A LOCAL EXISTENCE THEOREM OF THE SOLUTION OF THE CAUCHY PROBLEM FOR BBGKY CHAIN OF EQUATIONS REPRESENTED IN CUMULANT EXPANSIONS IN THE SPACE $E_\xi$

Abstract. It is proved convergence of solution in cumulant expansions of the initial value problem for BBGKY chain of equations of non-symmetrical one-dimensional system of particles which interact via a short-range potential in the space $E_\xi$ of the sequences of continuous bounded functions.

Keywords: Non-symmetrical particle systems, space of the sequences of continuous bounded functions, BBGKY chain of equations, cumulant.

Mathematics Subject Classification: 35Q30.

1. INTRODUCTION

State of non-symmetrical one-dimensional infinite particle system can be described by the solution of the Cauchy problem for the BBGKY chain of equations represented in the form of cumulant expansions in the space $E_\xi$ of the sequences of continuous bounded functions with the norm

$$
\|f\| = \sup_{n \geq 0} \xi^{-n} \sup_{x \in \mathbb{R}^n} |f_n(x_{-n_2}, \ldots, x_{n_1})| \exp\left[\beta \sum_{i=-n_2}^{n_1} \frac{p_i^2}{2}\right], \quad \xi, \beta > 0.
$$

The space $E_\xi$ is different from the one $L^1_\alpha$ in which the solution was constructed [1]. The cumulant structure of the solution solves the problem of divergence of each term of the series (the solution is represented through the functional series) for initial data belonging to the space $E_\xi$. 
In this paper we prove convergence of the considered solution for $F(0) \in E_\xi$ on the finite time interval using method of interaction region. This method was formulated in [2], and was used to prove existence theorems for three-dimensional systems in [3,4] and for non-symmetrical systems in [5].

2. FORMULATION OF THE PROBLEM

Consider a one-dimensional non-symmetrical system of particles of unit mass interacting with their nearest neighbors via the potential which satisfies the following conditions
\[
\begin{align*}
\Phi(q) & \in C^2(0,R], \\
\Phi(q) & = 0, \text{if } q \in (R, +\infty),
\end{align*}
\]
from which it follows that the conditions of stability and regularity are fulfilled, and the following relations hold
\[
\left| \sum_{i=-n_2}^{n_1-1} \Phi(q_i - q_{i+1}) \right| \leq B(n_1 + n_2), B > 0,
\]
\[
\sup_i |F_i| = D < +\infty,
\]
where $F_i$ is a force acting upon the $i$-th particle of its nearest neighbors side.

For the sake of simplicity we denote $n = n_1 + n_2, s = s_1 + s_2$.

The solution of the Cauchy problem for BBGKY chain of equations of non-symmetrical particle system in space $L^1_1$ is given by the formula [1]
\[
F_{|Y|}(t,Y) = \sum_{n=0}^{\infty} \sum_{n_1+n_2=n} \int_{(R^1 \times R^1)^{n_1+n_2}} d(X \setminus Y) U_{(n_2,n_1)}(t,XY) F_{|X|}(0,X),
\]
where $Y = (x_{-s_2}, \ldots, x_{s_1}), X = (x_{-(n_2+s_2)}, \ldots, x_{s_1+n_1}), X_Y = (x_{-(n_2+s_2)}, \ldots, x_{-(s_2+1)}, x_{s_1+1}, \ldots, x_{s_1+n_1})$. Cumulant $U_{(n_2,n_1)}(t,XY)$ has the form
\[
U_{(n_2,n_1)}(t,XY) = \sum_{P: X_Y = \bigcup_i X_i} (-1)^{|P|-1} \prod_{X_i \subset P} S_{|X_i|}(-t, X_i),
\]
where $\sum_P$ — the sum over all ordered divisions of partly ordered set $X_Y$ onto $|P|$ nonempty ordered subsets $X_i \subset X_Y$, which do not intersect each other, and set $Y$ belongs entirely to one of subsets $X_i$.

Our aim is to give a mathematical meaning to the series (3) for initial data $F(0)$ from the space $E_\xi$. It is easy to see that each integrand in the formula (3) with an arbitrary fixed $n$ differs from zero only in a bounded region of the configuration space, which will be called the interaction region and denoted by $\Omega_{n_1+n_2}(t)$. Thus, each term in (3) with a fixed $n$ is meaningful and the corresponding integrals will be convergent. To prove the existence of $F_{|Y|}(t,Y)$ it is only required to show that the series (3) converges.
3. THE ESTIMATE OF THE SOLUTION REPRESENTED IN CUMULANT EXPANSIONS IN NON-EXPLICIT FORM

Taking into account the invariance of the Gibbs distributions with respect to the acting of evolution operator $S(-t)$ and using the conditions (2) we have

$$S(-t, x_{k+1}, \ldots, x_{k+n}) \exp \left(-\beta \sum_{i=k+1}^{k+n} \frac{p_i^2}{2} \right) \leq \exp (2\beta B n) \exp \left(-\beta \sum_{i=k+1}^{k+n} \frac{p_i^2}{2} \right).$$

According to these inequalities and if initial data $F(0) \in E_\xi$ the following estimate holds

$$|F|_Y (t, Y) \leq \|F(0)\| \sum_{n=0}^{\infty} \sum_{n_1+n_2=n \atop n_1, n_2 \geq 0} \int_{R^n \times \Omega_n(t)} d(X \setminus Y)^n \times \exp (2\beta B(n + s)) \xi^{n+s} \exp \left(-\beta \sum_{i=-(n_2+s_2)}^{s_1+n_1} \frac{p_i^2}{2} \right).$$

(4)

Since the volume of the interaction region $V_n(t) \equiv V_n(t, p_{-(n_2+s_2)}, \ldots, p_{-(s_2+1)}, p_{n_1+1}, \ldots, p_{n_1+n_1})$ (4) is transformed in the following way

$$|F|_Y (t, Y) \leq \|F(0)\| \sum_{n=0}^{\infty} \sum_{n_1+n_2=n \atop n_1, n_2 \geq 0} 2^n \exp (2\beta B(n + s)) \xi^{n+s} \int_{R^n} dp_{-(n_2+s_2)} \ldots \times dp_{-(s_2+1)} dp_{n_1+1} \ldots dp_{n_1+n_1} V_n(t) \exp \left(-\beta \sum_{i=-(n_2+s_2)}^{s_1+n_1} \frac{p_i^2}{2} \right).$$

(5)

Denote by:

— $V_{n_1}(t) \equiv \text{the interaction region volume of the right-hand side particles } x_{s_1+1}, \ldots, x_{s_1+n_1},$

— $V_{n_2}(t) \equiv \text{the interaction region volume of the left-hand side particles } x_{-(n_2+s_2)}, \ldots, x_{-(s_2+1)},$

then $V_n(t) = V_{n_1}(t)V_{n_2}(t)$, besides denote $b = 2\xi \exp (2\beta B)$.

Therefore, the estimate (5) takes on the form

$$|F|_Y (t, Y) \leq \|F(0)\| \xi^{s} \exp (2\beta Bs) \exp \left(-\beta \sum_{i=-s_2}^{s_1} \frac{p_i^2}{2} \right) A_1 A_2,$$

(6)
where

\[ A_1 = \sum_{n_1=0}^{\infty} b_{n_1} \int_{R_{n_1}} dp_{s_1+1} \ldots dp_{s_1+n_1} V_{n_1}(t) \exp \left( -\beta \sum_{i=s_1+1}^{n_1} \frac{p_i^2}{2} \right) , \]

\[ A_2 = \sum_{n_2=0}^{\infty} b_{n_2} \int_{R_{n_2}} dp_{-(n_2+s_2)} \ldots dp_{-(s_2+1)} V_{n_2}(t) \exp \left( -\beta \sum_{i=-(n_2+s_2)}^{-(s_2+1)} \frac{p_i^2}{2} \right) . \]

For the sake of simplicity we shall prove convergence of \( A_1 \) (convergence of \( A_2 \) can be proved analogously).

4. ON THE ESTIMATE OF THE INTERACTION REGION VOLUME \( V_{n_1}(t) \) IN EXPLICIT FORM

We fix the position \( q_{s_1} \) of the last particle from the group \( Y = (x_{s_1}, \ldots, x_{s_1}) \) and denote this position by a conditional origin of reference system along a straight line.

Denote by \( \Upsilon_k \) the maximum interval such that \( k \) particles located within it can interact, then

\[ \Upsilon_k = \Upsilon_{k-1} + \Delta s_k, \]

\[ \Upsilon_{k+1} = \Upsilon_k + \Delta s_{k+1}, \]

where \( \Delta s_k \) is a maximum admissible distance by which the particle \( q_{s_1+k} \) increases the total run interval \( \Upsilon_{k-1} \) of the particles \( q_{s_1+1}, q_{s_1+2}, \ldots, q_{s_1+k-1} \).

Estimate \( \Delta s_k \) — the maximum relative run of the particle \( q_{s_1+k} \) during the time \( 2t \). If some particle (say \( k \)-particle) has the initial momentum \( |p_k| \), then we divide its motion into two parts during the time \( 2t \):

— other particles act with the force, limited by \( D \), upon this particle during the time \( \tau \). Thus, its velocity increases to \( |p_k| + D\tau \), but relative run of this particle to the other ones cannot be greater than \( R \);

— free run of this particle with constant velocity increases the total interval by

\[ p_{\text{max}}(2t - \tau) = (|p_k| + D\tau)(2t - \tau). \]

Find the maximum of the function

\[ f(\tau) = (p + D\tau)(2t - \tau) \rightarrow \max, \tau \in [0, 2t]. \]

\[ \frac{df}{d\tau} = D(2t - \tau) + (p + D\tau)(-1) = 0 \Rightarrow \tau = t - \frac{p}{2D} \]

\[ \begin{cases} \Delta s = R + 2pt, & \text{if } t \leq \frac{p}{2D}, \\ \Delta s = R + D \left( t + \frac{p}{4D} \right)^2, & \text{if } t \geq \frac{p}{2D}. \end{cases} \]

but if \( p \leq 2Dt \), then

\[ \Delta s \leq R + Dt^2 + pt + \frac{p}{4D}2Dt \leq R + Dt^2 + 2pt. \]
A local existence theorem of the solution of the Cauchy problem...

Thus, the two expressions can be estimated in the following way

$$\triangle s_k \leq R + 2|p_k|t + Dt^2.$$  

Now let us talk about the interaction region volume. For the sake of simplicity denote $V_{n_1} = V_{n_1}(t)$.

$$V_{n_1} = \int_0^{Z_1} dq_{s_1+1} \int_0^{Z_2} dq_{s_1+2} \cdots \int_0^{Z_{n_1}} dq_{s_1+n_1}, \quad (7)$$  

where

$$Z_k = \Upsilon_1 = \triangle s_0 + \triangle s_1,$$

$$\triangle s_0 = 2|p_0|t + 2Dt^2,$$

$$\triangle s_1 = R + 2|p_1|t + Dt^2,$$

$$|p_0| = |p_{s_1}|, \quad |p_1| = |p_{s_1+1}|,$$

and

$$Z_k = \min (\Upsilon_k, q_{s_1+k-1}+\triangle s_{k-1} + \triangle s_k - R),$$

where $Z_k \leq \Upsilon_k$, because the distance between $q_{s_1+k}$ and $q_{s_1}$ cannot be greater than $\Upsilon_k = \triangle s_0 + \triangle s_1 + \ldots + \triangle s_k$. From the other hand, if $q_{s_1+1}, q_{s_1+2}, \ldots, q_{s_1+k-1}$ are very close to the origin of reference system, then $q_{s_1+k}$ cannot be located near extreme point of $\Upsilon_k$, otherwise $q_{s_1+k-1}$ and $q_{s_1+k}$ are not able to interact; therefore, the maximum admissible distance between them is $\triangle s_{k-1} + \triangle s_k - R$.

Using these notations, from (7) we obtain

$$V_{n_1} \leq \int_0^{Z_1} dq_{s_1+1} \cdots \int_0^{Z_{n_1}} dq_{s_1+n_1} =$$

$$= \int_0^{Z_1} dq_{s_1+1} \cdots \int_0^{Z_{n_1-1}} dq_{s_1+n_1-2} (\triangle s_{n_1-1} + \triangle s_{n_1} - R) dq_{s_1+n_1-1},$$

or

$$V_{n_1} \leq V_{n_1-1}(\triangle s_{n_1-1} + \triangle s_{n_1} - R). \quad (8)$$

Finally, prove the convergence of the series

$$A_1 = \sum_{n_1=0}^{\infty} b^{n_1} g_{n_1}, \quad (9)$$

where

$$g_{n_1} = \int_{R^{n_1}} dp_{s_1+1} \cdots dp_{s_1+n_1} V_{n_1}(t) \exp \left( -\beta \sum_{i=s_1+1}^{s_1+n_1} \frac{p_i^2}{2} \right).$$
The last integral can be transformed in this way

\[
\begin{align*}
g_{n_1} &\leq \int_{R^{n_1}} dp_{s_1+1} \cdots dp_{s_1+n_1} V_{n_1-1}(t) \exp \left( -\beta \sum_{i=s_1+1}^{n_1-1} \frac{p_i^2}{2} \right) \\
&\times (\Delta s_{n_1-1} + \Delta s_{n_1} - R) \exp \left( -\beta \frac{p_{s_1+n_1}^2}{2} \right) = \\
&= \int_{R^{n_1-1}} dp_{s_1+1} \cdots dp_{s_1+n_1-1} V_{n_1-1}(t) \\
&\times \exp \left( -\beta \sum_{i=s_1+1}^{n_1-1} \frac{p_i^2}{2} \right) \int_{R} dp_{s_1+n_1} (\Delta s_{n_1-1} + \Delta s_{n_1} - R) \exp \left( -\beta \frac{p_{s_1+n_1}^2}{2} \right).
\end{align*}
\]

(10)

The last integral can be transformed in this way

\[
\int_{-\infty}^{\infty} (2t|p_{s_1+n_1-1}| + 2t|p_{s_1+n_1}| + 2t^2 D + R) \exp \left( -\beta \frac{p_{s_1+n_1}^2}{2} \right) dp_{s_1+n_1} = \\
= (2t|p_{s_1+n_1-1}| + 2t^2 D + R) I_0 + 2t I_1,
\]

where

\[
I_0 = \int_{-\infty}^{\infty} \exp \left( -\beta \frac{p^2}{2} \right) dp = \frac{\sqrt{2\pi}}{\beta},
\]

\[
I_1 = \int_{-\infty}^{\infty} |p| \exp \left( -\beta \frac{p^2}{2} \right) dp = \frac{2}{\beta},
\]

then (10) takes on the form

\[
\begin{align*}
&\int_{R^{n_1-1}} dp_{s_1+1} \cdots dp_{s_1+n_1-1} \left( (2t^2 D + R) I_0 + 2t I_1 \right) V_{n_1-1}(t) \exp \left( -\beta \sum_{i=s_1+1}^{n_1-1} \frac{p_i^2}{2} \right) + \\
&+ I_0 \int_{R^{n_1-1}} dp_{s_1+1} \cdots dp_{s_1+n_1-1} V_{n_1-1}(t) 2t |p_{s_1+n_1-1}| \exp \left( -\beta \sum_{i=s_1+1}^{n_1-1} \frac{p_i^2}{2} \right) = \\
&= \left( (2t^2 D + R) I_0 + 2t I_1 \right) g_{n_1-1} + \\
&+ I_0 \int_{R^{n_1-1}} dp_{s_1+1} \cdots dp_{s_1+n_1-1} V_{n_1-1}(t) 2t |p_{s_1+n_1-1}| \exp \left( -\beta \sum_{i=s_1+1}^{n_1-1} \frac{p_i^2}{2} \right).
\end{align*}
\]

(11)

Note, the function \( V_{n_1-1}(t) \) is no greater than linear one with respect to momentum \(|p_{s_1+n_1-1}|\), therefore, \( V_{n_1-1}(t) = \alpha + \gamma |p_{s_1+n_1-1}| \), where \( \alpha, \gamma \) do not depend on \( p_{s_1+n_1-1} \). Thus, the second term of the formula (11) has the form

\[
2t I_0 (MI_1 + NI_2), \quad \text{and} \quad g_1 = MI_0 + NI_1,
\]
A local existence theorem of the solution of the Cauchy problem... 167

where

\[ I_2 = \int_{-\infty}^{\infty} p^2 \exp \left( -\beta \frac{p^2}{2} \right) \, dp = \frac{1}{\beta} \sqrt{\frac{2\pi}{\beta}}. \]

\[ M = \int_{R^{n-2}} dp_{s_1+1} \ldots dp_{s_1+n_1-2} \alpha \exp \left( -\beta \sum_{i=s_1+1}^{s_1+n_1-2} \frac{p_i^2}{2} \right). \]

\[ N = \int_{R^{n-2}} dp_{s_1+1} \ldots dp_{s_1+n_1-2} \gamma \exp \left( -\beta \sum_{i=s_1+1}^{s_1+n_1-2} \frac{p_i^2}{2} \right). \]

It is obvious, that

\[ MI_1 + NI_2 < \kappa (MI_0 + NI_1), \]

where \( \kappa = \max \left( \frac{I_1}{I_0}, \frac{I_2}{I_1} \right) = \max \left( \sqrt{\frac{2}{\beta \pi}}, \sqrt{\frac{2\pi}{2\beta}} \right) = \frac{1}{2} \sqrt{\frac{2\pi}{\beta}} = \frac{I_0}{2}. \)

Therefore,

\[ 2tI_0 (MI_1 + NI_2) < 2tI_0 \kappa (MI_0 + NI_1) = 2tI_0 \kappa g_{n_1-1}. \]

Thus,

\[ g_{n_1} < ((2t^2 D + R)I_0 + 2tI_1 + 2t\kappa I_0) g_{n_1-1}. \]

The series (9) converges if

\[ b ((2t^2 D + R)I_0 + 2tI_1 + 2t\kappa I_0) < 1 \]

or

\[ 2\xi \exp(2\beta B) ((2t^2 D + R)I_0 + 2tI_1 + tI_0^2) < 1, \]

hence

\[ t < -\frac{2I_1 + I_0^2}{4DI_0} + \sqrt{\frac{1}{2DI_0} \left( \exp \left( -\frac{2\beta B}{2\xi} \right) - RI_0 \right) + \left( \frac{2I_1 + I_0^2}{4DI_0} \right)^2}. \]

Thus, the series (6) converges, and the series (3) — converges uniformly if

\[ 0 \leq t < 2t_0 = -\frac{2I_1 + I_0^2}{4DI_0} + \sqrt{\frac{1}{2DI_0} \left( \exp \left( -\frac{2\beta B}{2\xi} \right) - RI_0 \right) + \left( \frac{2I_1 + I_0^2}{4DI_0} \right)^2}. \] (12)

Thus, from the results, obtained above, we can formulate the following theorem.

**Theorem 1.** If the interaction potential satisfies the conditions (1) and the initial data \( F(0) \in E_\xi \), then the series (3) converges uniformly in \( Y \) on any compact if

\[ 0 \leq t < 2t_0 = -\frac{2I_1 + I_0^2}{4DI_0} + \sqrt{\frac{1}{2DI_0} \left( \exp \left( -\frac{2\beta B}{2\xi} \right) - RI_0 \right) + \left( \frac{2I_1 + I_0^2}{4DI_0} \right)^2}. \]
From this theorem the local existence theorem follows:

**Theorem 2.** If the interaction potential satisfies the conditions (1) and the initial data $F(0) \in E_\xi$, then the sequence $F(t)$ (3) exists and the series (3) converges uniformly on any compact on a finite time interval (12).

5. CONCLUSION

Using physical aspects of interaction for non-symmetrical one-dimensional system, we proved convergence of the series (3) on the time interval (12) (local existence of the sequence $F(t)$).

REFERENCES


Myhaylo O. Stashenko
Halyna M. Hubal
smo@univer.lutsk.ua

Volyn State University named after Lesya Ukrainka
Department of Mathematics
13 Voli av., Lutsk 43025, Ukraine

Received: May 27, 2004.