G) = 2k + 2$. If we want to find a maximum independent set of $G$, we must put there the end-vertices. Then the neighbours of these vertices, formed by vertices of degree 3 from the cycle $C_{2k+2}$, do not belong to this set. The rest of vertices from the cycle, i.e., vertices of degree 4 and 5, we can put into $\lceil \frac{k}{2} \rceil$ disjoint sets, where the induced subgraphs by these sets are complete graphs (if $k$ is even, then we have only graphs $K_4$ and if $k$ is odd, then except for the graphs $K_4$ we have one graph $K_2$). Only one vertex from each of these subgraphs can belong to a maximum independent set, so $\beta_0(G) = \lceil \frac{k}{2} \rceil + 2$ and $\gamma_{\text{con}}(G) - \beta_0(G) = 2k - \lceil \frac{k}{2} \rceil = \lfloor \frac{3k}{2} \rfloor$. Observe, that the maximum independent set in this graph is also the maximum minimal dominating set of $G$. Thus $\Gamma(G) = \beta_0(G) = \lceil \frac{k}{2} \rceil + 2$ and $\gamma_{\text{con}}(G) - \Gamma(G) = \lfloor \frac{3k}{2} \rfloor$.

3. EQUALITY $\gamma_c = \gamma_{\text{con}}$ FOR CUBIC GRAPHS

Now we study cubic graphs, for which $\gamma_c$ is equal to $\gamma_{\text{con}}$.

**Theorem 4.** If $G$ is a cubic graph and there exists a minimum connected dominating set $D$ in $G$ such that $\langle D \rangle$ is a star, then $\gamma_c(G) = \gamma_{\text{con}}(G)$.

**Proof.** Let $G$ be a cubic graph and let $D$ be a minimum connected dominating set of $G$ such that $\langle D \rangle$ is a star. If $\langle D \rangle = K_1$ or $\langle D \rangle = K_{1,1}$, then $D$ consists of one or two vertices and $D$ is convex. So we have required equality. If $\langle D \rangle = K_{1,2}$, then a minimum connected dominating set consists of three vertices, so in such a graph $G$ we have $\gamma_c(G) = 3$.

Graph $G$ is cubic, and since $\sum_{v \in V} \deg_G(v) = 2|E|$, it is of even order. For $n = 4$ we have $G = K_4$, so $\gamma_c(G) = \gamma_{\text{con}}(G) = 1$. It is easy to check, that for every cubic graph $G$ of order $n = 6$ we have $\gamma_c(G) = \gamma_{\text{con}}(G) = 2$, (except for a graph $G^*$ in Fig. 4, for which $\gamma_c(G^*) = 3$, but in this graph does not exist a minimum connected dominating set $D$ such that $\langle D \rangle$ is a star). Thus, since $\gamma_c(G) = 3$ and $G \neq G^*$, we have $n \geq 8$.

![Fig. 4. Graf $G^*$](image-url)
Let us denote vertices from a set $D$ successively $v_1, v_2, v_3$, and vertices from $V - D$ successively $v_4, v_5, \ldots, v_n$. Suppose, that $\gamma_c(G) < \gamma_{\text{con}}(G)$. Then $D$ is not a convex set and there exists a vertex belonging to $V - D$ (without loss of generality, a vertex $v_4$) such that it is a neighbour of $v_1$ and $v_3$. Each of vertices $v_5, \ldots, v_n$ must have a neighbour in $D$. So, since $G$ is cubic, we have $n \leq 7$, which gives a contradiction. Thus $D$ is a convex set in $G$ and $\gamma_c(G) = \gamma_{\text{con}}(G)$.

Now suppose that $\langle D \rangle = K_{1,3}$. We know that $n \geq 8$ and we consider two cases: $n = 8$ and $n > 8$. Let us denote by $v_1$ a vertex of degree 3 in $\langle D \rangle$ and the rest of vertices from $\langle D \rangle$ successively by $v_2, v_3, v_4$.

1) $n = 8$.

Vertices from $V - D$ we denote by $v_5, \ldots, v_8$. Suppose that $\gamma_c(G) < \gamma_{\text{con}}(G)$, that is a set $D$ is not convex. Then there exists a vertex in $V - D$ (without loss of generality, a vertex $v_5$) such that it has two neighbours in $D$, say, vertices $v_2, v_3$.

If any other vertex from $V - D$ has two neighbours in $D$, then $D - \{v_3\}$ (or $D - \{v_2\}$) is a connected dominating set, which gives a contradiction. Thus exactly one vertex from $V - D$ has two neighbours in $D$ (this is a vertex $v_5$).

If the vertex $v_5$ has three neighbours in $D$, then every of vertices $v_6, v_7, v_8$ has exactly one neighbour in $D$. Moreover, none of these vertices has a common neighbour in $D$ and the subgraph $\langle v_6, v_7, v_8 \rangle$ induced by $v_6, v_7, v_8$ is $K_3$. In such a graph $G$, vertices $v_4, v_6, v_7, v_8$ create a convex dominating set of cardinality $|D|$, which gives a contradiction. Thus, the vertex $v_5$ has exactly two neighbours in $D$, each of vertices $v_6, v_7, v_8$ has exactly one neighbour in $D$ and none of these vertices has a common neighbour in $D$. Then some of vertices belonging to $V - D$ must be neighbours of $v_4$, which gives a contradiction. Thus $D$ is a convex set.

2) $n > 8$.

If $n > 8$, then $n \geq 10$, because $n$ is even. Let us denote by $v_5, \ldots, v_n$ the vertices from $V - D$. Suppose that $D$ is not convex. Then there exists a vertex in $V - D$ having two neighbours in $D$. Thus, since $G$ is cubic, we have $n \leq 9$, which gives a contradiction.

\begin{theorem}
If $G$ is a cubic graph of order $n$ and there exists a minimum connected dominating set $D$ in $G$ such that $\langle D \rangle$ is a cycle $C_p$, then:

a) $p \geq 4$, $n \geq 8$; if $p = 3$, then $n = 6$ and $G = G^*$ or

b) $D$ is a perfect dominating set and every vertex from $D$ has exactly one private neighbour or

c) $n = 2p$ or

d) if $p < 6$, then $\gamma_c(G) = \gamma_{\text{con}}(G)$.
\end{theorem}
Weakly convex and convex domination numbers

Proof. Let \( G \) be a cubic graph of order \( n \) and let \( D \) be a minimum connected dominating set of \( G \) such that \( (D) \) is a cycle \( C_p \). Suppose that a) does not hold. Then \( p = 3, n \geq 8 \). Thus there exist at least 5 vertices in \( V - D \), and since \( G \) is cubic and \( p = 3 \), then \( |V - D| \leq 3 \), which gives a contradiction.

Suppose that b) does not hold. If \( D \) is not a perfect dominating set, then there exists a vertex \( v \in V - D \) such that it has at least two neighbours in \( D \), let us denote it \( u_1, u_2 \). Then \( D - \{u_1\} \) (or \( D - \{u_2\} \)) is a connected dominating set, which gives a contradiction. If there exists a vertex \( u \in D \) having two private neighbours, then \( d(u) > 3 \), which gives contradiction.

If a) and b) hold, then \( n = |D| + |V - D| = |D| + |D| = 2|D| = 2p \).

If \( p < 6 \), then \( p = 3 \) and \( G = G^* \) or \( p = 4 \) or \( p = 5 \). For \( p = 3 \) and \( G = G^* \) we have \( \gamma_c(G^*) = \gamma_{con}(G^*) = 3 \). If \( p = 4 \) or \( p = 5 \), then suppose that \( \gamma_c(G) < \gamma_{con}(G) \), that is \( D \) is not convex. Then there exists a vertex \( v \in V - D \) such that \( v \) is a neighbour of two non-adjacent vertices from \( D \), which contradicts b).

Theorem 6. If \( G \) is a cubic graph and there exists the minimum connected dominating set \( D \) in \( G \) such that \( \text{diam}((D)) \leq 2 \), then \( \gamma_c(G) = \gamma_{con}(G) \).

Proof. Let \( G \) be a cubic graph and let \( D \) be a minimum connected dominating set of \( G \) such that \( \text{diam}((D)) \leq 2 \). If \( \text{diam}((D)) = 1 \), then there exists a dominating vertex in \( G \) and a minimum connected and simultaneously convex dominating set consists of this vertex and we have required equality. Suppose that \( (D) \) has a diameter equal to 2. It is easy to observe, that \( |V((D))| \geq 3 \). If \( |V((D))| = 3 \), then \( (D) = C_3 \) or \( (D) = K_{1,2} \). So from Theorem 5 and Theorem 4 we have \( \gamma_c(G) = \gamma_{con}(G) \). If \( |V((D))| = 4 \), then:

a) \( (D) = K_{1,3} \). Then from Theorem 4 we have \( \gamma_c(G) = \gamma_{con}(G) \).

b) \( (D) = C_4 \). Then from Theorem 5 we have \( \gamma_c(G) = \gamma_{con}(G) \).

c) \( (D) = K_1 + (K_1 \cup K_2) \) Then \( |V - D| = 4 \), in the other case we would find in \( G \) a connected dominating set \( D_1 \) such that \( |D_1| < |D| \). Every vertex from \( V - D \) is a neighbour of exactly one vertex from \( D \) and it is easy to observe that \( D \) is convex. So we have required equality.

\( d) (D) = C_4 \cup \{e\} \), where \( e \) is any chord of the cycle \( C_4 \). Then \( |V - D| = 6 \), which gives \( n \leq 6 \), contradiction, because in cubic graphs we have \( \gamma_c \leq 3 \) for \( n \leq 6 \).

If \( |V((D))| = 5 \), then:

a) \( (D) = C_5 \). Then, from Theorem 5 we have \( \gamma_c(G) = \gamma_{con}(G) \).

b) \( (D) = C_5 \cup \{e\} \), where \( e \) is any chord of the cycle \( C_5 \). Then \( |V - D| \leq 3 \) and we find in \( G \) a connected dominating set \( D_1 \) such that \( |D_1| < |D| \), contradiction.

c) \( (D) = C_5 \cup \{e_1, e_2\} \), where \( e_1, e_2 \) are chords of a cycle \( C_5 \) having no common vertex. Then \( |V - D| = 1 \) and a vertex \( v \in V - D \) is of degree 1, which is impossible in cubic graphs.
If $|V(D)| = 6$, then every vertex in $D$ is of degree 3. Thus $D - \{v\}$, where $v$ is any vertex from $D$, is a connected dominating set, which gives a contradiction. If $\langle D \rangle$ has more than six vertices, then to have a diameter equal to two, a vertex of degree greater than three must exists in $G$, which is impossible in cubic graphs. \qed

**Theorem 7.** If $G$ is cubic and $\gamma(G) = \gamma_c(G)$, then $\gamma_c(G) = \gamma_{con}(G)$.

**Proof.** In [2] Arumugam and Paulraj showed that if $G$ is cubic and $\gamma(G) = \gamma_c(G)$, then $\gamma_c \leq 3$. Thus from Theorem 6 follows immediately that $\gamma_c(G) = \gamma_{con}(G)$. \qed

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