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**P_m-SATURATED GRAPHS WITH MINIMUM SIZE**

**Abstract.** By $P_m$ we denote a path of order $m$. A graph $G$ is said to be $P_m$-saturated if $G$ has no subgraph isomorphic to $P_m$ and adding any new edge to $G$ creates a $P_m$ in $G$. In 1986 L. Kászonyi and Zs. Tuza considered the following problem: for given $m$ and $n$ find the minimum size $\text{sat}(n;P_m)$ of $P_m$-saturated graph and characterize the graphs of $\text{Sat}(n;P_m)$ – the set of $P_m$-saturated graphs of minimum size. They have solved this problem for $n \geq a_m$ where $a_m = \begin{cases} 3 \cdot 2^{k-1} - 2 & \text{if } m = 2k, k > 2 \\ 2^{k+1} - 2 & \text{if } m = 2k + 1, k \geq 2 \end{cases}$. We define $b_m = \begin{cases} 3 \cdot 2^{k-2} & \text{if } m = 2k, k \geq 3 \\ 3 \cdot 2^{k-1} - 1 & \text{if } m = 2k + 1, k \geq 3 \end{cases}$ and give $\text{sat}(n;P_m)$ and $\text{Sat}(n;P_m)$ for $m \geq 6$ and $b_m \leq n < a_m$.

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1. **INTRODUCTION**

We deal with simple graphs without loops and multiple edges. As usual $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively, $|G|$, $e(G)$ the order and the size of $G$ and $d_G(v)$ the degree of $v \in V(G)$. For simplicity we shall suppose $|G| = n$. By $P_m$ we denote the path of order $m$ and by $K_r$ the complete graph on $r$ vertices. For vertex disjoint graphs $G$ and $H$ we denote by $G * H$ the graph with vertex set $V(G * H) = V(G) \cup V(H)$, and edge set $E(G * H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. The union $G \cup H$ of graphs $G$ and $H$ is defined by $V(G \cup H) = V(G) \cup V(H)$, $E(G \cup H) = E(G) \cup E(H)$, and we shall always suppose that the components of the union are vertex disjoint. A vertex of a graph is called penultimate if it is a neighbour of a vertex of degree 1 (pendent vertex).

A graph $G$ is $H$-saturated if there is no subgraph of $G$ isomorphic to $H$ and adding any new edge $e$ to $G$ creates $H$. We shall denote by $G \cup e$ the graph obtained
from $G$ by the addition of the edge $e$, supposing that the two end vertices of $e$ are in $G$. We define also:

$$sat(n; P_m) = \min\{e(G) : |G| = n, G \text{ is } P_m\text{-saturated}\},$$

$$Sat(n; P_m) = \{G : |G| = n, e(G) = sat(n; P_m), G \text{ is } P_m\text{-saturated}\}.$$

The best general reference here is [1]. The first and well known result considering $H$-saturated graphs with minimum size is the following theorem of P. Erdős, A. Hajnal and J.W. Moon, [6].

**Theorem 1.** The minimum size of $K_m$-saturated graph of order $n$

$$sat(n; K_m) = \left(\frac{m-2}{2}\right) + (m-2)(n-m+2), \quad (n \geq m \geq 2)$$

and the only $K_m$-saturated graph of size $sat(n; K_m)$ is $K_{m-2} \ast K_{n-m+2}$.

$P_m$-saturated graphs of order $n$ with minimum size for $m$ small have been characterized by L. Kászonyi and Zs. Tuza in [7].

**Theorem 2 (L. Kászonyi and Zs. Tuza).**

$$sat(n; P_3) = \left\lfloor \frac{n}{2} \right\rfloor ;$$

$$Sat(n; P_3) = \begin{cases} kK_2 & \text{if } n = 2k, \\ kK_2 \cup K_1 & \text{if } n = 2k + 1; \end{cases}$$

$$sat(n; P_4) = \begin{cases} k & \text{if } n = 2k, \\ k+2 & \text{if } n = 2k + 1; \end{cases}$$

$$Sat(n; P_4) = \begin{cases} kK_2 & \text{if } n = 2k, \\ (k-1)K_2 \cup K_3 & \text{if } n = 2k + 1; \end{cases}$$

$$sat(n; P_5) = n - \left\lceil \frac{n-2}{6} \right\rceil - 1 \quad (\text{for } n \geq 6).$$

Let us suppose that $m \geq 5$ is an integer. Then $A_m$ is the following tree. All nonpendent vertices of $A_m$ have their degree equal to three. If $m = 2k$, $k \geq 3$, then $A_m$ has one center and $k$ levels. If $m = 2k+1$, $k \geq 2$, then $A_m$ has two centers $v_1$, $v_2$ and each component of $G - \{v_1, v_2\}$ has $k-1$ levels. Observe that $|A_m| = a_m$, where

$$a_m = \begin{cases} 3 \cdot 2^{k-1} - 2 & \text{if } m = 2k, k > 2, \\ 2^{k+1} - 2 & \text{if } m = 2k + 1, k \geq 2. \end{cases}$$

For $m > n$ the only $P_m$-saturated graph of order $n$ is $K_n$. Hence $K_1$ and $K_2$ are the only $P_m$-saturated trees when $m > n$. It is easy to see that the trees $A_m$ are $P_m$-saturated. Moreover, every graph in which every component is isomorphic to $A_m$ or is obtained from $A_m$ by multiplying some branches, is $P_m$-saturated. In [7] Kászonyi and Tuza proved much more, namely the following theorem.

**Theorem 3 (L. Kászonyi and Zs. Tuza).** Let $m \geq 5$. If $T$ is a $P_m$-saturated tree then either $A_m \subset T$ or $T = K_i$ with $i \in \{1, 2\}$.
Theorem 3 implies easily:

**Corollary 1 (L. Káznonyi and Zs. Tuza).** Let \( m \geq 6 \) and let \( G = (V; E) \) be a \( P_m \)-saturated graph with minimum size such that \( |G| = n \geq a_m \). Then \( |E(G)| = n - \lfloor \frac{n}{a_m} \rfloor \) and every component of \( G \) is a tree containing \( A_m \).

In particular, for \( m \geq 6 \) and \( n \geq a_m \) we have \( \text{sat}(n; P_m) = n - \lfloor \frac{n}{a_m} \rfloor \). It is natural to try to find \( P_m \)-saturated graphs with minimum size for \( n < a_m \). In [3] we give a general upper bound for \( \text{sat}(n; P_m) \) for some \( m \) and \( n \).

**Theorem 4.** Let \( n \geq m \).

1. For \( m \) even, \( m \in \{22, 30, 38, 40, 42, 46, 48, 50\} \) or \( m \geq 54 \)

\[
\text{sat}(n; P_m) \leq n + \frac{m}{2} - 1.
\]

2. For \( m \) odd, \( m \in \{23, 31, 39, 41, 43, 47, 49, 51\} \) or \( m \geq 55 \)

\[
\text{sat}(n; P_m) \leq \frac{3n}{2}.
\]

In section 2 we introduce the notation, look more closely at \( P_m \)-saturated graphs with exactly one cycle and prove that \( \text{sat}(n; P_m) = n \) for \( 7 \leq m, b_m \leq n < a_m \) where

\[
b_m = \begin{cases} 
3 \cdot 2^{k-2} & \text{if } m = 2k, k \geq 3 \\
3 \cdot 2^{k-1} - 1 & \text{if } m = 2k + 1, k \geq 3
\end{cases}
\]

and we shall give \( \text{Sat}(n; m) \) for \( 6 \leq m \leq n, b_m \leq n < a_m \). Note that, by Theorem 3, every \( P_m \)-saturated graph \( G \) of order \( n \geq m \) such that \( 5 \leq n < a_m \), contains at least one cycle. Hence, it is clear that the minimum size of a connected \( P_m \)-saturated graph \( G \) of order \( |G| < a_m \) is at least \( |G| \).

The \( P_m \)-saturated graphs were considered by P. Erdős and T. Gallai in [5] who proved that the maximum size of \( P_m \)-saturated graphs of order \( n \) is equal to \( \frac{n(m-2)}{2} \) when \( n \equiv 0 \mod (m-1) \) and then the extremal graph is isomorphic with \( \frac{n}{m-1}K_{m-1} \). The maximum size of hamiltonian path saturated graphs of order \( n \) is equal to \( \left( \frac{n-1}{2} \right) \) for \( n \geq 2 \) (see also [2, 4, 8, 9]).

2. **\( P_m \)-SATURATED GRAPHS WITH EXACTLY ONE CYCLE**

The section starts with eight easy claims in which we shall assume that \( n \) and \( m \) are integers, \( n \geq m \geq 6 \) and \( G \) is a connected \( P_m \)-saturated graph with exactly one non trivial cycle \( C = (c_1, \ldots, c_k, c_1) \), \( k \geq 3 \). We denote by \( x_l, l = 1, 2, \ldots, p_1 \), the neighbours of \( c_1 \) such that \( x_l \notin V(C) \), by \( y_l, l = 1, 2, \ldots, p_2 \), the neighbours of \( c_2 \) such that \( y_l \notin V(C) \), and by \( z_l, l = 1, 2, \ldots, p_3 \), the neighbours of \( c_3 \) such that \( z_l \notin V(C) \). Let \( u \in V(G) \). For simplicity we shall denote by \( L_G(u) \) — the order of the
longest path in $G$ containing $u$, and by $L_v(u)$ — the order of the longest path in $G$ starting from $u$ and containing $v$ where $v \in N_G(u)$. We shall denote for $i = 1, 2, 3$ by $L_{ij}$ — the order of the longest path in $G$ starting from $c_i$, $i = 1, 2, 3$, not containing any other vertex from $V(C)$, and containing respectively $x_j$ for $i = 1$, $y_j$ for $i = 2$, and $z_j$ for $i = 3$, $j = 1, \ldots, p_i$. For $j = 1$ we denote $L_{11} = L_i$, for $i = 1, 2, \ldots, k$. For convenience we shall assume:

$$L_1 \geq L_{12} \geq \cdots \geq L_{1p_1},$$
$$L_2 \geq L_{22} \geq \cdots \geq L_{2p_2},$$
$$L_3 \geq L_{32} \geq \cdots \geq L_{3p_3}. \tag{2}$$

By $|v_1v_2\ldots v_r\ldots|$ we shall denote the order of a longest path containing the segment $v_1v_2\ldots v_r$.

Mean set $M_G$ is the set of all central vertices of paths of order $m - 1$ in $G$. Every $v \in M_G$ will be called mean vertex.


Proof. To obtain a contradiction suppose that $|C| \geq 4$. Without loss of generality we may assume $L_2 = \min\{L_i : i = 1, 2, \ldots, k\}$. The idea of the proof is to show it is impossible to create $P_m$ by connecting $c_1$ with $c_3$. Observe that $|\ldots x_1c_1c_2c_3z_1\ldots| < |\ldots x_1c_1c_2c_3c_4\ldots|$ and $|\ldots x_1c_1c_2c_3c_4\ldots| < |\ldots x_1c_1c_2c_3c_4\ldots|$. Hence the only possibility to create $P_m$ in $G \cup \{c_1c_3\}$ is to use the path $\ldots y_1c_2c_3c_4\ldots$ or $\ldots y_2c_1c_3c_4\ldots$. Thus we have either $m \leq |\ldots y_1c_2c_3c_4\ldots| \leq |\ldots z_1c_2c_3c_4\ldots|$ or $m \leq |\ldots y_1c_2c_3c_4\ldots| \leq |\ldots x_1c_1c_2c_3c_4\ldots|$. A contradiction. \hfill \Box

Claim 2. Mean set $G$ is either a singleton or a set of two adjacent vertices or else the set $C = \{c_1, c_2, c_3\}$ of the vertices of the cycle $C$.

Proof. It is easy to see that the mean vertices are adjacent. The claim follows. \hfill \Box

Claim 3. Let $C = (c_1, c_2, c_3, c_1)$. If $d_G(c_1) = 2$ and $d_G(c_2) \neq 2$ then $d_G(c_3) = 2$.

Proof. The proof will be divided into two steps.

By contradiction suppose first that $d_G(c_3) = 3$. By adding an edge $c_1z_1$ we create $P_m$ in $G \cup \{c_1z_1\}$ which contains the segment $z_1c_1c_2c_3$ or $c_3z_1c_1c_2$ (in case $L_3 = 2$). But replacing $z_1c_1c_2c_3$ or $c_3z_1c_1c_2$ by $z_1c_2c_1c_2$ we obtain $P_m$ which does not contain the new edge $c_1z_1$, a contradiction.

So we may suppose now that $d_G(c_3) \geq 4$. It is easy to observe that without loss of generality we may assume $L_2 \geq L_3$. We next show that by adding the edge $z_2c_2$ we will not create $P_m$. Observe first that $y_1c_2z_2c_3z_1$ can be replaced by $y_1c_2c_1c_3z_1$. The only way to create $P_m$ in $G \cup \{z_2c_2\}$ is to use the path $\ldots z_2c_2c_1c_3z_1\ldots$. Since $L_2 \geq L_3$ and (2) we have $|\ldots z_2c_2c_1c_3z_1\ldots| \leq |\ldots z_1c_3c_1c_2y_1\ldots|$, a contradiction. \hfill \Box

Claim 4. Let $C = (c_1, c_2, c_3, c_1)$. If $d_G(c_1) = 2$ and $d_G(c_3) = 2$ then $d_G(c_2) \geq 4$.

Claim 5. If $v \in V(G)$ is such a vertex that $d_G(v) = 2$ then $v \in V(C)$. 

Claim 6. Let \( u \in V(G) - V(C) \), \( d_G(u) \neq 1 \), \( N_G(u) = \{u_1, u_2, \ldots, u_p\} \) and \( L_{u_1}(u) \geq \cdots \geq L_{u_p}(u) \). Then:

1. \( L_{u_2}(u) = L_{u_3}(u) \),
2. \( L_G(u) = m - 1 \).

Proof. Suppose that the first assertion of the Claim 6 is false. Then \( L_{u_2}(u) > L_{u_3}(u) \). Since \( u \notin V(C) \), we have \( u_1u_2 \notin E(G) \). Observe first that to create \( P_m \) in \( G \cup \{u_1u_2\} \) we have to use \( u_1u_2uv_3 \). We obtain:

\[
m \leq L_{u_3}(u) + L_{u_3}(u)
\]  

(3)

Since \( G \) is \( P_m \)-saturated we have \( L_G(u) = L_{u_1}(u) + L_{u_2}(u) - 1 < m \) so

\[
L_{u_1}(u) + L_{u_2}(u) \leq m
\]  

(4)

We conclude from (3) and (4) that \( L_{u_2}(u) \leq L_{u_3}(u) \) and finally get a contradiction. To prove the second part of the Claim 6 we use inequalities (3), (4) and equality \( L_{u_2}(u) = L_{u_3}(u) \).

Claim 7. Let \( C = (c_1, c_2, c_3, c_1) \), \( d_G(c_i) \neq 2 \), for \( i = 1, 2, 3 \) and \( L_1 \leq L_2 \leq L_3 \). Then:

1. \( L_1 = L_2 \),
2. If \( L_1 \neq L_3 \) then \( d_G(c_3) \geq 4 \).

Proof. In the first part of the claim it is easily seen that it would be impossible to create \( P_m \) in \( G \cup \{c_2z_1\} \) if \( L_1 < L_2 \). For similar reasons we have \( d_G(c_3) \geq 4 \).

Claim 8.

1. If there is in \( G \) a vertex \( u \) such that \( L_G(u) < m - 1 \) then \( d_G(u) \leq 2 \).
2. If \( C = (c_1, c_2, c_3, c_1) \), \( d_G(c_1) = d_G(c_2) = 2 \) and \( L_G(c_1) < m - 1 \) then \( d_G(c_3) \geq 5 \).
3. If \( C = (c_1, c_2, c_3, c_1) \), \( d_G(c_1) = d_G(c_2) = 2 \) then \( L_G(c_1) = L_G(c_3) = m - 1 \).

Proof. We give the proof for first part only, the proofs of the others parts of the claim are left to the reader. Let \( u \in V(G) \) and \( L_G(u) < m - 1 \) and suppose that \( d_G(u) > 2 \). By Claim 6, we have \( u \in V(C) \). Observe that if the conditions \( d_G(c_i) > 2 \) and \( L_G(c_i) < m - 1 \) hold for exactly two vertices \( c_i \), \( i = 1, 2, 3 \), then for the remaining vertex \( c_j \) we have \( d_G(c_j) > 2 \) and \( L_G(c_j) = m - 1 \) (otherwise we have exactly one vertex \( c_i \) of degree 2 contrary to Claim 3). If \( d_G(c_1) > 2 \) and \( L_G(c_1) < m - 1 \) and \( d_G(c_i) = 2 \) or \( L_G(c_i) = m - 1 \) for \( i = 2, 3 \), then \( L_G(c_1) = L_G(c_2) \), a contradiction. Thus the proof falls into three cases.
Case 1. $d_G(c_i) > 2$, $L_G(c_i) < m - 1$ for $i = 1, 2, 3$.
From Claim 7 follows it is impossible that only one of the vertex $x_1, y_1, z_1$ have his degree equal to 1. If exactly two of vertices $x_1, y_1, z_1$ have his degree equal to 1, say $d_G(x_1) = d_G(y_1) = 1$, then by Claim 7 we have $d_G(c_3) \geq 4$. It is easy to see that if $d_G(z_1) \neq 1$ then $d_G(z_2) \neq 1$. From Claim 6 follows $L_G(z_1) = L_G(z_2) = m - 1$.
Observe that there exist vertex $u_1 \in N_G(z_1)$ and $u_2 \in N_G(z_2)$ and $u_1, u_2 \notin V(C)$ such that $L_{u_1}(z_1) \geq \left\lceil \frac{m-1}{2} \right\rceil$ and $L_{u_2}(z_2) \geq \left\lceil \frac{m-1}{2} \right\rceil$. We obtain the path containing segment $\ldots u_1z_1c_3z_2u_2 \ldots$ such that $\ldots u_1z_1c_3z_2u_2 \ldots \geq m - 1$, a contradiction. If $d_G(x_1) = d_G(y_1) = d_G(z_1) = 1$ then there is no vertex $u$ with $L_G(u) < m - 1$ (observe that $m = 6$). So we may suppose that $d_G(x_1) > 1$, $d_G(y_1) > 1$, $d_G(z_1) > 1$ (note that since $x_1, y_1, z_1$ are not on the cycle we have then $d_G(x_1) \geq 3$, $d_G(y_1) \geq 3$, $d_G(z_1) \geq 3$).
Let $v_1, v_2 \in N_G(x_1)$ and $v_1, v_2 \notin V(C)$, $w_1, w_2 \in N_G(y_1)$ and $w_1, w_2 \notin V(C)$. We shall assume $L_{v_1}(x_1) \geq L_{v_2}(x_1)$ and $L_{w_1}(y_1) \geq L_{w_2}(y_1)$. By Claim 6 we have $L_G(x_1) = L_G(y_1) = m - 1$. We must have $L_{v_1}(x_1) \geq \left\lceil \frac{m-1}{2} \right\rceil$ and $L_{w_1}(y_1) \geq \left\lceil \frac{m-1}{2} \right\rceil$. The path $\ldots v_1x_1c_3c_2y_1 \ldots$ has the order at least $m$, which is impossible.

Case 2. $d_G(c_i) > 2$, $L_G(c_i) < m - 1$ and $L_G(c_i) = m - 1$.
The proof is similar to the proof of the former case.

Case 3. $d_G(c_i) > 2$, $L_G(c_i) < m - 1$ and $L_G(c_i) = m - 1$ for $i = 2, 3$.
This case is easy, and we leave it to the reader.

Construction of the family $\mathcal{R}_m$. For every integer $m \geq 6$ we shall define a family of connected $P_m$-saturated graphs containing exactly one cycle. Let us consider the tree $A_m = (V'; E')$. $\mathcal{R}_m$ is the family of graphs which may be obtained in the following way. Let $u \in V'$ be a vertex of $A_m$ such that $d_{A_m}(u) = 3$ and let $N_{A_m}(u) = \{x, y, z\}$. If $d_{A_m}(x) \neq 1$, $d_{A_m}(y) \neq 1$ and $d_{A_m}(z) \neq 1$ then denote by $x_1, x_2$ the neighbours of $x$ such that $\{x_1, x_2\} = N_{A_m}(x) \setminus \{u\}$, by $y_1, y_2$ the neighbours of $y$ such that $\{y_1, y_2\} = N_{A_m}(y) \setminus \{u\}$ and by $z_1, z_2$ the neighbours of $z$ such that $\{z_1, z_2\} = N_{A_m}(z) \setminus \{u\}$. First define $A_m(u)$ by removing $u$ with incident edges and adding the edges $xy, xz, yz$. We shall consider three cases.

Case 1. $u$ is the only central vertex of $A_m$ (in particular $m$ is even). Define $R^1_m$ by removing from $A_m(u)$ the edges $xx_1, yy_1, zz_1$ and the components containing vertices $x_1, y_1, z_1$ (see Fig. 1).

Case 2. $d_{A_m}(u) = 3$ and $u$ is neither the only central nor penultimate vertex of $A_m$.
Without loss of generality we may suppose that $L_{x_1}(x) = L_{x_2}(x) = L_{y_1}(y) = L_{y_2}(y) < L_{z_1}(z)$. Then define $R^2_m(u)$ deleting from $A_m(u)$ the edges $xx_1, yy_1$ and the components containing $x_1$ and $y_1$ (see Fig. 2). By $\mathcal{R}^2_m$ we denote the set of all the graphs $R^2_m(u)$.

Case 3. $u$ is a penultimate vertex of $A_m$.
Let $v, w \in N_{A_m}(x_1), d_{A_m}(v) = d_{A_m}(w) = 1$. Define $R^3_m$ deleting from $A_m(u)$ the vertices $v, w$ with incident edges. Put $\mathcal{R}_m = \{R^1_m\} \cup \mathcal{R}^2_m \cup \{R^3_m\}$ (see Fig. 3). Note, that the least order of a graph $H \in \mathcal{R}_m$ is equal to $b_m$ defined by the formula (1).
Let $H'$ be a graph obtained from $R^1_n$ by addition of a number of pendent neighbours of the vertices of the triangle. Then $G = H' \cup K_2$ is a $P_n$-saturated graph such that $|E(G)| = |G| - 1$.

Note that, for $m \geq 6$, the union of $K_1$ and a graph $H \in R_m$ is never $P_m$-saturated.

**Lemma 1.** Let $n$ and $m$ be integers such that $6 \leq m \leq n$ and $b_m \leq n < a_m$. Then the minimum size of a connected $P_m$-saturated graph of order $n$ is equal to $n$.

**Proof.** Since $6 \leq n < a_m$, by Theorem 3, there is no $P_m$-saturated tree of order $n$. On the other hand, for every $n \geq b_m$ there is an unicyclic connected graph $G$ of order $n$ which is $P_m$-saturated (in fact, $G$ may be obtained from a graph $H \in R_m$ by addition of a suitable number of pendent vertices). The lemma follows.

**Lemma 2.** Let $G$ be a connected unicyclic $P_m$-saturated graph of order $n$, $6 \leq m \leq n < a_m$, such that there are two vertices $a, b$ of degree 2. Then $G$ contains $R^3_m$. 

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**Fig. 1**
Proof. By Claims 1 and 3 the only cycle of $G$ is a triangle $C$ containing the vertices $a$ and $b$, $C = (a, b, c, a)$, say. We shall consider two cases.

Case 1. $L_G(a) < m - 1$.
Observe that we have $L_G(a) = L_G(b)$. We shall prove that $G - \{a, b\}$ is $P_m$-saturated graph. Let $x, y \in V(G)$ be two nonadjacent vertices of $G$. Then there is a $P_m$ through $e = xy$. Without loss of generality we may suppose $P_m = (abc \ldots xy \ldots)$. Then by Claim 8 there is a neighbour $v$ of $c$ such that $v \not\in V(P_m)$.

If $d_G(v) > 1$ then we can replace the segment $\ldots abc \ldots$ of $P_m$ with $wve$ where $w \in N_G(v) - \{c\}$ to obtain a path of order $m$ passing through $e$.

If $d_G(v) = 1$ then the only path of order $m$ through $e' = av$ must have form $P_m = (vabc \ldots)$. But then there is a path of order $m - 1$ starting at $a$, a contradiction.

Case 2. $L_G(a) = m - 1$.
If there exists a vertex $x_2 \in N_G(c)$ such that $d_G(x_2) = 1$ and clearly $G' = (V', E')$ with $V' = V \cup \{d, x, y\}$, $E' = E - \{ab, ac, bc\} \cup \{bd, ad, cd, xx_2, yy_2\}$ is $P_m$-saturated graph without cycle. Thus $n = a_m - 3$ and $G$ is $R_3^m$. If non of neighbours of $c$ is equal to 1 then by similar method as before we can show that $G$ contains $R_3^m$. □
Lemma 3. Let $G$ be a connected, unicyclic and $P_m$-saturated graph of order $n$ without vertices of degree 2, where $6 \leq m \leq n < a_m$. Then $G$ has a subgraph $H \in \{R_m^1\} \cup \overline{R_m}^2$.

Proof. By Claim 1 we may suppose that $C = (c_1, c_2, c_3, c_1)$ and by Claim 7 it is sufficient to consider the following two cases.

Case 1. $L_1 = L_2 = L_3$.

Then by Claim 8 the vertices $c_1, c_2, c_3$ form the center of $G$ and $L_1 = L_2 = L_3 = \frac{m}{2} - 1$ (note that $m$ is even). Let $r_1$ be a neighbour of $c_1$ such that $L_{r_1}(c_1) = \frac{m}{2} - 1$. We shall prove that the component of $G' = (V, E - \{r_1c_1\})$ containing the vertex $r_1$ contains $T_H$ the complete binary tree with the root $r_1$ with $h = \frac{m}{2} - 2$ levels. By Claim 6, $r_1$ has two neighbours $x$ and $y$ different from $c_1$ such that $L_x(r_1) = L_y(r_1) = \frac{m}{2} - 2$. Choose one of these vertices as $r_2$. Having chosen $r_i$ with $2 \leq i < \frac{m}{2} - 1$ we have by Claim 6 two vertices $x'$ and $y'$ such that $L_{x'}(r_i) = L_{y'}(r_i) = \frac{m}{2} - (i + 1)$. We may choose arbitrary one of three vertices $r_{i+1}$ and the proof is finished in this case.

Case 2. $L_1 = L_2, L_1 < L_3$.

Let $P = (c_3z \ldots v)$ be a path with $v$ being a mean vertex of $G$. We apply the same method as in the proof of Case 1 to the graph $G'$ obtained from $G$ by the delition of the vertices of the path $P$ and the cycle $C$ with all incident edges.

By Claims 3, 4 and 5 and Lemmas 2 and 3, we have the following

Theorem 5. Let $6 \leq m \leq n < a_m$. Every connected and unicyclic $P_m$-saturated graph of order $n$ contains a subgraph $H \in \overline{R_m}$.

The following two lemmas are very easy to deduce.
Lemma 4. Let $u$ be a vertex of degree equal to two in a $P_m$-saturated graph $G$, $|G| \geq m \geq 4$. Then the neighbours of $u$ are adjacent.

Lemma 5. Let $m \geq 6$. Every union of $K_2$ and a graph $H \in \mathcal{R}_m$ is $P_m$-saturated if and only if $m = 6$ and $H = R^1_6$.

$\mathcal{H}$ is the family of graphs defined in Figure 4. Observe that for every graph $H \in \mathcal{H}$ the graphs $H$ and $H \cup K_2$ are $P_6$-saturated.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Figure4.png}
\caption{}
\end{figure}

Lemma 6. Let $m \geq 6$ and let $H$ be a connected $P_m$-saturated graph such that $|E(H)| = |H| + 1$. Then the graph $H \cup K_2$ is $P_m$-saturated if and only if $m = 6$ and $H \in \mathcal{H}$.

Proof. Denote by $u$ and $v$ the vertices of $K_2$. Let $C$ and $C'$ be two different and nontrivial cycles in $H$ and suppose that $H \cup K_2$ is $P_m$-saturated.

There are two possibilities: either $C$ and $C'$ intersect in at most one vertex or the intersection of $C$ and $C'$ is a non trivial path.

Thus we shall consider two cases.

Case 1. $|V(C) \cap V(C')| \leq 1$.

There is a unique path from $C$ to $C'$, say $P = (x_1, \ldots, x_p)$, such that $x_1 \in V(C)$, $x_p \in V(C')$ and $x_i \notin V(C) \cup V(C')$ for $i = 2, \ldots, p-1$ (possibly $p = 1$ and $V(C) \cap V(C') = \{x_1\}$). It is clear that the new edge $ux_1$ can not create any $P_m$ in $H \cup K_2$.

Case 2. $|V(C) \cap V(C')| \geq 2$.

Let the path $P'' = (z_1, \ldots, z_r)$ be the intersection of the cycles $C$ and $C'$. So, we have three paths, say $P = (x_1, \ldots, x_p)$, $P' = (y_1, \ldots, y_q)$ and $P''$ such that $x_1 = y_1 = z_1$ and $x_p = y_q = z_r$. At least two of the paths $P, P'$ or $P''$ have length at least two.

The graph $H \cup K_2$ is $P_m$-saturated, hence there is a path of order $m$ in the graph $H \cup K_2 \cup vx_1$, say $P''' = (u, v, x_1, \ldots)$. Since there is no $P_m$ in the graph $H$ we deduce easily that $P'''$ contains all, but at most one, vertices of two paths $P, P'$, $P''$ while the third of them contains at most three vertices. Without loss of generality we may suppose that either

$$P''' = (u, v, x_1, x_2, \ldots, x_p, y_{q-1}, \ldots, y_2, a_1, \ldots, a_t)$$
or

$$P''' = (u, v, x_1, x_2, \ldots, x_p, y_{q-1}, \ldots, y_3, a_1, \ldots, a_t)$$

where $$\{a_1, \ldots, a_t\} \cap (V(C) \cup V(C')) = \emptyset$$ and $$r \leq 3$$. For $$p \geq 3$$ and $$q \geq 3$$ we have, by Lemma 4, $$r = 2$$.

Let $$(b_1, \ldots, b_s)$$ be such a path of the graph $$H$$ that $$b_1 \in V(C) \cup V(C')$$ and $$b_2 \notin V(C) \cup V(C')$$ (by consequence $$b_1$$ is the only vertex of that path which is in $$V(C) \cup V(C')$$). Since $$H \cupvb_1$$ contains $$P_m$$ and there is no $$P_m$$ in $$H$$, we have $$s \leq 2$$. In particular $$t \leq 2$$ and every vertex $$x \in V(H) - (V(C) \cup V(C'))$$ is pendant and adjacent to a vertex of $$V(C) \cup V(C')$$. Therefore joining two vertices $$x, y \in V(C) \cup V(C')$$ we can not obtain any path of order greater than $$p + q$$. Thus $$m \leq p + q$$. Now it is very easy to deduce that $$H \in \mathcal{H}$$.

**Theorem 6.** Let $$n$$ and $$m$$ be integers such that $$6 \leq m \leq n$$, $$b_m \leq n < a_m$$ and let $$G \in \text{Sat}(n; P_m)$$. Then

$$e(G) = \begin{cases} 
  n - 1 & \text{for } m = 6, \ n = 8, 9 \\
  n & \text{for } m = 6, \ n = 6, 7 \\
  n & \text{for } m > 6.
\end{cases}$$

Moreover, if $$m > 6$$ then $$G$$ contains a subgraph $$H \in \mathcal{R}_m$$, if $$m = 6$$ then either $$G$$ is the graph depicted in Figure 5 or $$G = 2K_3$$.

![Fig. 5](image-url)
Proof. Let us suppose first that one component of $G$ is a tree. Since $n < a_m$ and by Theorem 3, the tree component of $G$ is either $K_2$ or $K_1$. By Lemma 1 we have $\text{sat}(n, P_m) \leq n$. Thus every other component of $G$ has at most two cycles. We conclude this case applying Theorem 5, Lemma 5 and Lemma 6.

So we may suppose that no component of $G$ is a tree. Then the theorem follows by Theorem 5.

Observe that since $2b_m > a_m$, for $n \geq m \geq 7$ every graph $G \in \text{Sat}(n; P_m)$ is connected.

Added in proof. We have been informed that L. W. Beineke, J. E. Dunbar and M. Frick in yet unpublished paper *Detour-saturated graphs* have given a complete characterization of unicyclic $P_m$-saturated graphs. We are very indebted for this information (and the preprint of the paper) to professor M. Frick. Thanks are due to professor M. Frick also for indicating an error in the previous version of our paper.

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$P_m$-saturated graphs with minimum size

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