Opuscula Math. 39, no. 1 (2019), 91–108
https://doi.org/10.7494/OpMath.2019.39.1.91 Opuscula Mathematica

OSCILLATION CRITERIA FOR EVEN ORDER NEUTRAL DIFFERENCE EQUATIONS

S. Selvarangam, S.A. Rupadevi, E. Thandapani, and S. Pinelas

Communicated by Marek Galewski

Abstract. In this paper, we present some new sufficient conditions for oscillation of even order nonlinear neutral difference equation of the form

$$\Delta^m(x_n + ax_{n-\tau_1} + bx_{n+\tau_2}) + p_n x^{\alpha}_{n-\sigma_1} + q_n x^{\beta}_{n+\sigma_2} = 0, \quad n \geq n_0 > 0,$$

where $m \geq 2$ is an even integer, using arithmetic-geometric mean inequality. Examples are provided to illustrate the main results.

Keywords: even order, neutral difference equation, oscillation, asymptotic behavior, mixed type.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

In this paper, we are concerned with the even order mixed type neutral difference equation of the form

$$\Delta^m(x_n + ax_{n-\tau_1} + bx_{n+\tau_2}) + p_n x^{\alpha}_{n-\sigma_1} + q_n x^{\beta}_{n+\sigma_2} = 0, \tag{1.1}$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, \ldots\}$, and n_0 is a nonnegative integer, subject to the following conditions:

(i) $\{p_n\}$ and $\{q_n\}$ are positive real sequences for all $n \in \mathbb{N}(n_0)$,
(ii) a and b are nonnegative real numbers, τ_1, τ_2, σ_1 and σ_2 are nonnegative integers,
(iii) α and β are ratios of odd positive integers and $m \geq 2$ is an even integer.

Let $\theta = \max\{\tau_1, \sigma_1\}$. By a solution of the equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, and satisfying the equation (1.1) for all $n \geq n_0$.

A nontrivial solution of the equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

Since the difference equations have important applications in population dynamics, biology, probability theory, computer science and many other fields, there is a permanent interest in obtaining sufficient conditions for the oscillation or nonoscillation of solutions of various types of even order/odd order difference equations, see references in this paper and their references.

For the oscillation of even order difference equations, see [1–3,6–8,12,13]. Regarding the higher order mixed type neutral difference equations, Agarwal and Grace [4], Agarwal, Bohner, Grace and O'Regan [7], and Grace [9], considered several higher order mixed type neutral difference equations and established sufficient conditions for the oscillation of all solutions.

In [7], Agarwal, Bohner, Grace and O'Regan considered the m^{th} order mixed type neutral difference the equation (1.1) with $\alpha = \beta = 1, p_n \equiv p$ and $q_n \equiv q$, and established some sufficient conditions for the oscillation of the equation (1.1). Motivated by this observation, in this paper we investigate the oscillatory behavior of solutions of the equation (1.1), and hence the results obtained in this paper complement and generalize that of in [1, 3–6, 8, 9, 12, 13].

In Section 2, we present some basic lemmas which will be used to prove the main results. In Section 3, we obtain sufficient conditions for the oscillation of all solutions of the equation (1.1) by using arithmetic-geometric mean inequality. Examples are provided in Section 4 to illustrate the main results.

2. SOME PRELIMINARY LEMMAS

In this section, we present some lemmas, which are useful in proving the main results. We write

$$z_n = x_n + a x_{n-\tau_1} + b x_{n+\tau_2}.$$

Lemma 2.1. *Let a, b, c are positive quantities not all equal. Then*

$$a^\alpha + b^\alpha + c^\alpha \geq \frac{1}{3^{\alpha-1}}(a+b+c)^\alpha \quad \text{if } \alpha \geq 1,$$
$$a^\alpha + b^\alpha + c^\alpha \geq (a+b+c)^\alpha \quad \text{if } \; 0 < \alpha \leq 1.$$

The proof is elementary and hence it is omitted.

Lemma 2.2 ([3]). *Let $\{u_n\}$ be a sequence of positive real numbers with $\{\Delta^m u_n\}$ be of constant sign eventually and not identically zero eventually. Then there exists integer $l \in \{0, 1, 2, \ldots, m\}$ with $m + l$ odd for $\Delta^m u_n \leq 0$, and $m + l$ even for $\Delta^m u_n \geq 0$ and for $N > 0$ such that*

$$\Delta^j u_n > 0 \quad \text{for} \quad j = 0, 1, 2, 3, \ldots, l-1$$

and

$$(-1)^{j+l} \Delta^j u_n > 0 \quad \text{for} \quad j = l, l+1, l+2, \ldots, m-1$$

for all $n \geq N$.

Lemma 2.3 ([11])**.** *Let $\{u_n\}$ be a sequence of positive real numbers with $\Delta^m u_n \le 0$ and not identically zero eventually. Then there exists a large integer N such that*

$$u_n \ge \frac{(n-N)^{m-1}}{(m-1)!}\Delta^{m-1}u_{2^{m-l-1}n}, \quad \textit{for} \quad n \ge N,$$

where l is defined as in Lemma 2.2. Further if $\{u_n\}$ is increasing, then

$$u_n \ge \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1}u_n \quad \textit{for all} \quad n \ge 2^{m-1}N. \tag{2.1}$$

Lemma 2.4. *Let m be an even positive integer, and let $\{x_n\}$ be a positive solution of the equation* (1.1)*. Then there exists an integer $n_1 \in \mathbb{N}(n_0)$ such that*

$$z_n > 0, \Delta z_n > 0, \Delta^{m-1}z_n > 0, \ \textit{and}\ \Delta^m z_n \le 0 \ \textit{for all}\ n \ge n_1.$$

Proof. Since $\{x_n\}$ is an eventually positive solution of the equation (1.1), there is an integer $n_1 \in \mathbb{N}(n_0)$ such that $x_n > 0$, $x_{n-\tau_1} > 0$ and $x_{n-\sigma_1} > 0$ for all $n \ge n_1$. Noting that $a \ge 0$, $b \ge 0$, we have $z_n > 0$ for all $n \ge n_1$, and

$$\Delta^m z_n = -p_n x^{\alpha}_{n-\sigma_1} - q_n x^{\beta}_{n+\sigma_2} \le 0, \quad n \ge n_1.$$

It follows that $\{\Delta^{m-1}z_n\}$ is decreasing and eventually of one sign. We claim that $\Delta^{m-1}z_n > 0$ for $n \ge n_1$. Otherwise, if there is an integer $n_2 \ge n_1$ such that $\Delta^{m-1}z_{n_2} \le 0$ for $n \ge n_2$, that is,

$$\Delta^{m-1}z_{n_2} = -c \quad (c > 0),$$

which implies that

$$\Delta^{m-1}z_n \le -c \quad \text{for} \quad n \ge n_2.$$

Summing the last inequality from n_2 to $n-1$, we have

$$\Delta^{m-2}z_n \le \Delta^{m-2}z_{n_2} - c(n-n_2).$$

Letting $n \to \infty$, we obtain $\lim_{n\to\infty}\Delta^{m-2}z_n = -\infty$, which implies that $\{z_n\}$ is eventually negative by Lemma 2.2. This contradiction shows that $\Delta^{m-1}z_n > 0$ for all $n \ge n_1$. Again from Lemma 2.2 and noting that m is even, we have $\Delta z_n > 0$ for all $n \ge n_1$. This completes the proof. □

3. OSCILLATION RESULTS

In this section, we obtain some sufficient conditions for all the solutions of the equation (1.1) to be oscillatory. From the form of the equation (1.1) the assumption of existence of a positive solution leads to contradiction since the proof for the opposite case is similar.

For our convenience, we introduce the following notations:

$$P_n = \min\{p_{n-\tau_1}, p_n, p_{n+\tau_2}\},$$
$$Q_n = \min\{q_{n-\tau_1}, q_n, q_{n+\tau_2}\},$$

and

$$R_n = K_1 P_n + K_2 Q_n,$$

where K_1 and K_2 are some positive constants.

Theorem 3.1. *Assume that $\alpha < 1 < \beta$. If the first order difference inequality*

$$\Delta w_n + \frac{A_n}{(1+d_1+d_2)} \frac{\lambda}{(m-1)!} (n-\sigma_1)^{m-1} w_{n+\tau_1-\sigma_1} \leq 0, \tag{3.1}$$

where

$$A_n = \eta_1^{-\eta_1} \eta_2^{-\eta_2} P_n^{\eta_1} \left(\frac{Q_n}{3^{\beta-1}} \right)^{\eta_2}, \quad \eta_1 = \frac{\beta-1}{\beta-\alpha}, \quad \eta_2 = \frac{1-\alpha}{\beta-\alpha},$$

$$d_1 = \begin{cases} a^\alpha & \text{if } a \leq 1, \\ a^\beta & \text{if } a \geq 1 \end{cases} \quad \text{and} \quad d_2 = \begin{cases} b^\alpha & \text{if } b \leq 1, \\ b^\beta & \text{if } b \geq 1 \end{cases}$$

has no positive solution for some $\lambda \in (0,1)$ and for all $n \geq n_0$, then every solution of the equation (1.1) *is oscillatory.*

Proof. Assume that $\{x_n\}$ is a positive solution of the equation (1.1). Then there exists an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-\sigma_1} > 0$ and $x_{n-\tau_1} > 0$ for all $n \geq n_1$. By the definition of z_n we have $z_n > 0$ for all $n \geq n_1$. Now from the equation (1.1), we obtain

$$\Delta^m z_n = -p_n x_{n-\sigma_1}^\alpha - q_n x_{n+\sigma_2}^\beta \leq 0$$

for all $n \geq n_1$. From Lemma 2.4 we have $\Delta z_n > 0$ for all $n \geq n_1$.
Now we discuss the different cases for a and b.
Case 1. Suppose $a \leq 1$ and $b \leq 1$. Then from the equation (1.1), we have

$$a^\alpha \Delta^m z_{n-\tau_1} + a^\alpha p_{n-\tau_1} x_{n-\sigma_1-\tau_1}^\alpha + a^\alpha q_{n-\tau_1} x_{n+\sigma_2-\tau_1}^\beta = 0, \quad n \geq n_1, \tag{3.2}$$

and

$$b^\alpha \Delta^m z_{n+\tau_2} + b^\alpha p_{n+\tau_2} x_{n-\sigma_1+\tau_2}^\alpha + b^\alpha q_{n+\tau_2} x_{n+\sigma_2+\tau_2}^\beta = 0, \quad n \geq n_1. \tag{3.3}$$

Now combining equations (1.1), (3.2) and (3.3), we obtain

$$\begin{aligned} &\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) \\ &+ P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ &+ Q_n(x_{n+\sigma_2}^\beta + a^\alpha x_{n+\sigma_2-\tau_1}^\beta + b^\alpha x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Since $a \leq 1$, $b \leq 1$ and $\beta > \alpha$, the last inequality becomes

$$\begin{aligned} &\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) \\ &+ P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ &+ Q_n(x_{n+\sigma_2}^\beta + a^\beta x_{n+\sigma_2-\tau_1}^\beta + b^\beta x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1}z_n + a^\alpha\Delta^{m-1}z_{n-\tau_1} + b^\alpha\Delta^{m-1}z_{n+\tau_2}) + P_n z^\alpha_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}}z^\beta_{n+\sigma_2} \le 0, \quad n \ge n_1. \tag{3.4}$$

Case 2. Suppose $a \ge 1$ and $b \ge 1$. Then from the equation (1.1), we have

$$a^\beta\Delta^m z_{n-\tau_1} + a^\beta p_{n-\tau_1}x^\alpha_{n-\sigma_1-\tau_1} + a^\beta q_{n-\tau_1}x^\beta_{n+\sigma_2-\tau_1} = 0, \quad n \ge n_1, \tag{3.5}$$

and

$$b^\beta\Delta^m z_{n+\tau_2} + b^\beta p_{n+\tau_2}x^\alpha_{n-\sigma_1+\tau_2} + b^\beta q_{n+\tau_2}x^\beta_{n+\sigma_2+\tau_2} = 0, \quad n \ge n_1. \tag{3.6}$$

Now combining equations (1.1), (3.5) and (3.6), we obtain

$$\begin{aligned}&\Delta(\Delta^{m-1}z_n + a^\beta\Delta^{m-1}z_{n-\tau_1} + b^\beta\Delta^{m-1}z_{n+\tau_2})\\ &+ P_n(x^\alpha_{n-\sigma_1} + a^\beta x^\alpha_{n-\sigma_1-\tau_1} + b^\beta x^\alpha_{n-\sigma_1+\tau_2})\\ &+ Q_n(x^\beta_{n+\sigma_2} + a^\beta x^\beta_{n+\sigma_2-\tau_1} + b^\beta x^\beta_{n+\sigma_2+\tau_2}) \le 0, \quad n \ge n_1.\end{aligned}$$

Since $a \ge 1$, $b \ge 1$ and $\beta > \alpha$, the last inequality becomes

$$\begin{aligned}&\Delta(\Delta^{m-1}z_n + a^\beta\Delta^{m-1}z_{n-\tau_1} + b^\beta\Delta^{m-1}z_{n+\tau_2})\\ &+ P_n(x^\alpha_{n-\sigma_1} + a^\alpha x^\alpha_{n-\sigma_1-\tau_1} + b^\alpha x^\alpha_{n-\sigma_1+\tau_2})\\ &+ Q_n(x^\beta_{n+\sigma_2} + a^\beta x^\beta_{n+\sigma_2-\tau_1} + b^\beta x^\beta_{n+\sigma_2+\tau_2}) \le 0, \quad n \ge n_1.\end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1}z_n + a^\beta\Delta^{m-1}z_{n-\tau_1} + b^\beta\Delta^{m-1}z_{n+\tau_2}) + P_n z^\alpha_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}}z^\beta_{n+\sigma_2} \le 0, \quad n \ge n_1. \tag{3.7}$$

Case 3. Now suppose $a \le 1$, and $b \ge 1$. Then from the equation (1.1), we have

$$a^\alpha\Delta^m z_{n-\tau_1} + a^\alpha p_{n-\tau_1}x^\alpha_{n-\sigma_1-\tau_1} + a^\alpha q_{n-\tau_1}x^\beta_{n+\sigma_2-\tau_1} = 0, \quad n \ge n_1, \tag{3.8}$$

and

$$b^\beta\Delta^m z_{n+\tau_2} + b^\beta p_{n+\tau_2}x^\alpha_{n-\sigma_1+\tau_2} + b^\beta q_{n+\tau_2}x^\beta_{n+\sigma_2+\tau_2} = 0, \quad n \ge n_1. \tag{3.9}$$

Combining equations (1.1), (3.8) and (3.9), we obtain

$$\begin{aligned}&\Delta(\Delta^{m-1}z_n + a^\alpha\Delta^{m-1}z_{n-\tau_1} + b^\beta\Delta^{m-1}z_{n+\tau_2})\\ &+ P_n(x^\alpha_{n-\sigma_1} + a^\alpha x^\alpha_{n-\sigma_1-\tau_1} + b^\beta x^\alpha_{n-\sigma_1+\tau_2})\\ &+ Q_n(x^\beta_{n+\sigma_2} + a^\alpha x^\beta_{n+\sigma_2-\tau_1} + b^\beta x^\beta_{n+\sigma_2+\tau_2}) \le 0, \quad n \ge n_1.\end{aligned}$$

Since $a \le 1$, $b \ge 1$ and $\beta > \alpha$, the last inequality becomes

$$\begin{aligned}&\Delta(\Delta^{m-1}z_n + a^\alpha\Delta^{m-1}z_{n-\tau_1} + b^\beta\Delta^{m-1}z_{n+\tau_2})\\ &+ P_n(x^\alpha_{n-\sigma_1} + a^\alpha x^\alpha_{n-\sigma_1-\tau_1} + b^\alpha x^\alpha_{n-\sigma_1+\tau_2})\\ &+ Q_n(x^\beta_{n+\sigma_2} + a^\beta x^\beta_{n+\sigma_2-\tau_1} + b^\beta x^\beta_{n+\sigma_2+\tau_2}) \le 0, \quad n \ge n_1.\end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1}z_n + a^{\alpha}\Delta^{m-1}z_{n-\tau_1} + b^{\beta}\Delta^{m-1}z_{n+\tau_2}) + P_n z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n+\sigma_2} \le 0, \quad n \ge n_1. \tag{3.10}$$

Case 4. Suppose $a \ge 1$, and $b \le 1$. Then from the equation (1.1), we have

$$a^{\beta}\Delta^m z_{n-\tau_1} + a^{\beta} p_{n-\tau_1} x^{\alpha}_{n-\sigma_1-\tau_1} + a^{\beta} q_{n-\tau_1} x^{\beta}_{n+\sigma_2-\tau_1} = 0, \quad n \ge n_1, \tag{3.11}$$

and

$$b^{\alpha}\Delta^m z_{n+\tau_2} + b^{\alpha} p_{n+\tau_2} x^{\alpha}_{n-\sigma_1+\tau_2} + b^{\alpha} q_{n+\tau_2} x^{\beta}_{n+\sigma_2+\tau_2} = 0, \quad n \ge n_1. \tag{3.12}$$

Now combining equations (1.1), (3.11), (3.12) and $\beta > \alpha$, we obtain

$$\begin{aligned}
&\Delta(\Delta^{m-1}z_n + a^{\beta}\Delta^{m-1}z_{n-\tau_1} + b^{\alpha}\Delta^{m-1}z_{n+\tau_2}) \\
&+ P_n(x^{\alpha}_{n-\sigma_1} + a^{\beta} x^{\alpha}_{n-\sigma_1-\tau_1} + b^{\alpha} x^{\alpha}_{n-\sigma_1+\tau_2}) \\
&+ Q_n(x^{\beta}_{n+\sigma_2} + a^{\beta} x^{\beta}_{n+\sigma_2-\tau_1} + b^{\alpha} x^{\beta}_{n+\sigma_2+\tau_2}) \le 0, \quad n \ge n_1.
\end{aligned}$$

In view of $a \ge 1, b \le 1$ and $\beta > \alpha$, the last inequality becomes

$$\begin{aligned}
&\Delta(\Delta^{m-1}z_n + a^{\beta}\Delta^{m-1}z_{n-\tau_1} + b^{\alpha}\Delta^{m-1}z_{n+\tau_2}) \\
&+ P_n(x^{\alpha}_{n-\sigma_1} + a^{\alpha} x^{\alpha}_{n-\sigma_1-\tau_1} + b^{\alpha} x^{\alpha}_{n-\sigma_1+\tau_2}) \\
&+ Q_n(x^{\beta}_{n+\sigma_2} + a^{\beta} x^{\beta}_{n+\sigma_2-\tau_1} + b^{\beta} x^{\beta}_{n+\sigma_2+\tau_2}) \le 0, \quad n \ge n_1.
\end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1}z_n + a^{\beta}\Delta^{m-1}z_{n-\tau_1} + b^{\alpha}\Delta^{m-1}z_{n+\tau_2}) + P_n z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n+\sigma_2} \le 0, \quad n \ge n_1. \tag{3.13}$$

Now the inequalities (3.4), (3.7), (3.10) and (3.13) can be written as

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + P_n z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n+\sigma_2} \le 0, \quad n \ge n_1. \tag{3.14}$$

Since $\{z_n\}$ is increasing, the inequality (3.14) becomes

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + P_n z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n-\sigma_1} \le 0, \quad n \ge n_1. \tag{3.15}$$

Let $u_1\eta_1 = P_n z^{\alpha}_{n-\sigma_1}$ and $u_2\eta_2 = \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n-\sigma_1}$. Using the arithmetic-geometric mean inequality

$$\frac{u_1\eta_1 + u_2\eta_2}{\eta_1+\eta_2} \ge (u_1^{\eta_1} u_2^{\eta_2})^{\frac{1}{\eta_1+\eta_2}},$$

and the fact that $\eta_1 + \eta_2 = 1$, we get

$$P_n z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n-\sigma_1} \ge \eta_1^{-\eta_1}\eta_2^{-\eta_2} P_n^{\eta_1} \left(\frac{Q_n}{3^{\beta-1}}\right)^{\eta_2} z_{n-\sigma_1} = A_n z_{n-\sigma_1}, \quad n \ge n_1. \tag{3.16}$$

Now using (3.16) in (3.15), we obtain

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + A_n z_{n-\sigma_1} \le 0 \tag{3.17}$$

for all $n \ge n_1$. Using (2.1) in (3.17), we obtain

$$\begin{aligned}&\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2})\\ &+ A_n\frac{\lambda}{(m-1)!}(n-\sigma_1)^{m-1}\Delta^{m-1}z_{n-\sigma_1} \le 0\end{aligned} \tag{3.18}$$

for all $n \ge n_1$. By setting $\Delta^{m-1}z_n = y_n$, we see that $y_n > 0$ and $\Delta y_n \le 0$, and the inequality (3.18) becomes

$$\Delta(y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2}) + A_n\frac{\lambda}{(m-1)!}(n-\sigma_1)^{m-1}y_{n-\sigma_1} \le 0 \tag{3.19}$$

for all $n \ge n_1$. Now by denoting $y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2} = w_n$, and using the monotonicity of y_n, we get

$$w_n \le (1+d_1+d_2)y_{n-\tau_1} \quad \text{for all} \quad n \ge n_1.$$

Using the last inequality in the inequality (3.19), we see that $\{w_n\}$ is a positive solution of the inequality

$$\Delta w_n + \frac{A_n}{(1+d_1+d_2)}\frac{\lambda}{(m-1)!}(n-\sigma_1)^{m-1}w_{n+\tau_1-\sigma_1} \le 0, \quad n \ge n_1,$$

which is a contradiction to (3.1). This completes the proof. □

Theorem 3.2. *Assume that $\beta < 1 < \alpha$. If the first order difference inequality*

$$\Delta w_n + \frac{B_n}{(1+d_3+d_4)}\frac{\lambda}{(m-1)!}(n-\sigma_1)^{m-1}w_{n+\tau_1-\sigma_1} \le 0, \tag{3.20}$$

where

$$B_n = \eta_1^{-\eta_1}\eta_2^{-\eta_2}\left(\frac{P_n}{3^{\alpha-1}}\right)^{\eta_1}Q_n^{\eta_2}, \quad \eta_1 = \frac{\alpha-1}{\alpha-\beta}, \quad \eta_2 = \frac{1-\beta}{\alpha-\beta},$$

$$d_3 = \begin{cases} a^\alpha & \textit{if}\, a \ge 1,\\ a^\beta & \textit{if}\, a \le 1\end{cases} \quad \textit{and} \quad d_4 = \begin{cases} b^\alpha & \textit{if}\, b \ge 1,\\ b^\beta & \textit{if}\, b \le 1\end{cases}$$

has no positive solution for some $\lambda \in (0,1)$ and for all $n \ge n_0$, then every solution of the equation (1.1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 3.1, and hence the details are omitted. □

Theorem 3.3. *Assume that $\alpha < 1 < \beta$ hold. If*

$$\sum_{n=n_1}^{\infty} R_n = \infty, \tag{3.21}$$

then every solution of the equation (1.1) *is oscillatory.*

Proof. Assume that $\{x_n\}$ is a positive solution of the equation (1.1). Then proceeding as in the proof of Theorem 3.1, we obtain (3.15). Since $\{z_n\}$ is positive increasing there exists $M > 0$ such that $z_n \geq M$ for all $n \geq n_1$. Therefore from the inequality (3.15), we obtain

$$\Delta(\Delta^{m-1} z_n + d_1 \Delta^{m-1} z_{n-\tau_1} + d_2 \Delta^{m-1} z_{n+\tau_2}) + P_n M^\alpha + \frac{Q_n M^\beta}{3^{\beta-1}} \leq 0, \quad n \geq n_1,$$

that is,

$$\Delta(\Delta^{m-1} z_n + d_1 \Delta^{m-1} z_{n-\tau_1} + d_2 \Delta^{m-1} z_{n+\tau_2}) + R_n \leq 0, \quad n \geq n_1.$$

Now taking summation from n_1 to $n-1$ and letting $n \to \infty$, we get

$$\sum_{s=n_1}^{\infty} R_s \leq \Delta^{m-1} z_{n_1} + d_1 \Delta^{m-1} z_{n_1-\tau_1} + d_2 \Delta^{m-1} z_{n_1+\tau_2} < \infty,$$

which is a contradiction to (3.21). This completes the proof. □

Theorem 3.4. *Assume that $1 < \alpha < \beta$. If the first order difference inequality*

$$\Delta w_n + \frac{C_n}{(1 + d_1 + d_2)} \frac{\lambda}{(m-1)!} (n - \sigma_1)^{m-1} w_{n+\tau_1-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \leq 0, \tag{3.22}$$

where $C_n = \frac{\eta_1^{-\eta_1} \eta_2^{-\eta_2}}{3^{\beta-1}} P_n^{\eta_1} Q_n^{\eta_2}$, $\eta_1 = \frac{\alpha-1}{\beta-1}$, $\eta_2 = \frac{\beta-\alpha}{\beta-1}$, d_1 and d_2 are as in Theorem 3.1, has no positive solution for some $\lambda \in (0,1)$ and for all $n \geq n_0$, then every solution of the equation (1.1) *is oscillatory.*

Proof. Proceeding as in Theorem 3.1, we see that $z_n > 0$ and $\Delta z_n > 0$ for all $n \geq n_1$. Now we discuss the different cases for a and b.

Suppose $a \leq 1$ and $b \leq 1$. Then from the equation (1.1), we get

$$a^\alpha \Delta^m z_{n-\tau_1} + a^\alpha p_{n-\tau_1} x_{n-\sigma_1-\tau_1}^\alpha + a^\alpha q_{n-\tau_1} x_{n+\sigma_2-\tau_1}^\beta = 0, \quad n \geq n_1 \tag{3.23}$$

and

$$b^\alpha \Delta^m z_{n+\tau_2} + b^\alpha p_{n+\tau_2} x_{n-\sigma_1+\tau_2}^\alpha + b^\alpha q_{n+\tau_2} x_{n+\sigma_2+\tau_2}^\beta = 0, \quad n \geq n_1. \tag{3.24}$$

Now combining equations (1.1), (3.23) and (3.24), we obtain

$$\begin{aligned} &\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) \\ &+ P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ &+ Q_n(x_{n+\sigma_2}^\beta + a^\alpha x_{n+\sigma_2-\tau_1}^\beta + b^\alpha x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Since $a \leq 1, b \leq 1$ and $\beta > \alpha$, the last inequality yields

$$\begin{aligned} &\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) \\ &+ P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ &+ Q_n(x_{n+\sigma_2}^\beta + a^\beta x_{n+\sigma_2-\tau_1}^\beta + b^\beta x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Now, using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1}z_n + a^\alpha \Delta^{m-1}z_{n-\tau_1} + b^\alpha \Delta^{m-1}z_{n+\tau_2}) + \frac{P_n}{3^{\beta-1}} z^\alpha_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^\beta_{n+\sigma_2} \le 0, \quad n \ge n_1.$$

The proof for the other case of a and b are similar to that of in Theorem 3.1. Therefore for all cases of a and b, we have the inequality

$$\Delta(\Delta^{m-1}z_n + d_1 \Delta^{m-1}z_{n-\tau_1} + d_2 \Delta^{m-1}z_{n+\tau_2}) + \frac{P_n}{3^{\beta-1}} z^\alpha_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^\beta_{n+\sigma_2} \le 0, \quad n \ge n_1. \tag{3.25}$$

Since $\{z_n\}$ is increasing, the inequality (3.25) becomes

$$\Delta(\Delta^{m-1}z_n + d_1 \Delta^{m-1}z_{n-\tau_1} + d_2 \Delta^{m-1}z_{n+\tau_2}) + \frac{P_n}{3^{\beta-1}} z^\alpha_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^\beta_{n-\sigma_1} \le 0, \quad n \ge n_1. \tag{3.26}$$

Let $u_1\eta_1 = \frac{P_n}{3^{\beta-1}} z^\alpha_{n-\sigma_1}$ and $u_2\eta_2 = \frac{Q_n}{3^{\beta-1}} z^\beta_{n-\sigma_1}$. Using the arithmetic-geometric mean inequality, and the fact $\eta_1 + \eta_2 = 1$, we get

$$\frac{P_n}{3^{\beta-1}} z^\alpha_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^\beta_{n-\sigma_1} \ge \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{3^{\beta-1}} P_n^{\eta_1} Q_n^{\eta_2} z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} = C_n z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1}. \tag{3.27}$$

Now using (3.27) in (3.26), we obtain

$$\Delta(\Delta^{m-1}z_n + d_1 \Delta^{m-1}z_{n-\tau_1} + d_2 \Delta^{m-1}z_{n+\tau_2}) + C_n z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} \le 0 \tag{3.28}$$

for all $n \ge n_1$. From (2.1) and (3.28), we obtain

$$\begin{aligned} &\Delta(\Delta^{m-1}z_n + d_1 \Delta^{m-1}z_{n-\tau_1} + d_2 \Delta^{m-1}z_{n+\tau_2}) \\ &+ C_n \frac{\lambda}{(m-1)!}(n-\sigma_1)^{m-1} \Delta^{m-1} z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} \le 0 \end{aligned} \tag{3.29}$$

for all $n \ge n_1$. By setting $\Delta^{m-1}z_n = y_n$, we see that $y_n > 0$ and $\Delta y_n \le 0$, and the inequality (3.29) becomes

$$\Delta(y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2}) + C_n \frac{\lambda}{(m-1)!}(n-\sigma_1)^{m-1} y^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} \le 0 \tag{3.30}$$

for all $n \ge n_1$. Now by denoting $y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2} = w_n$, and using the monotonicity of y_n, we get

$$w_n \le (1 + d_1 + d_2) y_{n-\tau_1} \quad \text{for all} \quad n \ge n_1.$$

From the last inequality and (3.30), we see that $\{w_n\}$ is a positive solution of the inequality

$$\Delta w_n + \frac{C_n}{(1+d_1+d_2)} \frac{\lambda}{(m-1)!}(n-\sigma_1)^{m-1} w^{\alpha\eta_1+\beta\eta_2}_{n+\tau_1-\sigma_1} \le 0, \quad n \ge n_1,$$

which is a contradiction to (3.22). This completes the proof. □

Theorem 3.5. *Assume that $1 < \beta < \alpha$. If the first order difference inequality*

$$\Delta w_n + \frac{D_n}{(1+d_3+d_4)(m-1)!}(n-\sigma_1)^{m-1} w_{n+\tau_1-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \leq 0, \tag{3.31}$$

where $D_n = \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{3^{\alpha-1}} P_n^{\eta_1} Q_n^{\eta_2}$, $\eta_1 = \frac{\beta-1}{\alpha-1}$, and $\eta_2 = \frac{\alpha-\beta}{\alpha-1}$, d_3 and d_4 are as in Theorem 3.2, has no positive solution for some $\lambda \in (0,1)$ and for all $n \geq n_0$, then every solution of the equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 3.4, and hence the details are omitted. □

Theorem 3.6. *Assume that $1 < \alpha < \beta$ holds. If*

$$\sum_{n=n_1}^{\infty} R_n = \infty, \tag{3.32}$$

then every solution of the equation (1.1) is oscillatory.

Proof. Assume that $\{x_n\}$ is a positive solution of the equation (1.1). Then proceeding as in the proof of Theorem 3.4, we obtain (3.26). Since $\{z_n\}$ is positive increasing there exists $M > 0$ such that $z_n \geq M$ for all $n \geq n_1$. Then from the inequality (3.26), we obtain

$$\Delta(\Delta^{m-1} z_n + d_1 \Delta^{m-1} z_{n-\tau_1} + d_2 \Delta^{m-1} z_{n+\tau_2}) + \frac{P_n}{3^{\beta-1}} M^{\alpha} + \frac{Q_n}{3^{\beta-1}} M^{\beta} \leq 0, \quad n \geq n_1,$$

that is,

$$\Delta(\Delta^{m-1} z_n + d_1 \Delta^{m-1} z_{n-\tau_1} + d_2 \Delta^{m-1} z_{n+\tau_2}) + R_n \leq 0, \quad n \geq n_1.$$

Now taking summation from n_1 to $n-1$ and letting $n \to \infty$, we get

$$\sum_{s=n_1}^{\infty} R_s \leq \Delta^{m-1} z_{n_1} + d_1 \Delta^{m-1} z_{n_1-\tau_1} + d_2 \Delta^{m-1} z_{n_1+\tau_2} < \infty,$$

which is a contradiction to (3.32). This completes the proof. □

Theorem 3.7. *Assume that $\alpha < \beta < 1$. If the first order difference inequality*

$$\Delta w_n + \frac{E_n}{(1+d_1+d_2)} \frac{\lambda}{(m-1)!}(n-\sigma_1)^{m-1} w_{n+\tau_1-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \leq 0, \tag{3.33}$$

where $E_n = \eta_1^{-\eta_1}\eta_2^{-\eta_2} P_n^{\eta_1} Q_n^{\eta_2}$, $\eta_1 = \frac{\beta-\alpha}{1-\alpha}$, $\eta_2 = \frac{1-\beta}{1-\alpha}$, d_1 and d_2 are as in Theorem 3.1, has no positive solution for some $\lambda \in (0,1)$ and for all $n \geq n_0$, then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a positive solution of the equation (1.1). Then proceeding as in Theorem 3.1, we have $z_n > 0$ and $\Delta z_n \geq 0$ for all $n \geq n_1$. Now we discuss the different cases for a and b.

From the equation (1.1), we get

$$a^\alpha \Delta^m z_{n-\tau_1} + a^\alpha p_{n-\tau_1} x^\alpha_{n-\sigma_1-\tau_1} + a^\alpha q_{n-\tau_1} x^\beta_{n+\sigma_2-\tau_1} = 0, \quad n \geq n_1, \tag{3.34}$$

and

$$b^\alpha \Delta^m z_{n+\tau_2} + b^\alpha p_{n+\tau_2} x^\alpha_{n-\sigma_1+\tau_2} + b^\alpha q_{n+\tau_2} x^\beta_{n+\sigma_2+\tau_2} = 0, \quad n \geq n_1. \tag{3.35}$$

Now combining equations (1.1), (3.34) and (3.35), we obtain

$$\begin{aligned}&\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2})\\ &+ P_n(x^\alpha_{n-\sigma_1} + a^\alpha x^\alpha_{n-\sigma_1-\tau_1} + b^\alpha x^\alpha_{n-\sigma_1+\tau_2})\\ &+ Q_n(x^\beta_{n+\sigma_2} + a^\alpha x^\beta_{n+\sigma_2-\tau_1} + b^\alpha x^\beta_{n+\sigma_2+\tau_2}) \leq 0, \quad n \geq n_1.\end{aligned}$$

Since $a \leq 1, b \leq 1$ and $\alpha < \beta < 1$, the last inequality becomes

$$\begin{aligned}&\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2})\\ &+ P_n(x^\alpha_{n-\sigma_1} + a^\alpha x^\alpha_{n-\sigma_1-\tau_1} + b^\alpha x^\alpha_{n-\sigma_1+\tau_2})\\ &+ Q_n(x^\beta_{n+\sigma_2} + a^\beta x^\beta_{n+\sigma_2-\tau_1} + b^\beta x^\beta_{n+\sigma_2+\tau_2}) \leq 0, \quad n \geq n_1.\end{aligned}$$

Now, using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) + P_n z^\alpha_{n-\sigma_1} + Q_n z^\beta_{n+\sigma_2} \leq 0, \quad n \geq n_1.$$

The proof for the other case of a and b are similar to that of Theorem 3.1. Therefore for all cases of a and b, we have the inequality

$$\Delta(\Delta^{m-1} z_n + d_1 \Delta^{m-1} z_{n-\tau_1} + d_2 \Delta^{m-1} z_{n+\tau_2}) + P_n z^\alpha_{n-\sigma_1} + Q_n z^\beta_{n+\sigma_2} \leq 0, \quad n \geq n_1. \tag{3.36}$$

Since $\{z_n\}$ is increasing, the inequality (3.36) becomes

$$\Delta(\Delta^{m-1} z_n + d_1 \Delta^{m-1} z_{n-\tau_1} + d_2 \Delta^{m-1} z_{n+\tau_2}) + P_n z^\alpha_{n-\sigma_1} + Q_n z^\beta_{n-\sigma_1} \leq 0, \quad n \geq n_1. \tag{3.37}$$

Now set $u_1\eta_1 = P_n z^\alpha_{n-\sigma_1}$, $u_2\eta_2 = Q_n z^\beta_{n-\sigma_1}$, $\eta_1 = \frac{\beta-\alpha}{1-\alpha}$ and $\eta_2 = \frac{1-\beta}{1-\alpha}$. Then by the arithmetic-geometric mean inequality

$$\frac{u_1\eta_1 + u_2\eta_2}{\eta_1 + \eta_2} \geq (u_1^{\eta_1} u_2^{\eta_2})^{\frac{1}{\eta_1+\eta_2}}$$

implies that

$$P_n z^\alpha_{n-\sigma_1} + Q_n z^\beta_{n-\sigma_1} \geq \eta_1^{-\eta_1} \eta_2^{-\eta_2} P_n^{\eta_1} Q_n^{\eta_2} z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} = E_n z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1}, \quad n \geq n_1. \tag{3.38}$$

Combining (3.37) and (3.38), we obtain

$$\Delta(\Delta^{m-1} z_n + d_1 \Delta^{m-1} z_{n-\tau_1} + d_2 \Delta^{m-1} z_{n+\tau_2}) + E_n z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} \leq 0.$$

From the last inequality by taking $\Delta^{m-1}z_n = y_n$, we see that $y_n > 0$ and $\Delta y_n \le 0$, and

$$\Delta(y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2}) + E_n z_{n-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \le 0. \tag{3.39}$$

Now let $y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2} = w_n$. Then $w_n > 0$ and using $\Delta y_n \le 0$, we get

$$w_n \le (1 + d_1 + d_2) y_{n-\tau_1} \quad \text{for all} \quad n \ge n_1. \tag{3.40}$$

Combining (3.39) and (3.40), we see that $\{w_n\}$ is a positive solution of the inequality

$$\Delta w_n + E_n \frac{\lambda(n-\sigma_1)^{m-1}}{(m-1)!(1+d_1+d_2)} w_{n-\sigma_1+\tau_1}^{\alpha\eta_1+\beta\eta_2} \le 0, \quad n \ge n_1,$$

which is a contradiction to (3.33). This completes the proof of the theorem. □

Theorem 3.8. *Assume that $\beta < \alpha < 1$. If the first order difference inequality*

$$\Delta w_n + \frac{E_n}{(1+d_3+d_4)} \frac{\lambda}{(m-1)!}(n-\sigma_1)^{m-1} w_{n+\tau_1-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \le 0, \tag{3.41}$$

where $\eta_1 = \frac{\alpha-\beta}{1-\beta}$, $\eta_2 = \frac{1-\alpha}{1-\beta}$, d_3 and d_4 are as in Theorem 3.2, and E_n is as defined in Theorem 3.7, has no positive solution, then every solution of the equation (1.1) *is oscillatory.*

Proof. The proof is similar to Theorem 3.7 and hence it is omitted. □

Theorem 3.9. *Assume that $\alpha < \beta < 1$ holds. If*

$$\sum_{n=n_1}^{\infty} R_n = \infty \tag{3.42}$$

holds, then every solution of the equation (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a positive solution of the equation (1.1). Then proceeding as in the proof of Theorem 3.7, we deduce the inequality (3.37). Since $\{z_n\}$ is positive increasing there exists $M > 0$ such that $z_n \ge M$ for all $n \ge n_1$. Then from the inequality (3.37), we obtain

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + P_n M^\alpha + Q_n M^\beta \le 0, \quad n \ge n_1,$$

that is,

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + R_n \le 0, \quad n \ge n_1.$$

Now taking summation from n_1 to $n-1$ and letting $n \to \infty$, we get

$$\sum_{s=n_1}^{\infty} R_s \le \Delta^{m-1}z_{n_1} + d_1\Delta^{m-1}z_{n_1-\tau_1} + d_2\Delta^{m-1}z_{n_1+\tau_2} < \infty,$$

which is a contradiction to (3.42). This completes the proof. □

Corollary 3.10. *Assume that* $\alpha < 1 < \beta$ *and* $\sigma_1 > \tau_1$ *hold. If*

$$\liminf_{n\to\infty} \sum_{s=n-(\sigma_1-\tau_1)}^{n-1} A_s(s-\sigma_1)^{m-1} > \frac{(1+d_1+d_2)(m-1)!}{\lambda}\left(\frac{\sigma_1-\tau_1}{\sigma_1-\tau_1+1}\right)^{\sigma_1-\tau_1+1}, \tag{3.43}$$

then every solution of the equation (1.1) *is oscillatory.*

Proof. By Theorem 7.5.1 of [10], the condition (3.43) guarantees that the first order difference inequality (3.1) has no positive solution. Now the result follows from Theorem 3.1. □

Corollary 3.11. *Assume that* $\beta < 1 < \alpha$ *and* $\sigma_1 > \tau_1$ *hold. If*

$$\liminf_{n\to\infty} \sum_{n-(\sigma_1-\tau_1)}^{n-1} B_s(s-\sigma_1)^{m-1} > \frac{(1+d_3+d_4)(m-1)!}{\lambda}\left(\frac{\sigma_1-\tau_1}{\sigma_1-\tau_1+1}\right)^{\sigma_1-\tau_1+1}, \tag{3.44}$$

then every solution of the equation (1.1) *is oscillatory.*

Proof. By Theorem 7.5.1 of [10], the condition (3.44) guarantees that the first order difference inequality (3.20) has no positive solution. Now the result follows from Theorem 3.2. □

Note that for $\beta > \alpha > 1$, $\eta_1 = \frac{\alpha-1}{\beta-1}$ and $\eta_2 = \frac{\beta-\alpha}{\beta-1}$, imply $\alpha\eta_2 + \beta\eta_2 > 1$. Now using Theorem 3.4, we have the following corollary.

Corollary 3.12. *Assume that* $1 < \alpha < \beta$ *and* $\sigma_1 > \tau_1$ *hold. If there exists a* $\mu > 0$ *such that* $\mu > \frac{1}{\sigma_1-\tau_1}\ln(\alpha\eta_1+\beta\eta_2)$, *and*

$$\liminf_{n\to\infty} C_n(n-\sigma_1)^{m-1}\exp(-e^{\mu n}) > 0, \tag{3.45}$$

then every solution of the equation (1.1) *is oscillatory.*

Proof. By Theorem 2 of [14], condition (3.45) guarantees that the first order difference inequality (3.22) has no positive solution. Now the result follows from Theorem 3.4. □

Corollary 3.13. *Assume that* $1 < \beta < \alpha$ *and* $\sigma_1 > \tau_1$ *hold. If there exists a* $\mu > 0$ *such that* $\mu > \frac{1}{\sigma_1-\tau_1}\ln(\alpha\eta_1+\beta\eta_2)$, *and*

$$\lim\inf_{n\to\infty} D_n(n-\sigma_1)^{m-1}\exp(-e^{\mu n}) > 0, \tag{3.46}$$

then every solution of the equation (1.1) *is oscillatory.*

Proof. By Theorem 2 of [14], condition (3.46) guarantees that the first order difference inequality (3.31) has no positive solution. Now the result follows from Theorem 3.5. □

Note that for $\alpha < \beta < 1$, $\eta_1 = \frac{\beta-\alpha}{1-\alpha}$ and $\eta_2 = \frac{1-\beta}{1-\alpha}$, we have $\alpha\eta_2 + \beta\eta_2 < 1$. Now using Theorem 3.7, we have the following corollary.

Corollary 3.14. *Assume that* $\alpha < \beta < 1$ *hold. If*

$$\lim_{n\to\infty} \inf \sum_{n=n_0}^{\infty} E_s(s-\sigma_1)^{m-1} = \infty, \tag{3.47}$$

then every solution of the equation (1.1) *is oscillatory.*

Proof. By Theorem 1 of [14], condition (3.47) guarantees that the first order difference inequality (3.33) has no positive solution. Now the result follows from Theorem 3.7. □

Note that for $\beta < \alpha < 1$, $\eta_1 = \frac{\alpha-\beta}{1-\beta}$ and $\eta_2 = \frac{1-\alpha}{1-\beta}$, we have $\alpha\eta_2 + \beta\eta_2 < 1$. Now using Theorem 3.8, we have the following Corollary.

Corollary 3.15. *Assume that* $\beta < \alpha < 1$ *hold. If*

$$\lim_{n\to\infty} \inf \sum_{n=n_0}^{\infty} E_s(s-\sigma_1)^{m-1} = \infty, \tag{3.48}$$

then every solution of the equation (1.1) *is oscillatory.*

Proof. By Theorem 1 of [14], the condition (3.48) guarantees that the first order difference inequality (3.41) has no positive solution. Now the result follows from Theorem 3.8. □

Theorem 3.16. *Assume that conditions* $\alpha < 1 < \beta$ *and* $\sigma_1 \leq \tau_1$ *hold. Further assume that there exists real valued function* $H : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ *such that*

$$\begin{aligned} H_{n,n} &= 0 \quad \textit{for} \quad n \geq n_0 > 0, \\ H_{n,s} &> 0 \quad \textit{for} \quad n > s \geq n_0, \\ \Delta_2 H_{n,s} &\leq 0 \quad \textit{for} \quad n > s \geq n_0, \end{aligned}$$

where

$$\Delta_2 H_{n,s} = H_{n,s+1} - H_{n,s}.$$

If

$$\lim_{n\to\infty} \sup \frac{1}{H_{n,n_1}} \sum_{s=n_1}^{n-1} A_s H_{n,s} = \infty, \quad n \geq n_1 \geq n_0, \tag{3.49}$$

then every solution of (1.1) *is oscillatory.*

Proof. Assume that $\{x_n\}$ is a positive solution of the equation (1.1). Then there exists an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-\sigma_1} > 0$ and $x_{n-\tau_1} > 0$ for all $n \geq n_1$. Then by Lemma 2.4, we have $\Delta z_n > 0$ for all $n \geq n_1 \geq n_0$. Now define a function

$$w_n = \frac{\Delta^{m-1} z_n}{z_{n-\tau_1}}$$

for all $n \geq n_1$. Then $w_n > 0$ for all $n \geq n_1$, and

$$\Delta w_n = \frac{\Delta^m z_n}{z_{n-\tau_1}} - \frac{\Delta^{m-1} z_{n+1}}{z_{n-\tau_1} z_{n+1-\tau_1}} \Delta z_{n-\tau_1} \leq \frac{\Delta^m z_n}{z_{n-\tau_1}}, \quad n \geq n_1.$$

Similarly by defining v_n and u_n for all $n \geq n_1$, respectively, by

$$v_n = \frac{\Delta^{m-1} z_{n-\tau_1}}{z_{n-\tau_1}}, \quad n \geq n_1,$$

and

$$u_n = \frac{\Delta^{m-1} z_{n+\tau_2}}{z_{n-\tau_1}}, \quad n \geq n_1,$$

we obtain $v_n > 0$ and $u_n > 0$ for all $n \geq n_1$, and

$$\Delta v_n \leq \frac{\Delta^m z_{n-\tau_1}}{z_{n-\tau_1}},$$

and

$$\Delta u_n \leq \rho_n \frac{\Delta^m z_{n+\tau_2}}{z_{n-\tau_1}}$$

for all $n \geq n_1$. Now combining these inequalities, we obtain

$$\Delta w_n + a^\beta \Delta v_n + b^\beta \Delta u_n \leq \frac{1}{z_{n-\tau_1}} [\Delta^m z_n + a^\beta \Delta^m z_{n-\tau_1} + b^\beta \Delta^m z_{n+\tau_2}]$$

for all $n \geq n_1$. Now using (3.17) and the monotonicity of z_n, the last inequality becomes

$$\Delta w_n + a^\beta \Delta v_n + b^\beta \Delta u_n \leq -A_n.$$

Replacing n by s and multiplying the last inequality by $H_{n,s}$ and then summing the resulting inequality from n_1 to $n-1$, we have

$$\sum_{s=n_1}^{n-1} A_s H_{n,s} \leq - \sum_{s=n_1}^{n-1} [\Delta w_s + a^\beta \Delta v_s + b^\beta \Delta u_s] H_{n,s}.$$

Now using summation by parts we get

$$\begin{aligned} \sum_{s=n_1}^{n-1} A_s H_{n,s} &\leq H_{n,n_1}[w_{n_1} + a^\beta v_{n_1} + b^\beta u_{n_1}] + \sum_{s=n_1}^{n-1} [w_{s+1} + a^\beta v_{s+1} + b^\beta u_{s+1}] \Delta_2 H_{n,s} \\ &\leq [w_{n_1} + a^\beta v_{n_1} + b^\beta u_{n_1}] H_{n,n_1} \end{aligned}$$

or

$$\frac{1}{H(n,n_1)} \sum_{s=n_1}^{n-1} A_s H_{n,s} \leq [w_{n_1} + a^\beta v_{n_1} + b^\beta u_{n_1}].$$

Taking $\limsup$ as $n \to \infty$, in the last inequality, we obtain

$$\lim_{n\to\infty} \sup \frac{1}{H_{n,n_1}} \sum_{s=n_1}^{n-1} A_s H_{n,s} < \infty,$$

which is a contradiction to (3.49). The theorem is now proved. □

4. EXAMPLES

In this section, we provide some examples to illustrate the main results.

Example 4.1. Consider an even order neutral difference equation

$$\Delta^m\Big(x_n+\frac{1}{2}x_{n-1}+\frac{1}{3}x_{n+2}\Big)+nx_{n-2}^{\frac{1}{3}}+\frac{1}{n}x_{n+1}^3=0,\quad n\geq 1, \tag{4.1}$$

where $m\geq 2$ is an even integer.

Here $a=\frac{1}{2}$, $b=\frac{1}{3}$, $\tau_1=1$, $\tau_2=2$, $\sigma_1=2$, $\sigma_2=1$, $p_n=n$, $q_n=\frac{1}{n}$, $\alpha=\frac{1}{3}$ and $\beta=3$. A simple calculation shows that $P_n=(n-1)$, $Q_n=\frac{1}{n+2}$, $\eta_1=\frac{3}{4}$, $\eta_2=\frac{1}{4}$, $d_1=\left(\frac{1}{2}\right)^{1/3}$, $d_2=\left(\frac{1}{3}\right)^{1/3}$ and $A_n=\frac{4}{3^{5/4}}\frac{(n-1)^{3/4}}{(n+2)^{1/4}}$. Further calculation shows that

$$\liminf_{n\to\infty}\sum_{s=n-1}^{n-1}\frac{4}{3^{5/4}}\frac{(s-1)^{3/4}}{(s+2)^{1/4}}(s-2)^{(m-1)}=\liminf_{n\to\infty}\frac{4}{3^{5/4}}\frac{(n-2)^{3/4}}{(n+1)^{1/4}}(n-3)^{(m-1)}=\infty$$

for all $m\geq 2$. Hence all conditions of Corollary 3.10 are satisfied and therefore every solution of the equation (4.1) is oscillatory.

Example 4.2. Consider an even order neutral difference equation

$$\Delta^m(x_n+2x_{n-1}+3x_{n+2})+nx_{n-2}^3+\frac{1}{n}x_{n+1}^{1/3}=0,\quad n\geq 1. \tag{4.2}$$

Here $a=2$, $b=3$, $\tau_1=1$, $\tau_2=2$, $\sigma_1=2$, $\sigma_2=1$, $p_n=n$, $q_n=\frac{1}{n}$, $\alpha=3$ and $\beta=\frac{1}{3}$. A simple calculation shows that $P_n=(n-1)$, $Q_n=\frac{1}{n+2}$, $\eta_1=\frac{3}{4}$, $\eta_2=\frac{1}{4}$, $d_3=8$, $d_4=27$ and $B_n=\frac{4}{3^{5/4}}(n-1)^{3/4}\frac{1}{(n+2)^{1/4}}$. Also we see that

$$\lim_{n\to\infty}\inf\sum_{s=n-1}^{n-1}\frac{4}{3^{5/4}}\frac{(s-1)^{3/4}}{(s+2)^{1/4}}(s-2)^{(m-1)}=\lim_{n\to\infty}\inf\frac{4}{3^{5/4}}\frac{(n-2)^{3/4}}{(n+1)^{1/4}}(n-3)^{(m-1)}=\infty$$

for all $m\geq 2$. Therefore all conditions of Corollary 3.11 are satisfied and therefore every solution of the equation (4.2) is oscillatory.

Example 4.3. Consider an even order neutral difference equation

$$\Delta^m(x_n+3x_{n-1}+3x_{n+2})+\frac{2^m}{n}x_{n-2}^3+2^m\left(\frac{n+1}{n}\right)x_{n+3}^5=0,\quad n\geq 1, \tag{4.3}$$

where $m\geq 2$ is an even integer.

Here $a=b=3$, $\tau_1=1$, $\tau_2=2$, $\sigma_1=2$, $\sigma_3=3$, $p_n=\frac{2^n}{n}$, $q_n=2^m\left(\frac{n+1}{n}\right)$, $\alpha=3$ and $\beta=5$. Further $P_n=\frac{2^m}{n+2}$, $Q_n=2^m\left(\frac{n+3}{n+2}\right)$, $R_n=\frac{2^m}{(n+2)}(k_1+k_2(n+3))$ and

$$\sum_{n=1}^{\infty}R_n=\sum_{n=1}^{\infty}\frac{2^m}{n+2}(k_1+k_2(n+3))=\infty.$$

Therefore, by Theorem 3.6, every solution of the equation (4.3) is oscillatory. In fact $\{x_n\}=\{(-1)^n\}$ is one such oscillatory solution of equation (4.3).

Remark 4.4.

1. The established results are presented in a form which is essentially new and include some of the existing results as special cases.
2. The existing results [4, 7, 9] cannot to be applied to equations (4.1), (4.2) and (4.3) since $\alpha \neq 1$ and $\beta \neq 1$.
3. The results of this paper may be extended to equation of the form

$$\Delta\left(a_n(\Delta^{m-1}(x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2}))\right) + q_n x^{\alpha}_{n-\sigma_1} + p_n x^{\beta}_{n+\sigma_2} = 0,$$

when $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$ or $\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$, and the details are left to the reader.

Acknowledgements
The publication was supported by the Ministry of Education and Science of the Russian Federation (the agreement number N.02.a03.21.0008).
The authors thank the referee for his/her suggestions and corrections that improved the content of the paper.

REFERENCES

[1] R. Ali-Hamouri, A. Zein, *Oscillation criteria for certain even order neutral delay differential equations*, Inter. J. Diffential Equ. **2014** (2014), Article ID 437278, 5 pp.

[2] R.P. Agarwal, *Difference Equations and Inequalities*, Theory, Methods and Applications, 2nd ed., Marcel Dekker, New York, 2000.

[3] R.P. Agarwal, S.R. Grace, *Oscillation criteria for certain higher order difference equations*, Math. Sci. Res. J. **6** (2002), 60–64.

[4] R.P. Agarwal, S.R. Grace, E.A. Bohner, *On the oscillation of higher order neutral difference equations of mixed type*, Dynam. Systems Appl. **11** (2002), 459–470.

[5] R.P. Agarwal, S.R. Grace, D. O'Regan, *On the oscillation of higher order difference equations*, Soochow J. Math. **31** (2005), 245–259.

[6] R.P. Agarwal, E. Thandapani, P.J.Y. Wong, *Oscillation of higher order neutral difference equations*, Appl. Math. Lett. **10** (1997), 71–78.

[7] R.P. Agarwal, M. Bohner, S.R. Grace, D. O'Regan, *Discrete Oscillation Theory*, Hindawi, New York, 2005.

[8] Y. Bolat, O. Akin, *Oscillatory behavior of higher order neutral type functional difference equation with oscillating coefficient*, Appl. Math. Lett. **17** (2004), 1073–1078.

[9] S.R. Grace, *Oscillation of certain neutral difference equations of mixed type*, J. Math. Anal. Appl. **224** (1998), 241–254.

[10] I. Gyori, G. Ladas, *Oscillation Theorey of Delay Differential Equations with Applications*, Claredon Press, Oxford, 1991.

[11] M. Migda, *On the discrete version of generalized Kiguradzes lemma*, Fasci. Math. **35** (2005), 1–7.

[12] D. Seghar, S. Selvarangam, E. Thandapani, *Oscillation criteria for even order nonlinear neutral difference equation*, Differ. Equ. Appl. **6** (2014), 441–453.

[13] E. Thandapani, *Oscillation theorems for higher order nonlinear difference equations*, Indian J. Pure Appl. Math. **25** (1994), 519–524.

[14] X.H. Tang, Y.J. Liu, *Oscillation for nonlinear delay difference equations*, Tamkang J. Math. **32** (2001), 275–280.

S. Selvarangam
selvarangam.9962@gmail.com

Department of Mathematics
Presidency College Chennai – 600 005, India

S.A. Rupadevi

Department of Mathematics
Presidency College Chennai – 600 005, India

E. Thandapani
ethandapani@yahoo.co.in

University of Madras
Ramanujan Institute for Advanced Study in Mathematics
Chennai – 600 005, India

S. Pinelas
sandra.pinelas@gmail.com

RUDN University
6 Miklukho-Maklaya Str., 117198, Moscow, Russia

Received: February 11, 2018.
Revised: July 7, 2018.
Accepted: July 7, 2018.