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FLAT STRUCTURE AND POTENTIAL VECTOR FIELDS RELATED WITH ALGEBRAIC SOLUTIONS TO PAINLEVÉ VI EQUATION

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Abstract. A potential vector field is a solution of an extended WDVV equation which is a generalization of a WDVV equation. It is expected that potential vector fields corresponding to algebraic solutions of Painlevé VI equation can be written by using polynomials or algebraic functions explicitly. The purpose of this paper is to construct potential vector fields corresponding to more than thirty non-equivalent algebraic solutions.

Keywords: flat structure, Painlevé VI equation, algebraic solution, potential vector field.

Mathematics Subject Classification: 34M56, 33E17, 35N10, 32S25.

1. INTRODUCTION

In [13], the authors generalized the theory of Frobenius manifolds and WDVV equation so that an extended WDVV equation in the semisimple and three dimensional case is equivalent to the Painlevé VI equation. A solution to the extended WDVV equation is called a potential vector field[1]. The Painlevé VI equation is given by

$$\frac{d^2w}{dt^2} = \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right)\left(\frac{dw}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{w-t}\right)\frac{dw}{dt} + \frac{w(w-1)(w-t)}{2t^2(t-1)^2}\left((\theta_\infty-1)^2 - \frac{\theta_x^2 t}{w^2} + \frac{\theta_y^2(t-1)}{(w-1)^2} + \frac{(1-\theta_z^2)t(t-1)}{(w-t)^2}\right). \tag{1.1}$$

[1] A potential vector field is defined in [19]. A similar notion called "a local vector potential" is introduced by Yu. Manin [21].

The equation (1.1) has four parameters $\theta = (\theta_x, \theta_y, \theta_z, \theta_\infty)$. In general, solutions to (1.1) are transcendental functions of t. However for some values of the parameter, solutions to (1.1) can be expressed via polynomials, classical functions or algebraic functions. A solution $w = w(t)$ to (1.1) means algebraic if there is a polynomial $P(u, v)$ of u, v such that $P(t, w) = 0$. It is expected that potential vector fields corresponding to algebraic solutions can be written by using polynomials or algebraic functions explicitly. The purpose of this paper is to construct potential vector fields corresponding to many of algebraic solutions.

Algebraic solutions to Painlevé VI equation are studied and constructed by many authors, for example, K. Iwasaki [11], N.J. Hitchin [9,10], B. Dubrovin [6], B. Dubrovin and M. Mazzocco [7], P. Boalch [2–5], F.V. Andreev – A.V. Kitaev [1], A.V. Kitaev [15–17], A.V. Kitaev and R. Vidunas [18,31]. After these efforts, a classification of such solutions was accomplished by O. Lisovyy and Y. Tykhyy (cf. [20]) showing that there is no solution except those constructed so far. As is mentioned in [20], algebraic solutions are related to various mathematical structures, including, for example, Frobenius manifolds [6], symmetry groups of regular polyhedra [7], [10], complex reflection groups [2], Grothendiek's dessins d'enfants and their deformations [1, 16, 17]. Among others we focus our attention to the relationship between algebraic solutions and Frobenius manifolds or their generalization studied in [12, 13].

At the end of 1970's, K. Saito introduced the notion of flat structure in order to study structures of the parameter spaces of the versal families of isolated hypersurface singularities [26] (see also [27]). After his pioneering work with the collaborators T. Yano and the third author of this paper, B. Dubrovin [6] unified both the flat structure formulated by K. Saito and the WDVV equation which arose from the 2D topological field theory as the Frobenius manifold structure and as an application, he derived a one-parameter family of the Painlevé VI, written by Painlevé VIμ $(\mu \in \mathbb{C})$ from the three dimensional Frobenius manifolds[2)]. Solutions to the Painevé VIμ are transcendental, but some of them become algebraic. In the formulation by B. Dubrovin, the existence of prepotentials plays a central role in the construction of Frobenius manifold structure. He obtained four non-cubic polynomial prepotentials in the three dimensional Frobenius manifolds. They are given by

$$\begin{aligned} F &= \frac{x_1x_3^2 + x_2^2x_3}{2} + \frac{x_1^2x_2^2}{4} + \frac{x_1^5}{60}, && A_3 \text{ case}, \\ F &= \frac{x_1x_3^2 + x_2^2x_3}{2} + \frac{x_1x_2^3}{6} + \frac{x_1^3x_2^2}{6} + \frac{x_1^7}{210}, && B_3 \text{ case}, \\ F &= \frac{x_1x_3^2 + x_2^2x_3}{2} + \frac{x_1^2x_2^3}{6} + \frac{x_1^5x_2^2}{20} + \frac{x_1^{11}}{3960}, && H_3 \text{ case}, \\ F &= \frac{x_1x_3^2 + x_2^2x_3}{2} + x_2^4. \end{aligned} \tag{1.2}$$

(The last one is essentially reduced to two dimensional case (cf. [6]).) It is underlined here that K. Saito already recognized the first three prepotentials when he formulated

2) For a general theory on Frobenius manifolds and WDVV equation, see, for example, Sabbah [23], Hertling [8].

the flat structure (unpublished). Dubrovin [6] constructed algebraic solutions corresponding to these prepotentials. In [7], Dubrovin and Mazzocco classified algebraic solutions to Painlevé VIμ, showing that algebraic solutions to Painlevé VIμ are in one-to-one correspondence with the regular polyhedra or star-polyhedra in the three dimensional space. As a result, they constructed five algebraic solutions to Painlevé VIμ and among them, three are done by using the polynomial prepotentials above. As to the remaining two solutions obtained by Dubrovin and Mazzocco, prepotentials are not polynomials and their explicit forms are given in the main text of this paper.

We now explain an idea how an algebraic solution of Painlevé VIμ arose from WDVV equation based on Dubrovin [6] and [13], restricting to the case $n = 3$. For this purpose, we give a brief explanation of the definition of WDVV equation and prepotentials. Let $F(x) = F(x_1, x_2, x_3)$ be a function of the following form

$$F = \frac{1}{2}x_1x_3^2 + \frac{1}{2}x_2^2x_3 + F_0(x_1, x_2).$$

Assume that $F(x)$ is weighted homogeneous. This means that there are non-zero numbers $d_1, d_2, , d_3, d$ such that $EF = dF$, where $E = \sum_{j=1}^{3} d_jx_j\partial_{x_j}$ is an Euler vector field. We also assume for simplicity that each d_j is a rational number[3)] and that[4)] $0 < d_1 < d_2 < d_3 = 1$. Using F, we define $g_j = \partial_{x_{4-j}}F$ $(j = 1, 2, 3)$ and put $P = (g_1, g_2, g_3)$. By differentiating P with respect to x_1, x_2, x_3, we obtain 3-vectors $\partial_{x_j}P$ and construct a matrix $C = (C_{ij})$ by

$$C = \begin{pmatrix} \partial_{x_1}P \\ \partial_{x_2}P \\ \partial_{x_3}P \end{pmatrix}.$$

In particular $C_{ij} = \partial_{x_i}g_j$ and $\partial_{x_3}P = (x_1, x_2, x_3)$. Moreover we put $\tilde{B}^{(k)} = \partial_{x_k}C$. It follows from the definition that $(\tilde{B}^{(k)})_{ij} = \partial_{x_k}\partial_{x_i}g_j$. As a consequence, $(\tilde{B}^{(k)})_{ij} = (\tilde{B}^{(i)})_{kj}$. It is clear from the definition that $C - x_3I_3$ is independent of x_3. This implies that $\tilde{B}^{(3)} = I_3$. We introduce the matrix $T = EC$ for later consideration.

If all the matrices $\tilde{B}^{(j)}$ $(j = 1, 2, 3)$ commute with each other, the function F is called a prepotential or free energy (cf. [6]). Since $\tilde{B}^{(3)} = I_3$, the commutativity condition is reduced to the unique equation $\tilde{B}^{(1)}\tilde{B}^{(2)} = \tilde{B}^{(2)}\tilde{B}^{(1)}$. Writing down the matrix entries of $\tilde{B}^{(1)}\tilde{B}^{(2)} = \tilde{B}^{(2)}\tilde{B}^{(1)}$, we obtain non-linear differential equations for F. The collection of such differential equations is called a WDVV equation. As a consequence, a prepotential is a solution of WDVV equation. Moreover in this case, $x = (x_1, x_2, x_3)$ is called a flat coordinate.

We assume that $F = F(x_1, x_2, x_3)$ is a prepotential and keep the notation introduced above, define a diagonal matrix $B_\infty^{(3)}$ by $B_\infty^{(3)} = \mathrm{diag}(r + d_1, r + d_2, r + d_3)$ for some constant $r \in \mathbf{C}$ and also define 3×3 matrices $B^{(j)}$ $(j = 1, 2, 3)$ by

$$B^{(j)} = -T^{-1}\tilde{B}^{(j)}B_\infty^{(3)}. \tag{1.3}$$

3) In general, we need to consider d_j a complex number in order to treat transcendental solutions. However our interest in this paper is restricted to algebraic solutions.

4) Sometimes we treat the case where $d_1 = d_2$.

Using $B^{(j)}$, we introduce a system of differential equations

$$\partial_{x_j} Y = B^{(j)} Y \quad (j = 1, 2, 3). \tag{1.4}$$

The system (1.4) is integrable. We put $T_0 = x_3 I_3 - T$. Since $T - x_3 I_3$ does not depend on x_3 and since $B^{(3)} = -T^{-1} B_\infty^{(3)}$, the differential equation

$$\partial_{x_3} Y = B^{(3)} Y \tag{1.5}$$

turns out to be

$$(x_3 I_3 - T_0)\partial_{x_3} Y = -\frac{1}{d_3} B_\infty^{(3)} Y. \tag{1.6}$$

Since (1.6) is an ordinary differential equation with respect to the variable x_3, (1.6) is called an ordinary differential equation of Okubo type. In this sense, the system (1.4) is one of generalizations of Okubo type ordinary differential equation to several variables case.

We are in a position to explain the relationship between the system of differential equations (1.4) and solutions to Painlevé VI equation. Since $h = \det T$ is a cubic polynomial of x_3, let $p_j(x')$ $(j = 1, 2, 3)$ be defined by $h(x) = \prod_{i=1}^{3}(x_3 - p_i(x'))$, where $x' = (x_1, x_2)$. We consider $B^{(3)} = -T^{-1}\tilde{B}^{(3)} B_\infty^{(3)}$ as before. It follows from the definition that if $i \neq j$, the (i, j)-entry of $hB^{(3)}$ is a linear function of x_3. Noting this, we define $p_{ij}(x')$ $(i \neq j)$ by the condition that $x_3 = p_{ij}(x')$ is the zero of the (i, j)-entry of $hB^{(3)}$. Then it is easy to show that $p_{ij} = \frac{\det(T)(T^{-1})_{ij}}{T_{ij}}\big|_{x_3=0}$. It can be shown that $w_{ij} = \frac{p_{ij}(x') - p_1(x')}{p_2(x') - p_1(x')}$ is a solution to Painlevé VI equation as a function of $t = \frac{p_3(x') - p_1(x')}{p_2(x') - p_1(x')}$.

It is underlined here that some of the arguments so far go well forgetting the existence of a WDVV equation. This leads us to the following study. We start with introducing weighted homogeneous functions $h_1(x), h_2(x), h_3(x)$ such that $Eh_j = (d_j + d_3)h_j$ $(j = 1, 2, 3)$ and that

$$h_j = \begin{cases} x_j x_3 + h_j^{(0)}(x') & (j = 1, 2), \\ \frac{1}{2} x_3^2 + h_3^{(0)}(x') & (j = 3) \end{cases} \tag{1.7}$$

with functions $h_j^{(0)}(x')$ of $x' = (x_1, x_2)$. If there is a solution $F = F(x)$ to a WDVV equation, $h_j = \partial_{x_{4-j}} F$ $(j = 1, 2, 3)$ satisfy (1.7). We return to our situation. Using $h_j(x)$ $(j = 1, 2, 3)$, we define a 3×3 matrix C such that $C_{ij} = \partial_{x_i} h_j$. It is easy to see that $C_{3,j} = x_j$ $(j = 1, 2, 3)$. We define matrices $\tilde{B}^{(j)} = \partial_{x_j} C$ $(j = 1, 2, 3)$ and $T = \sum_{j=1}^{3} d_j x_j \partial_{x_j} C = \sum_{j=1}^{3} d_j x_j \tilde{B}^{(j)}$. If $\tilde{B}^{(1)}\tilde{B}^{(2)} = \tilde{B}^{(2)}\tilde{B}^{(1)}$, then $\vec{h} = (h_1, h_2, h_3)$ is called a potential vector field and (x_1, x_2, x_3) is a flat coordinate. Similar to the case of prepotentials, a potential vector field is a solution to a certain system of non-linear differential equations arising from the commutativity of matrices $\tilde{B}^{(j)}$ $(j = 1, 2, 3)$. In this sense, $\tilde{B}^{(1)}\tilde{B}^{(2)} = \tilde{B}^{(2)}\tilde{B}^{(1)}$ is called an extended WDVV equation for $\vec{h}$. It is possible to construct a flat structure from the existence of a potential vector field $\vec{h}$.

One of the reasons why we are led to the study on flat structures not necessary having a prepotential is to understand the relationship between a special kind of holonomic systems of three variables with singularities along free divisors and algebraic solutions to Painlevé VI equation. In this direction, systems of differential equations of Okubo type in several variables play an important role in attaching a flat structure to an algebraic solution to Painlevé VI equation. A polynomial potential vector field means a potential vector field $P = (h_1, h_2, h_3)$ whose entries h_1, h_2, h_3 are polynomials of a flat coordinate. An algebraic potential vector field means a potential vector field $P = (h_1, h_2, h_3)$ such that h_j $(j = 1, 2, 3)$ are algebraic functions of a flat coordinate. The following problem is also basic.

Problem A. Classify all the polynomial and algebraic potential vector fields.

It is not clear whether there is an algebraic potential vector field corresponding to any algebraic solution to Painlevé VI or not. Then Problem A would imply a new answer to the classification of algebraic solutions to Painlevé VI equations. This suggests that problem A is not easy to solve. An easier but still interesting problem is

Problem B. For a given algebraic solution to Painlevé VI equation, construct a potential vector field and its associated flat coordinate.

The main objective in this paper is Problem B. We will construct potential vector fields corresponding to many algebraic solutions obtained by the authors cited above. The methods employed in this paper are not systematic. We will explain one of ideas how we get potential vector fields from algebraic solutions to Painlevé VI equation. In spite that our purpose is to construct potential vector fields for all the forty five equivalent classes of algebraic solutions to Painlevé VI equation given in [20], there are many obstructions to accomplish it. We think that this reflects the difficulty of the classification of algebraic solutions. We start with a pair (w, t), where t is a variable and $w = w(t)$ is an algebraic solution to Painlevé VI equation. Let $w = w(s)$, $t = t(s)$ be their parametric representations. Our first job is to construct a free divisor in the three dimensional affine space which connects with $t = t(s)$. The second job is to construct systems of differential equations of rank two whose singularities are contained in the free divisor. Applying the argument in [13, Appendix B] to the system of rank two, we can construct an Okubo type system in several variables of three variables. Then a potential vector field is deduced in a natural manner. The details are explained in §2 by taking an example of algebraic solutions. It is underlined here that the first author (M. K.) found a systematic method to construct a potential vector field from an algebraic solution available at least to genus zero and genus one algebraic solutions. We show prepotentials for algebraic solutions obtained by Dubrovin and Mazzocco [7] in §3. The main purpose of this paper is to construct polynomial and algebraic potential vector fields corresponding to algebraic solutions to Painlevé VI equation. We obtained more than thirty such potential vector fields and the results are collected in §4.

2. A METHOD OF CONSTRUCTING A POTENTIAL VECTOR FIELD FROM AN ALGEBRAIC SOLUTION TO PAINLEVÉ VI EQUATION

In this section, we explain a method to construct potential vector fields from algebraic solutions to Painlevé VI equation. Before entering into the main text, we explain other ideas to construct them. As was explained in §1, Dubrovin [6] constructed polynomial prepotentials by solving WDVV equation in three variables directly. Similarly, it is possible to find polynomial potential vector fields by solving extended WDVV equations directly. It seems difficult to treat non polynomial case by this idea. The other one is to use Belyi maps introduced by A. V. Kitaev (cf. [16]). Since it is a little complicated to explain this idea, we don't discuss its details here.

We treat here solution 24 in [3] (same as Solution 11 in [20]) defined by

$$w = \frac{s(s+4)(3s^4-2s^3-2s^2+8s+8)}{8(s-1)(s+1)^2(s^2+4)}, \quad t = \frac{s^5(s+4)^3}{4(s-1)(s+1)^3(s^2+4)^2}. \tag{2.1}$$

Its parameter is $(\frac{1}{2}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5})$.

2.1. DETERMINATION OF A FREE DIVISOR IN THIS CASE

In this subsection, the notion of free divisor is freely used. The readers who want to know its definition and basic properties, see K. Saito [25].

We start with finding $u = \dfrac{as+b}{cs+d}$ and a polynomial $P(u)$ such that $\dfrac{P(-u)}{P(u)}$ coincides with one of $t, 1-t, \dfrac{1}{t}, \dfrac{1}{1-t}, \dfrac{t}{t-1}, \dfrac{t-1}{t}$, where t is given in (2.1). In this case, an easy computation shows that

$$u = \frac{s+4}{s}, \quad P(u) = (u-5)(u+3)^3(u^2-2u+5)^2$$

are required ones. In fact, it follows that

$$1-t = \frac{(-2+s)^3(2+3s)(2+2s+s^2)^2}{4(-1+s)(1+s)^3(4+s^2)^2} = \frac{P(-u)}{P(u)}.$$

Using $P(u)$, we define a weighted homogeneous polynomial $f_0(x_1, x_2, x_3)$ by

$$f_0(x_1, x_2, x_3) = x_3(x_3 - x_1^8 P(\sqrt{x_2}/x_1))(x_3 - x_1^8 P(-\sqrt{x_2}/x_1)).$$

From the definition,

$$f_0 = x_3(x_3^2 + Q_1(x_1, x_2)x_3 + Q_2(x_1, x_2)),$$

where

$$\begin{aligned} Q_1 &= -2(3375x_1^8 + 180x_1^6x_2 + 10x_1^4x_2^2 + 20x_1^2x_2^3 - x_2^4), \\ Q_2 &= 11390625x_1^{16} + 1215000x_1^{14}x_2 + 99900x_1^{12}x_2^2 - 123544x_1^{10}x_2^3 + 550x_1^8x_2^4 + 40x_1^6x_2^5 \\ &\quad + 380x_1^4x_2^6 - 40x_1^2x_2^7 + x_2^8. \end{aligned}$$

Since $x_3 = 0$ satisfies $f_0 = 0$, we put $z_1 = 0$. Let $z_2 = z_2(x_1, x_2)$ be an algebraic function of x_1, x_2 such that $x_3 = z_2$ is a non-trivial solution of $f_0 = 0$ as an equation of x_3. Putting

$$z_3 = -6750x_1^8 - 360x_1^6x_2 - 20x_1^4x_2^2 - 40x_1^2x_2^3 + 2x_2^4 - z_2, \tag{2.2}$$

we find that $f_0 = (x_3 - z_1)(x_3 - z_2)(x_3 - z_3)$ and

$$z_2^2 + Q_1z_2 + Q_2 = 0. \tag{2.3}$$

We are going to show that $f_0 = 0$ is a free divisor. For this purpose, we define a matrix M by

$$M = \begin{pmatrix} x_1 & 2x_2 & 8x_3 \\ -\frac{x_1}{15}\begin{pmatrix} 225x_1^6 + 37x_1^4x_2 \\ -141x_1^2x_2^2 + 7x_2^3 \end{pmatrix} & \frac{1}{3}\begin{pmatrix} 10125x_1^8 + 810x_1^6x_2 \\ +68x_1^4x_2^2 + 6x_1^2x_2^3 \\ -x_2^4 + 3x_3 \end{pmatrix} & \frac{8x_3}{3}\begin{pmatrix} -45x_1^6 - 5x_1^4x_2 \\ -15x_1^2x_2^2 + x_2^3 \end{pmatrix} \\ \frac{1}{3}\begin{pmatrix} 3375x_1^8 - 90x_1^6x_2 \\ -28x_1^4x_2^2 + 74x_1^2x_2^3 \\ -3x_2^4 + 3x_3 \end{pmatrix} & \frac{20x_1x_2}{3}\begin{pmatrix} 2025x_1^6 + 141x_1^4x_2 \\ +11x_1^2x_2^2 - x_2^3 \end{pmatrix} & -\frac{80x_1x_3}{3}\begin{pmatrix} 675x_1^6 + 27x_1^4x_2 \\ +x_1^2x_2^2 + x_2^3 \end{pmatrix} \end{pmatrix}. \tag{2.4}$$

Let V_1, V_2, V_3 be vector fields defined by

$$(V_1, V_2, V_3)^t = M(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})^t.$$

Then it is easy to show that

$$\det(M) = 8f_0, \quad V_1f_0 = 24f_0, \quad V_jf_0 = 0\,(j = 2, 3),$$

which means that $f_0 = 0$ is a free divisor.

2.2. CONSTRUCTION OF HOLONOMIC SYSTEMS OF RANK TWO

Let $R = \mathbb{C}[x_1, x_2, x_3]$ be the coordinate ring of $\mathbb{C}^3$ and put $\mathcal{L} = R[V_1, V_2, V_3]$. Then it is easy to see that $\mathcal{L}$ is a Lie algebra over R. In fact we have the following relations among V_1, V_2, V_3:

$$\begin{aligned} [V_1, V_2] &= 6V_2, \\ [V_1, V_3] &= 7V_3, \\ [V_2, V_3] &= k_1(x)V_1 + k_2(x)V_2 + k_3(x)V_3, \end{aligned} \tag{2.5}$$

where k_j are weighted homogeneous polynomials contained in R.

It is possible to consider a special kind of left $\mathcal{L}$-modules defined similarly as those treated in [29]. To formulate the content precisely, we consider an unknown function $u = u(x_1, x_2, x_3)$ such that $V_1u = q_0u$ for a constant q_0 and define a left R-module $\mathcal{N}$

generated by u and V_2u, namely, $\mathcal{N} = Ru + RV_2u$. We assume that $\mathcal{L}\mathcal{N} = \mathcal{N}$. Then we have a system of differential equations

$$\begin{cases} V_1u = q_0u, \\ (V_2)^2u = q_1(x)u + q_2(x)V_2u, \\ V_3u = q_3(x)u + q_4(x)V_2u, \end{cases} \tag{2.6}$$

where $q_0 \in \mathbb{C}$, $q_i(x) \in R$ $(i = 1, 2, 3, 4)$ (cf. [29, (7)]). We assume that each $q_i(x)$ is weighted homogeneous and are going to rewrite the system (2.6) to that for a vector-valued function. As a result, we obtain a system of differential equations of the form

$$V_j\vec{u} = A_j\vec{u}\ (j = 1, 2, 3), \tag{2.7}$$

where $\vec{u} = \begin{pmatrix} u \\ V_2u \end{pmatrix}$ and

$$A_1 = \begin{pmatrix} q_0 & 0 \\ 0 & 6 + q_0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ a_{21}^{(2)} & a_{22}^{(2)} \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_{11}^{(3)} & a_{12}^{(3)} \\ a_{21}^{(3)} & a_{22}^{(3)} \end{pmatrix}. \tag{2.8}$$

Note that $a_{21}^{(2)}, a_{22}^{(2)}, a_{11}^{(3)}, a_{12}^{(3)}, a_{21}^{(3)}, a_{22}^{(3)} \in R$ and they are weighted homogeneous.

With the help of the relations (2.5), we obtain the following compatibility conditions on the matrices A_1, A_2, A_3:

$$\begin{cases} [A_1, A_2] + [V_1, A_2] - 6A_2 = O, \\ [A_1, A_3] + [V_1, A_3] - 7A_3 = O, \\ [A_2, A_3] - [V_2, A_3] + [V_3, A_2] + k_1(x)A_1 + k_2(x)A_2 + k_3(x)A_3 = O. \end{cases} \tag{2.9}$$

The first and the second relations of (2.9) imply that each matrix entry of A_2, A_3 is weighted homogeneous. On the other hand, the third one implies non-trivial equations among the matrix entries of A_2, A_3.

The relations (2.9) are rewritten in a familiar form which we are going to explain. Since $(V_1, V_2, V_3)^t = M(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})^t$, we introduce 2×2 matrices B_1, B_2, B_3 by

$$(B_1, B_2, B_3)^t = M^{-1}(A_1, A_2, A_3)^t. \tag{2.10}$$

Then f_0B_j has polynomial entries and the system (2.7) is equivalent to

$$\partial_{x_j}\vec{u} = B_j\vec{u} \quad (j = 1, 2, 3). \tag{2.11}$$

The integrability condition for (2.11) is

$$[B_i, B_j] + \frac{\partial B_i}{\partial x_j} - \frac{\partial B_j}{\partial x_i} = O \quad (\text{for all } i, j). \tag{2.12}$$

It is clear that (2.12) is also an integrability condition for $\mathcal{N}$.

If there exist three matrices A_1, A_2, A_3 of the forms (2.8) satisfying the condition (2.9), we obtain a system of the form (2.7). By direct computation, we find that there are three cases of A_1, A_2, A_3 satisfying (2.9).

The result is given as follows.

(a) The first case:
The matrix entries of the matrices A_2, A_3 are defined as follows.

$$\begin{cases} a_{21}^{(2)} = \dfrac{12}{625}(3378675x_1^{12} - 493050x_1^{10}x_2 + 338525x_1^8x_2^2 + 30420x_1^6x_2^3 \\ \qquad - 5795x_1^4x_2^4 + 390x_1^2x_2^5 - 13x_2^6 + 1000x_1^4x_3), \\ a_{22}^{(2)} = -\dfrac{18}{25}(45x_1^6 + 5x_1^4x_2 + 15x_1^2x_2^2 - x_2^3), \\ a_{11}^{(3)} = -\dfrac{192}{5}x_1^3(5x_1^4 - 12x_1^2x_2 - x_2^2), \\ a_{12}^{(3)} = -10x_1, \\ a_{21}^{(3)} = \dfrac{24x_1}{125}(4781325x_1^{12} + 2416650x_1^{10}x_2 - 218525x_1^8x_2^2 + 87180x_1^6x_2^3 \\ \qquad + 6755x_1^4x_2^4 - 630x_1^2x_2^5 + 13x_2^6 + 1400x_1^4x_3 + 400x_1^2x_2x_3), \\ a_{22}^{(3)} = -\dfrac{12x_1}{5}(1945x_1^6 + 273x_1^4x_2 + 19x_1^2x_2^2 + 3x_2^3). \end{cases} \tag{2.13}$$

(b) The second case:
The matrix entries of A_2, A_3 are defined as follows.

$$\begin{cases} a_{21}^{(2)} = \dfrac{12}{625}(5012175x_1^{12} + 1628850x_1^{10}x_2 + 433725x_1^8x_2^2 + 26220x_1^6x_2^3 \\ \qquad - 1095x_1^4x_2^4 - 510x_1^2x_2^5 + 27x_2^6 + 1500x_1^4x_3 - 500x_1^2x_2x_3), \\ a_{22}^{(2)} = -\dfrac{2}{25}(45x_1^6 + 5x_1^4x_2 + 15x_1^2x_2^2 - x_2^3), \\ a_{11}^{(3)} = \dfrac{16x_1}{5}(605x_1^6 + 141x_1^4x_2 + 23x_1^2x_2^2 - x_2^3), \\ a_{12}^{(3)} = \dfrac{10x_1}{3}, \\ a_{21}^{(3)} = \dfrac{24x_1}{125}(8996975x_1^{12} + 3718650x_1^{10}x_2 + 858325x_1^8x_2^2 + 94140x_1^6x_2^3 \\ \qquad + 4105x_1^4x_2^4 - 790x_1^2x_2^5 + 19x_2^6 + 2850x_1^4x_3 + 600x_1^2x_2x_3 - 50x_2^2x_3), \\ a_{22}^{(3)} = -\dfrac{4x_1}{5}(3095x_1^6 + 591x_1^4x_2 + 93x_1^2x_2^2 - 3x_2^3). \end{cases} \tag{2.14}$$

(c) The third case:
The matrix entries of A_2, A_3 are defined as follows.

$$\begin{cases} a_{21}^{(2)} = -\dfrac{4}{5625}(68831775x_1^{12} - 4603950x_1^{10}x_2 + 6840925x_1^8x_2^2 + 558060x_1^6x_2^3 \\ \qquad\qquad - 60535x_1^4x_2^4 - 30x_1^2x_2^5 + 91x_2^6 + 19500x_1^4x_3 - 4500x_1^2x_2x_3), \\ a_{22}^{(2)} = \dfrac{34}{75}(45x_1^6 + 5x_1^4x_2 + 15x_1^2x_2^2 - x_2^3), \\ a_{11}^{(3)} = \dfrac{16x_1}{15}(9x_1^2 - x_2)(315x_1^4 + 62x_1^2x_2 - x_2^2), \\ a_{12}^{(3)} = 30x_1, \\ a_{21}^{(3)} = -\dfrac{8x_1}{375}(36330525x_1^{12} + 16102350x_1^{10}x_2 - 617825x_1^8x_2^2 \\ \qquad + 535860x_1^6x_2^3 + 50795x_1^4x_2^4 - 4770x_1^2x_2^5 \\ \qquad + 121x_2^6 + 7350x_1^4x_3 + 2600x_1^2x_2x_3 - 150x_2^2x_3), \\ a_{22}^{(3)} = \dfrac{4x_1}{15}(135x_1^6 - 513x_1^4x_2 + 301x_1^2x_2^2 + 13x_2^3). \end{cases} \tag{2.15}$$

Remark 2.1. It is not clear in general whether for a given free divisor arose from an algebraic solution to Painlevé VI equation, there exists a system of the form (2.6) or not. This is one of the reasons why the method explained in this section doesn't go well for an arbitrary algebraic solution.

2.3. CONSTRUCTION OF A 3×3 MATRIX

Our next target is to construct an Okubo type system in three variables from the system of the form (2.8) by using the argument explained in [13, Appendix B].

We now treat the case where the entries of the 2×2 matrices A_2, A_3 are defined in (2.13). Let B_1, B_2, B_3 be 2×2 matrices defined by the relation (2.10). As an analogue of (2.16) in [13, Appendix B], we construct an integrable Pfaffian system of rank 2 by using B_j $(j = 1, 2, 3)$

$$dZ = \Big(\sum_{i=1}^{3} \Gamma^{(i)} dx_i\Big) Z \tag{2.16}$$

which satisfies the following conditions:

(E1) $\Gamma^{(i)}$ $(i = 1, 2, 3)$ are 2×2 matrices whose entries are rational functions of x_1, x_2, x_3.

(E2) There are 2×2 matrices $\Gamma_j^{(i)}$ such that the matrices $\Gamma^{(i)}$ $(i = 1, 2, 3)$ take the forms

$$\Gamma^{(i)} = \sum_{j=1}^{3} \frac{\Gamma_j^{(i)}}{x_3 - z_j(x_1, x_2)}, \quad i = 1, 2, 3,$$

and that each entry of $\Gamma_j^{(i)}$ is a function of x_1, x_2 and independent of x_3. (As was explained before in this section, z_1, z_2, z_3 are defined by $f_0 = \prod_{i=1}^{3}(x_3 - z_i)$.)

(E3) $\operatorname{rank} \Gamma_j^{(3)} = 1$ $(j = 1, 2, 3)$.

(E4) $\Gamma_\infty = -\sum_{j=1}^{3}\Gamma_j^{(3)}$ is a diagonal matrix whose entries are constants.

For this purpose, we define

$$P_0 = \begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix}$$

and put

$$B_i' = (P_0 B_i + \partial_{x_i} P_0)P_0^{-1} - \frac{s_2(\partial_{x_i} f_0)}{f_0} I_2 \quad (i = 1, 2),$$
$$B_3' = (P_0 B_3 + \partial_{x_3} P_0)P_0^{-1} - \left(\frac{s_1}{x_3} + \frac{s_2(\partial_{x_3} f_0)}{f_0}\right) I_2,$$

where φ is a polynomial of x_1, x_2, x_3 and s_1, s_2 are constants and all of φ, s_1, s_2 are to be determined. Moreover we put

$$G_{i,j} = \lim_{x_3 \to z_j} (x_3 - z_j) B_i'.$$

Then φ is so chosen that

$$B_i' = \sum_{j=1}^{3} \frac{G_{i,j}}{x_3 - z_j} \quad (i = 1, 2, 3)$$

hold. In this case, by direct computation, we find that $\varphi = -\frac{32}{5} x_1^4 (7x_1^2 + 3x_2)$.

Since $G_{3,j}$ is a 2×2 matrix, $\det(G_{3,j}) = 0$ means that the rank of $G_{3,j}$ is 1. We assume that $f_0 = 0$ has no multiple root as a cubic equation with respect to x_3. On the one hand, $\det(G_{3,1}) = 0$ implies that

$$(24 + 25q_0 - 600s_1)(234 + 25q_0 - 600s_1) = 0$$

and as a consequence, we take s_1 as one of

$$s_1 = \frac{25q_0 + 24}{600}, \quad \frac{25q_0 + 234}{600}.$$

On the other hand, $\det(G_{3,j}) = 0$ $(j = 2, 3)$ imply that

$$(33 + 25q_0 - 600s_2)(63 + 25q_0 - 600s_2) = 0$$

and s_2 is one of

$$s_2 = \frac{25q_0 + 33}{600}, \quad \frac{25q_0 + 63}{600}.$$

In this manner, s_1, s_2 are determined. We treat the case

$$s_1 = \frac{25q_0 + 24}{600}, \quad s_2 = \frac{25q_0 + 63}{600}.$$

Then we put

$$\Gamma^{(i)} = B_i'|_{s_1 = \frac{25q_0+24}{600}, s_2 = \frac{25q_0+63}{600}} \quad (i = 1, 2, 3)$$

and write them of the form

$$\Gamma^{(i)} = \sum_{j=1}^{3} \frac{\Gamma_j^{(i)}}{x_3 - z_j(x)}, \quad i = 1, 2, 3.$$

Therefore, the conditions (E1)–(E4) hold for $\Gamma^{(i)}$ ($i = 1, 2, 3$). Next we define

$$\Gamma_\infty = -(\Gamma_1^{(3)} + \Gamma_2^{(3)} + \Gamma_3^{(3)}).$$

In this case $\Gamma_\infty = \mathrm{diag}[\frac{1}{4}, -\frac{1}{2}]$.

The computation beginning from now on is complicated. We are going to accomplish the computation step by step. Since the rank of $\Gamma_j^{(3)}$ is 1, there exist vectors $\vec{b}_j = (b_{1j}, b_{2j})$, $\vec{a}_j = (a_{j1}, a_{j2})$ ($j = 1, 2, 3$) satisfying the equations

$$\Gamma_j^{(3)} = -\vec{b}_j^{\,t} \cdot \vec{a}_j \Gamma_\infty \quad (j = 1, 2, 3). \tag{2.17}$$

Note that if a_{j1} is given, a_{j2}, b_{1j}, b_{2j} are determined uniquely. Then, as is explained in [13, Appendix B], it follows that

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = I_2.$$

Let (b_{31}, b_{32}, b_{33}), (a_{13}, a_{23}, a_{33}) are vectors satisfying

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = I_3.$$

This shows that if a_{13} is given, $a_{23}, a_{33}, b_{31}, b_{31}, b_{33}$ are determined uniquely. As a result, we have still undetermined constants $a_{11}, a_{21}, a_{31}, a_{13}$ and the remaining constants are done. The 3×3 matrix which we need is the one given by

$$S_0 = \sum_{j=1}^{3} \frac{1}{x_3 - z_j} \begin{pmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix} \begin{pmatrix} a_{j1} & a_{j2} & a_{j3} \end{pmatrix}.$$

An Okubo type system is defined by using S_0.

To find a potential vector field, we need $S = S_0^{-1}$ which plays the role of the matrix T in the introduction. Concretely we find that

$$S = \sum_{j=1}^{3} (x_3 - z_j) \begin{pmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix} \begin{pmatrix} a_{j1} & a_{j2} & a_{j3} \end{pmatrix}.$$

Let S_{ij} be the (i,j)-entry of S. Then concrete forms of S_{ij} are as follows:

$$S_{11} = \frac{1}{25}(78875x_1^8 - 700x_1^6x_2 - 750x_1^4x_2^2 + 420x_1^2x_2^3 - 21x_2^4 + 25x_3),$$

$$S_{12} = \frac{1}{2}(-5x_1^2 + x_2),$$

$$S_{13} = -\frac{4a_{31}(5x_1^2 - x_2)(55x_1^4 + 10x_1^2x_2 - x_2^2)(525x_1^6 + 1125x_1^4x_2 + 15x_1^2x_2^2 - x_2^3)}{25a_{11}(105x_1^4 - 10x_1^2x_2 + x_2^2)},$$

$$S_{21} = -\frac{4}{625}(33996875x_1^{14} + 704375x_1^{12}x_2 - 8388625x_1^{10}x_2^2 + 1022875x_1^8x_2^3 - 61775x_1^6x_2^4 - 12075x_1^4x_2^5 + 1365x_1^2x_2^6 - 39x_2^7),$$

$$S_{22} = \frac{1}{50}(45125x_1^8 - 2500x_1^6x_2 - 850x_1^4x_2^2 + 220x_1^2x_2^3 - 11x_2^4 + 50x_3),$$

$$S_{23} = -4a_{31}(5x_1^2 - x_2)(55x_1^4 + 10x_1^2x_2 - x_2^2)(12980625x_1^{12} - 1500750x_1^{10}x_2 + 342375x_1^8x_2^2 + 220700x_1^6x_2^3 - 18225x_1^4x_2^4 + 1170x_1^2x_2^5 - 39x_2^6) /625a_{11}(105x_1^4 - 10x_1^2x_2 + x_2^2),$$

$$S_{31} = -a_{11}(105x_1^4 - 10x_1^2x_2 + x_2^2)(89375x_1^{10} - 9375x_1^8x_2 + 51750x_1^6x_2^2 - 750x_1^4x_2^3 + 75x_1^2x_2^4 - 3x_2^5)/\{50a_{31}(5x_1^2 - x_2)(55x_1^4 + 10x_1^2x_2 - x_2^2)\},$$

$$S_{32} = -\frac{a_{11}(325x_1^4 + 30x_1^2x_2 - 3x_2^2)(105x_1^4 - 10x_1^2x_2 + x_2^2)}{16a_{31}(5x_1^2 - x_2)(55x_1^4 + 10x_1^2x_2 - x_2^2)},$$

$$S_{33} = \frac{1}{50}(134625x_1^8 + 21900x_1^6x_2 + 3350x_1^4x_2^2 + 940x_1^2x_2^3 - 47x_2^4 + 50x_3).$$

It is noted here that rational functions S_{ij} depend only on the ratio a_{31}/a_{11}. To simplify the computation, we introduce the function $\psi = \psi(x_1, x_2)$ by the relation

$$a_{31} = a_{11}\psi\frac{105x_1^4 - 10x_1^2x_2 + x_2^2}{(5x_1^2 - x_2)(55x_1^4 + 10x_1^2x_2 - x_2^2)}.$$

Then

$$S_{13} = -\frac{4}{25}\psi(525x_1^6 + 1125x_1^4x_2 + 15x_1^2x_2^2 - x_2^3),$$

$$S_{23} = -\frac{4}{625}\psi(12980625x_1^{12} - 1500750x_1^{10}x_2 + 342375x_1^8x_2^2 + 220700x_1^6x_2^3 - 18225x_1^4x_2^4 + 1170x_1^2x_2^5 - 39x_2^6),$$

$$S_{31} = -\frac{(89375x_1^{10} - 9375x_1^8x_2 + 51750x_1^6x_2^2 - 750x_1^4x_2^3 + 75x_1^2x_2^4 - 3x_2^5)}{50\psi},$$

$$S_{32} = -\frac{(325x_1^4 + 30x_1^2x_2 - 3x_2^2)}{16\psi}.$$

We continue the computation. Define

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and a matrix $S' = g(S^t)g$. Let $E = \frac{1}{8}(x_1\partial_{x_1} + 2x_2\partial_{x_2} + 8x_3\partial_{x_3})$ be an Euler operator and assume that each matrix entries of S' is weighted homogeneous with respect to E. That is, if S'_{ij} is the (i,j)-entry of S', then $ES'_{ij} = \frac{1}{\gamma_{ij}} \cdot S'_{ij}$ for some non-zero constant γ_{ij}. It follows from the definition that $\gamma_{11} = \gamma_{22} = \gamma_{33} = 1$, $\gamma_{31} = 4$, $\gamma_{13} = \frac{4}{7}$ and that $\psi(x_1, x_2)$ is also weighted homogeneous.

Moreover we introduce a 3×3 matrix $S'' = (S''_{ij})$ such that $S''_{ij} = \gamma_{ij} S'_{ij}$. This implies that $ES'' = S'$. Then the extended WDVV equation in this case is

$$[\partial_{x_1} S'', \partial_{x_2} S''] = O. \tag{2.18}$$

We are going to solve (2.18), or equivalently, to determine the constants γ_{ij} and the function $\psi(x_1, x_2)$ by (2.18). Then we obtain

$$\begin{cases} \gamma_{21} = \frac{16}{15\gamma_{12}},\ \gamma_{23} = \frac{8}{15\gamma_{12}},\ \gamma_{32} = \frac{5\gamma_{12}}{2}, \\ \partial_{x_1}\psi = \partial_{x_2}\psi = 0. \end{cases} \tag{2.19}$$

The latter implies that ψ is a constant. Then from the definition of S_{13}, S_{23}, S_{31}, S_{32}, we find that

$$\gamma_{21} = \frac{4}{3},\ \gamma_{23} = \frac{2}{3},\ \gamma_{12} = \frac{4}{5},\ \gamma_{32} = 2,$$

which implies a solution to (2.18). Let C be the matrix obtained from S'', by the substitution $\gamma_{21} = \frac{4}{3}$, $\gamma_{23} = \frac{2}{3}$, $\gamma_{12} = \frac{4}{5}$, $\gamma_{32} = 2$ and ψ is a constant. As a result, we find that

$$\begin{aligned} &C_{11} = S_{11}, && C_{12} = \frac{4}{5}S_{31}, && C_{13} = \frac{4}{7}S_{21}, \\ &C_{21} = \frac{4}{3}S_{13}, && C_{22} = S_{33}, && C_{23} = \frac{2}{3}S_{23}, \\ &C_{31} = 4S_{12}, && C_{32} = 2S_{32}, && C_{33} = S_{22}. \end{aligned}$$

2.4. A FLAT COORDINATE AND A POTENTIAL VECTOR FIELD CORRESPONDING TO C

Put

$$\begin{aligned} y_1 &= C_{31} = 2(x_2 - 5x_1^2), \\ y_2 &= C_{32} = -4(325x_1^4 + 30x_1^2x_2 - 3x_2^2), \\ y_3 &= C_{33} = \frac{1}{50}(45125x_1^8 - 2500x_1^6x_2 - 850x_1^4x_2^2 + 220x_1^2x_2^3 - 11x_2^4 + 50x_3), \end{aligned} \tag{2.20}$$

where we take $\psi = \frac{1}{32}$. Then $y = (y_1, y_2, y_3)$ is a flat coordinate in this case and $d(y_1) = \frac{1}{4}$, $d(y_2) = \frac{1}{2}$, $d(y_3) = 1$. There is a relation among y_1, y_2, x_1; $3y_1^2 - y_2 - 1600x_1^4 = 0$. For this reason, we put $z = 40x_1^2$ and regard z as an algebraic function of y_1, y_2 defined by

$$3y_1^2 - y_2 - z^2 = 0. \tag{2.21}$$

It follows from the definition that each matrix entry of C is a polynomial of y_1, z, y_3 or an algebraic function of y_1, y_2, y_3 and that $\partial_{y_1}C$, $\partial_{y_2}C$, $\partial_{y_3}C$ commute each other and in particular, $\partial_{y_3}C = I_3$.

We are going to determine a potential vector field (h_1, h_2, h_3), where

$$\begin{aligned} h_1 &= y_1 y_3 + h_1^{(0)}, \\ h_2 &= y_2 y_3 + h_2^{(0)}, \\ h_3 &= \frac{1}{2} y_3^2 + h_3^{(0)}. \end{aligned} \tag{2.22}$$

We may assume that each $h_j^{(0)}$ is a function of y_1, z. The definition of a potential vector field shows

$$\begin{pmatrix} \partial_{y_1} h_1 & \partial_{y_1} h_2 & \partial_{y_1} h_3 \\ \partial_{y_2} h_1 & \partial_{y_2} h_2 & \partial_{y_2} h_3 \\ \partial_{y_3} h_1 & \partial_{y_3} h_2 & \partial_{y_3} h_3 \end{pmatrix} = C. \tag{2.23}$$

Since $d(h_1) = \frac{5}{4}$, it follows that

$$\frac{5}{4} h_1 = \frac{1}{4}(y_1 \partial_{y_1} + 2y_2 \partial_{y_2} + 4y_3 \partial_{y_3}) h_1 = \frac{1}{4}(y_1 C_{11} + 2y_2 C_{21} + 4y_3 C_{31}),$$

which implies that

$$h_1 = \frac{1}{5}(y_1 S_{11} + \frac{8}{3} y_2 S_{13} + 4y_3(4S_{12})).$$

As a consequence, we have

$$h_1 = y_1 y_3 - \frac{27 y_1^5}{4000} - \frac{y_1^3 z^2}{1200} + \frac{y_1 z^4}{800} + \frac{z^5}{3750}.$$

The determination of h_2, h_3 is accomplished by an argument similar to the case h_1 and the result is as follows:

$$h_2 = y_2 y_3 - \frac{37 y_1^6}{1000} + \frac{9 y_1^4 z^2}{200} - \frac{3 y_1^2 z^4}{200} - \frac{4 y_1 z^5}{625} - \frac{z^6}{1000},$$

$$h_3 = \frac{y_3^2}{2} + \frac{897 y_1^8}{4480000} - \frac{13 y_1^6 z^2}{160000} - \frac{3 y_1^4 z^4}{320000} + \frac{y_1^3 z^5}{187500} + \frac{7 y_1^2 z^6}{800000} + \frac{y_1 z^7}{437500} + \frac{13 z^8}{48000000}.$$

2.5. OKUBO TYPE SYSTEM OF DIFFERENTIAL EQUATIONS ASSOCIATED TO THE ALGEBRAIC SOLUTION

We are going to construct an Okubo type system from now on starting from the potential vector field (h_1, h_2, h_3) determined in subsection 2.4. The matrix C is defined by (2.23). We put

$$E = \frac{1}{4}(y_1 \partial_{y_1} + 2y_2 \partial_{y_2} + 4y_3 \partial_{y_3})$$

and define $T = EC$. Moreover, we put

$$B_\infty^{(3)} = \operatorname{diag}\left[-r + \frac{2d_1 - d_2 - d_3}{3}, -r + \frac{-d_1 + 2d_2 - d_3}{3}, -r + \frac{-d_1 - d_2 + 2d_3}{3} \right],$$

where $d_1 = \frac{1}{4}$, $d_2 = \frac{1}{2}$, $d_3 = 1$. Then $B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{1}{3}, -r - \frac{-1}{12}, -r + \frac{5}{12}\right]$ in this case. On the other hand we define

$$\tilde{B}^{(i)} = \partial_{y_i} C \ (i = 1, 2, 3).$$

Then as is remarked before, we find that $\tilde{B}^{(i)}$ $(i = 1, 2, 3)$ commute each other and $\tilde{B}^{(3)} = I_3$. Using these matrix we define a system of differential equations for Y:

$$T(dY) = -\Big(\sum_{i=1}^{3} \tilde{B}^{(i)} dy_i\Big) B_\infty^{(3)} Y.$$

This system is called an Okubo type system of differential equations.

2.6. ALGEBRAIC SOLUTION OBTAINED FROM T

By direct computation, $\det(T)$ coincides with

$$\begin{aligned} f_1 = {} & (55y_1^4 + 4000y_3 - 10y_1^2z^2 - z^4) \\ & \times(38025y_1^8 - 1560000y_1^4y_3 + 16000000y_3^2 - 35100y_1^6z^2 \\ & \quad + 720000y_1^2y_3z^2 - 15600y_1^5z^3 + 320000y_1y_3z^3 + 4590y_1^4z^4 \\ & \quad + 72000y_3z^4 + 2080y_1^3z^5 - 620y_1^2z^6 - 240y_1z^7 + z^8) \end{aligned}$$

up to a non-zero constant factor. Introducing an algebraic function q_2 defined, by the equation

$$q_2^2 - 20y_1z - 5z^2 = 0,$$

we find that solutions of $f_1 = 0$ as a cubic equation of y_3 are p_1, p_2, p_3 defined by

$$\begin{aligned} p_1 &= (-55y_1^4 + 10y_1^2z^2 + z^4)/4000, \\ p_2 &= (195y_1^4 - 90y_1^2z^2 - 40y_1z^3 - 9z^4 - 4q_2z^2(4y_1 + z))/4000, \\ p_3 &= (195y_1^4 - 90y_1^2z^2 - 40y_1z^3 - 9z^4 + 4q_2z^2(4y_1 + z))/4000. \end{aligned}$$

Define

$$p_{3,1} = \left.\frac{\det(T)(T^{-1})_{3,1}}{T_{3,1}}\right|_{y_3=0}$$

and

$$t = \frac{p_3 - p_1}{p_2 - p_1}, \quad w_{3,1} = \frac{p_{3,1} - p_1}{p_2 - p_1}.$$

Then $t, w_{3,1}$ are functions of y_1, z. By the relation $q_2^2 - 20y_1z - 5z^2 = 0$, they are also regarded as functions of q_2, z and

$$\begin{aligned} t &= \frac{(q_2 - 3z)^3(q_2 + 5z)(q_2^2 + 2q_2z + 5z^2)^2}{(q_2 - 5z)(q_2 + 3z)^3(q_2^2 - 2q_2z + 5z^2)^2}, \\ w_{3,1} &= \frac{(-q_2 + 3z)(q_2 + 5z)(q_2^2 + 3z^2)(q_2^2 + 2q_2z + 5z^2)}{(q_2 + 3z)^2(-q_2^2 + 5z^2)(q_2^2 - 2q_2z + 5z^2)}. \end{aligned} \tag{2.24}$$

At last putting $q_2 = -zs$, we conclude that

$$t = \frac{(-5+s)(3+s)^3(5-2s+s^2)^2}{(-3+s)^3(5+s)(5+2s+s^2)^2}, \quad w_{3,1} = \frac{(-5+s)(3+s)(3+s^2)(5-2s+s^2)}{(-3+s)^2(-5+s^2)(5+2s+s^2)}.$$

It will be shown in §4.11 that $(t, w_{3,1})$ is an algebraic solution to Painlevé VI equation which is equivalent to (2.1).

3. PREPOTENTIALS CORRESPONDING TO ALGEBRAIC SOLUTIONS OBTAINED BY DUBROVIN AND MAZZOCCO

Since we discussed in [14] algebraic solutions to Painlevé VI equation constructed by Dubrovin and Mazzocco [7], we only give prepotentials corresponding to such solutions.

Icosahedral solution (H_3)

In this case,

$$d(y_1) = \frac{1}{5},\ d(y_2) = \frac{3}{5},\ d(y_3) = 1$$

and there is a prepotential defined by

$$F = \frac{y_2^2 y_3 + y_1 y_3^2}{2} + \frac{y_1^{11}}{3960} + \frac{y_1^5 y_2^2}{20} + \frac{y_1^2 y_2^3}{6}.$$

This prepotential is obtained by Dubrovin [6].

Great icosahedral solution $(H_3)'$

Let (y_1, y_2, y_3) be a flat coordinate and their weights are given by

$$d(y_1) = \frac{3}{5},\ d(y_2) = \frac{4}{5},\ d(y_3) = 1.$$

We introduce an algebraic function z of y_1, y_2 defined by the relation

$$y_2 + y_1 z + z^4 = 0.$$

It is clear from the definition that $d(z) = \frac{1}{5}$. In this case, we consider the algebraic function of (y_1, y_2, y_3) defined by

$$F = \frac{y_2^2 y_3 + y_1 y_3^2}{2} - \frac{y_1^4 z}{18} - \frac{7y_1^3 z^4}{72} - \frac{17 y_1^2 z^7}{105} - \frac{2y_1 z^{10}}{9} - \frac{64 z^{13}}{585}.$$

It is provable that F is a solution of a WDVV equation. Indeed, we first define

$$C = \begin{pmatrix} \partial_{y_1}\partial_{y_3} F & \partial_{y_1}\partial_{y_2} F & \partial_{y_1}^2 F \\ \partial_{y_2}\partial_{y_3} F & \partial_{y_2}^2 F & \partial_{y_2}\partial_{y_1} F \\ \partial_{y_3}^2 F & \partial_{y_3}\partial_{y_2} F & \partial_{y_3}\partial_{y_1} F \end{pmatrix}.$$

Then $\partial_{y_i} C$ $(i = 1, 2, 3)$ commute each other. This condition is equivalent to that F is a solution to a WDVV equation.

Great dodecahedron solution $(H_3)''$

Let (y_1, y_2, y_3) be a flat coordinate and their weights are given by

$$d(y_1) = \frac{1}{3},\ d(y_2) = \frac{2}{3},\ d(y_3) = 1.$$

We introduce an algebraic function z of y_1, y_2 defined by the relation

$$-y_1^2 + y_2 + z^2 = 0.$$

It is clear from the definition that $d(z) = \frac{1}{3}$. In this case, we consider the algebraic function of (y_1, y_2, y_3) defined by

$$F = \frac{y_2^2 y_3 + y_1 y_3^2}{2} + \frac{4063 y_1^7}{1701} + \frac{19 y_1^5 z^2}{135} - \frac{73 y_1^3 z^4}{27} + \frac{11 y_1 z^6}{9} - \frac{16 z^7}{35}.$$

Then F is also a solution to a WDVV equation.

4. POTENTIAL VECTOR FIELDS CORRESPONDING TO ALGEBRAIC SOLUTIONS TO PAINLEVÉ VI EQUATION

It is interesting to construct potential vector fields corresponding to all the algebraic solutions to Painlevé VI equation. The purpose of this section is to report on the determination of potential vector fields corresponding to some of algebraic solutions. Before showing the results, we prepare some notation. Let (y_1, y_2, y_3) be a coordinate system with weights $d(y_1), d(y_2), d(y_3)$. For simplicity we put $d_j = d(y_j)$ $(j = 1, 2, 3)$. We always assume that $0 < d_1 \le d_2 < d_3 = 1$. If z is an algebraic function of y_1, y_2 defined by the algebraic equation $\varphi(y_1, y_2, z) = 0$, where $\varphi(u_1, u_2, u_3)$ is a weighted homogeneous polynomial, then $d(z)$ is the weight of z determined by the equation. Below we define $h_j(y_1, y_2, y_3)$ $(j = 1, 2, 3)$ are polynomials (or algebraic functions) and $P = (h_1, h_2, h_3)$ is a potential vector field. Let $E = \sum_{j=1}^{3} d_j y_j \partial_{y_j}$ be an Euler vector field. The matrices C, T is defined by $C = \begin{pmatrix} \partial_{y_1} P \\ \partial_{y_2} P \\ \partial_{y_3} P \end{pmatrix}$, $T = EC$. It is clear that $\partial_{y_3} C = I_3$ and the condition that P is a potential vector field implies that $(\partial_{y_1} C)(\partial_{y_2} C) = (\partial_{y_2} C)(\partial_{y_1} C)$. The diagonal matrix $B_\infty^{(3)}$ is defined by

$$B_\infty^{(3)} = \mathrm{diag}\left[-r + \frac{2d_1 - d_2 - d_3}{3}, -r + \frac{-d_1 + 2d_2 - d_3}{3}, -r + \frac{-d_1 - d_2 + 2d_3}{3} \right], \tag{4.1}$$

where $r \in \mathbf{C}$.

We define

$$h(y_1, y_2, y_3) = \det(T). \tag{4.2}$$

From the definition, h is a cubic polynomial of y_3. Then $h = \prod_{j=1}^{3}(y_3 - p_j(y_1, y_2))$, where $p_j = p_j(y_1, y_2)$ $(j = 1, 2, 3)$ are roots of $h = 0$. It is shown by Sabbah [23] that $h = 0$ is a free divisor in a three dimensional space. We define $B^{(3)} = -T^{-1}B_{\infty}^{(3)}$. Assume that r is generic. Then it follows that if $i \neq j$, the (i, j)-entry of $hB^{(3)}$ is a linear function of y_3, that is, it takes of the form $k_{ij}(y_1, y_2)(y_3 - p_{ij}(y_1, y_2))$, where k_{ij}, p_{ij} are functions of y_1, y_2. By an easy computation, we have

$$p_{ij} = \left.\frac{h \cdot (T^{-1})_{ij}}{T_{ij}}\right|_{y_3=0}. \tag{4.3}$$

We introduce t and w_{ij} by

$$t = \frac{p_3 - p_1}{p_2 - p_1}, \quad w_{ij} = \frac{p_{ij} - p_1}{p_2 - p_1}. \tag{4.4}$$

Then $w = w_{ij}$ is an algebraic solution to Painlevé VI equation (1.1) with the variable t. Note that (1.1) has the parameter $\theta = (\theta_x, \theta_y, \theta_z, \theta_\infty)$. Bäcklund transformations map solutions of a given Painlevé VI equation to solutions of the same equation with differential parameter $\theta' = (\theta'_x, \theta'_y, \theta'_z, \theta'_\infty)$. The list of fundamental Bäcklund transformations for Painlevé VI equation is given in Table 1 (cf. [22], see also [20]):

Table 1.

	θ_x	θ_y	θ_z	θ_∞	w	t
s_x	$-\theta_x$	θ_y	θ_z	θ_∞	w	t
s_y	θ_x	$-\theta_y$	θ_z	θ_∞	w	t
s_z	θ_x	θ_y	$-\theta_z$	θ_∞	w	t
s_∞	θ_x	θ_y	θ_z	$2 - \theta_\infty$	w	t
s_δ	$\theta_x - \delta$	$\theta_y - \delta$	$\theta_z - \delta$	$\theta_\infty - \delta$	$w + \delta/p$	t
r_x	$\theta_\infty - 1$	θ_z	θ_y	$\theta_x + 1$	t/w	t
r_y	θ_z	$\theta_\infty - 1$	θ_x	$\theta_y + 1$	$(w - t)/(w - 1)$	t
r_z	θ_y	θ_x	$\theta_\infty - 1$	$\theta_z + 1$	$t(w - 1)/(w - t)$	t
P_{xy}	θ_y	θ_x	θ_z	θ_∞	$1 - w$	$1 - t$
P_{yz}	θ_x	θ_z	θ_y	θ_∞	w/t	$1/t$

Here $\delta = \frac{1}{2}(\theta_x + \theta_y + \theta_z + \theta_\infty)$ and $p = \frac{1}{2}\left\{\frac{t(t-1)w'}{w(w-1)(w-t)} - \left(\frac{\theta_x}{w} + \frac{\theta_y}{w-1} + \frac{\theta_z+1}{w-t}\right)\right\}$. We moreover need Bäcklund transformations defined by combinations of those defined above:

$$t_x = s_x s_\delta (s_y s_z s_\infty s_\delta)^2, \quad t_y = s_y s_\delta (s_x s_z s_\infty s_\delta)^2,$$
$$t_z = s_z s_\delta (s_x s_y s_\infty s_\delta)^2, \quad t_\infty = s_\infty s_\delta (s_x s_y s_z s_\delta)^2.$$

In this section, an algebraic solution LTn means "solution n" in Lisovyy-Tykhyy [20, pp. 156–162].

4.1. SOLUTION 20 OBTAINED BY P. BOALCH [3] (LT1)

In this case,

$$d(y_1) = \frac{1}{15},\ d(y_2) = \frac{1}{5},\ d(y_3) = 1,$$

$$\begin{aligned}
h_1 &= \frac{1}{13} y_1 (5y_1^{12} y_2 + 130 y_1^6 y_2^3 - 351 y_2^5 + 13 y_3),\\
h_2 &= \frac{1}{102} (5y_1^{18} - 255 y_1^{12} y_2^2 - 3825 y_1^6 y_2^4 + 459 y_2^6 + 102 y_2 y_3),\\
h_3 &= \frac{1}{58} (10 y_1^{30} + 1450 y_1^{24} y_2^2 - 8700 y_1^{18} y_2^4 + 234900 y_1^{12} y_2^6\\
&\quad + 587250 y_1^6 y_2^8 + 23490 y_2^{10} + 29 y_3^2),
\end{aligned}$$

$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{16}{45}, -r - \frac{2}{9}, -r + \frac{26}{45}\right],$$

$$\begin{aligned}
h = {} & (10 y_1^{12} y_2 - 100 y_1^6 y_2^3 - 54 y_2^5 - y_3)\\
& \times (y_1^{15} - 5 y_1^{12} y_2 + 30 y_1^9 y_2^2 + 90 y_1^6 y_2^3 + 225 y_1^3 y_2^4 + 27 y_2^5 - y_3)\\
& \times (y_1^{15} + 5 y_1^{12} y_2 + 30 y_1^9 y_2^2 - 90 y_1^6 y_2^3 + 225 y_1^3 y_2^4 - 27 y_2^5 + y_3).
\end{aligned}$$

It is easy to show that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned}
p_1 &= 10 y_1^{12} y_2 - 100 y_1^6 y_2^3 - 54 y_2^5,\\
p_2 &= 10 y_1^{12} y_2 - 100 y_1^6 y_2^3 - 54 y_2^5 + (-y_1^3 + y_2)^3 (y_1^3 + 9 y_2)^2,\\
p_3 &= 10 y_1^{12} y_2 - 100 y_1^6 y_2^3 - 54 y_2^5 + (y_1^3 + y_2)^3 (-y_1^3 + 9 y_2)^2.
\end{aligned}$$

The algebraic solution $(t, w_{3,1})$ has the parameter $\theta = (\frac{4}{15}, \frac{1}{5}, \frac{1}{5}, \frac{14}{15})$ and is uniformized by

$$t = \frac{(1+s)^3 (9-s)^2}{(1-s)^3 (9+s)^2}, \quad w_{3,1} = -\frac{4(1+s)(9-s)}{(9+s)(1-s)^2},$$

where $s = \frac{y_1^3}{y_2}$. To construct an algebraic solution equivalent to Solution 1, it is better to treat another algebraic solution

$$(t_a, w_a) = \left(\frac{p_2 - p_1}{p_3 - p_1}, \frac{p_{3,2} - p_1}{p_3 - p_1}\right) = \left(\frac{(1-s)^3 (9+s)^2}{(1+s)^3 (9-s)^2}, \frac{(1-s)(9+s)(3+s^2)}{3(9-s)(1+s)^2}\right).$$

Then the substitution $s = -\frac{r+5}{r-1}$ shows $(t_a, \frac{w_a - t_a}{w_a - 1})$ coincides with the solution 20 obtained by P. Boalch.

4.2. SOLUTION BY KITAEV [16, (3.4)] (LT2)

In this case,

$$d(y_1) = \frac{1}{5},\ d(y_2) = \frac{2}{5},\ d(y_3) = 1,$$

$$
\begin{aligned}
h_1 &= \frac{1}{60}(y_1^6 + 30y_1^2y_2^2 + 20y_2^3 + 60y_1y_3),\\
h_2 &= \frac{1}{20}y_2(3y_1^5 + 10y_1^3y_2 + 10y_1y_2^2 + 20y_3),\\
h_3 &= \frac{1}{480}(y_1^{10} + 60y_1^6y_2^2 + 120y_1^4y_2^3 + 180y_1^2y_2^4 + 24y_2^5 + 240y_3^2),
\end{aligned}
$$

$$
B_\infty^{(3)} = \operatorname{diag}\left[-r - \frac{1}{3}, -r - \frac{2}{15}, -r + \frac{7}{15}\right],
$$

$$
\begin{aligned}
h = &-\frac{1}{8000}(y_1^5 + 10y_1y_2^2 - 20y_3)\\
&\times(9y_1^{10} + 60y_1^8y_2 - 220y_1^6y_2^2 - 600y_1^4y_2^3 - 60y_1^2y_2^4 - 256y_2^5\\
&\quad + 120y_1^5y_3 + 400y_1^3y_2y_3 + 1200y_1y_2^2y_3 + 400y_3^2).
\end{aligned}
$$

Introduce an algebraic function q_2 of y_1, y_2 by the relation $y_2 = (-5y_1^2 + q_2^2)/4$. Then $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$
\begin{aligned}
p_1 &= \frac{1}{160}(133y_1^5 - 50y_1^3q_2^2 + 5y_1q_2^4),\\
p_2 &= \frac{1}{160}\{(133y_1^5 - 50y_1^3q_2^2 + 5y_1q_2^4) - 4(3y_1 + q_2)^3(2y_1 - q_2)^2\},\\
p_3 &= \frac{1}{160}\{(133y_1^5 - 50y_1^3q_2^2 + 5y_1q_2^4) - 4(3y_1 - q_2)^3(2y_1 + q_2)^2\}.
\end{aligned}
$$

The algebraic solution $(t, w_{3,2})$ has the parameter $\theta = (\frac{19}{45}, \frac{13}{45}, \frac{13}{45}, \frac{3}{5})$ and is uniformized by

$$
t = \frac{(2+s)^2(3-s)^3}{(2-s)^2(3+s)^3}, \quad w_{3,2} = -\frac{3(2+s)(3-s)}{(3+s)^2(2-s)},
$$

where $s = \frac{q_2}{y_1}$. To construct an algebraic solution equivalent to the solution obtained by Kitaev, it is better to treat another algebraic solution

$$
(t_a, w_a) = \left(\frac{p_3 - p_2}{p_1 - p_2}, \frac{p_{3,2} - p_2}{p_1 - p_2}\right) = \left(-\frac{2s^3(5-s^2)}{(2-s)^2(3+s)^3}, -\frac{s^2(1+s)}{(2-s)(3+s)^2}\right).
$$

Then $\left(t_a, \frac{w_a - t_a}{w_a - 1}\right)$ coincides with the solution obtained by A. Kitaev [16, p. 195, (3.4)].

Remark 4.1. The potential vector field (h_1, h_2, h_3) is already given in [12, §7.1] and as was mentioned there the polynomial h is related with E_{13}-singularity in the sense of Arnol'd.

4.3. SOLUTION 4.1.1A BY ANDREEV AND KITAEV [1, P.162] (LT3)

In this case, z is an algebraic function of y_1, y_2 defined by the relation

$$
y_2 - z(y_1^2 + z^2) = 0.
$$

$$
d(y_1) = \frac{1}{6},\ d(y_2) = \frac{1}{2},\ d(y_3) = 1,\ d(z) = \frac{1}{6},
$$

$$h_1 = \frac{1}{105}(105y_1y_3 - 210y_1^6 z - 70y_1^4 z^3 + 378y_1^2 z^5 + 270z^7),$$
$$h_2 = \frac{1}{5}(5y_1^9 + 5y_2y_3 + 20y_1^7 z^2 + 30y_1^5 z^4 - 12y_1^3 z^6 - 27y_1 z^8),$$
$$h_3 = \frac{1}{550}(-4800y_1^{12} + 275y_3^2 + 35200y_1^{10} z^2 + 88000y_1^8 z^4 + 98560y_1^6 z^6 + 73920y_1^4 z^8 + 23936y_1^2 z^{10} - 3520z^{12}),$$
$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{7}{18}, -r - \frac{1}{18}, -r + \frac{4}{9}\right].$$

It is easy to show that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$p_1 = -\frac{16}{15}(15y_1^5 z + 20y_1^3 z^3 + 9y_1 z^5),$$
$$p_2 = \frac{4}{5}(10y_1^5 z + 20y_1^3 z^3 + 18y_1 z^5 + 5\sqrt{-3}(y_1 - z)^2(y_1 + z)^2(y_1^2 + z^2)),$$
$$p_3 = \frac{4}{5}(10y_1^5 z + 20y_1^3 z^3 + 18y_1 z^5 - 5\sqrt{-3}(y_1 - z)^2(y_1 + z)^2(y_1^2 + z^2)).$$

The algebraic solution $(t, w_{3,2})$ has the parameter $\theta = (\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2})$ and is uniformized by

$$t = -\frac{(-3+s)^3(1+s)^3}{(-1+s)^3(3+s)^3}, \quad w_{3,2} = \frac{3(-3+s)(1+s)^2}{s(-1+s)(3+s)^2}.$$

Since $\frac{t}{w_{3,2}} = -\frac{s(-3+s)^2(1+s)}{3(-1+s)^2(3+s)}$, we conclude that $(t, t/w_{3,2})$ coincides with tetrahedral solution 6 of [4] which is equivalent to the solution 4.1.1.A obtained by Andreev and Kitaev [1] (cf. [20]).

4.4. OCTAHEDRAL SOLUTION 7 BY BOALCH [4] (LT4)

In this case, z is an algebraic function of y_1, y_2 defined by

$$y_1^8 - y_2 - 7z^2 = 0,$$

$$d(y_1) = \frac{1}{24},\ d(y_2) = \frac{1}{3},\ d(y_3) = 1,\ d(z) = \frac{1}{6},$$

$$h_1 = \frac{103}{1612875}y_1^{25} + y_1y_3 - \frac{1}{14025}y_1 z^2(45y_1^{16} - 255y_1^8 z^2 + 816y_1^4 z^3 - 425z^4),$$
$$h_2 = -\frac{3}{7130}y_1^{32} + y_2y_3 + \frac{1}{2530}z^2(164y_1^{24} - 1242y_1^{16} z^2 + 2944y_1^{12} z^3 - 3036y_1^8 z^4 + 161z^6),$$
$$h_3 = \frac{5847}{150421150}y_1^{48} + \frac{1}{2}y_3^2 - \frac{1}{417450}z^2(162y_1^{40} - 435y_1^{32} z^2 - 1920y_1^{28} z^3 - 2404y_1^{24} z^4 + 17664y_1^{20} z^5 - 24771y_1^{16} z^6 + 20608y_1^{12} z^7 - 13662y_1^8 z^8 - 253z^{10}).$$

$$B_\infty^{(3)} = \mathrm{diag}\left[-r-\frac{5}{4}, -r-\frac{3}{8}, -r+\frac{13}{8}\right].$$

It is straightforward to show that

$$h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3),$$

where

$$\begin{aligned}
p_1 &= (-47y_1^{24} + 207y_1^{16}z^2 - 69y_1^8z^4 - 1104y_1^4z^5 - 115z^6)/3795,\\
p_2 &= (45y_1^{24} - 345y_1^{16}z^2 + 2576y_1^{12}z^3 - 3657y_1^8z^4 + 2208y_1^4z^5 - 115z^6)/3795,\\
p_3 &= (45y_1^{24} - 345y_1^{16}z^2 - 368y_1^{12}z^3 + 759y_1^8z^4 + 253z^6)/3795.
\end{aligned}$$

The algebraic solution $(t, w_{3,2})$ corresponding to

$$\begin{aligned}
p_{3,2} = (&45y_1^{32} + 260y_1^{24}z^2 + 2668y_1^{20}z^3 - 8694y_1^{16}z^4\\
&+ 5244y_1^{12}z^5 + 1288y_1^8z^6 - 7728y_1^4z^7 + 805z^8)/3795(y_1^8 - 7z^2)
\end{aligned}$$

is uniformized by

$$t = \frac{(-1+3u)^2(2+3u)^4}{(1+6u)(4-12u+27u^2)^2}, \quad w_{3,2} = \frac{2(-1+3u)(2+3u)(4-6u+63u^2)}{(4-12u+27u^2)(-4+63u^2)},$$

where $u = \frac{2z}{3y_1^4}$. Substituting $u = -\frac{2(1+s)}{3(-2+s)}$, we find that $P_{yz}P_{xy}P_{yz}$ transforms $(t, w_{3,2})$ to octahedral solution 7 obtained by Boalch [4].

4.5. SOLUTION BY KITAEV [16, P.191] (LT5)

$$d(y_1) = \frac{1}{12}, \quad d(y_2) = \frac{1}{3}, \quad d(y_3) = 1,$$

$$\begin{aligned}
h_1 &= \frac{1}{78}y_1(40y_1^{12} + 858y_1^4y_2^2 - 715y_2^3 + 78y_3),\\
h_2 &= \frac{1}{12}y_2(128y_1^{12} - 528y_1^8y_2 + 528y_1^4y_2^2 - 11y_2^3 + 12y_3),\\
h_3 &= \frac{1}{184}(2048y_1^{24} + 97152y_1^{16}y_2^2 - 259072y_1^{12}y_2^3 + 400752y_1^8y_2^4\\
&\quad - 133584y_1^4y_2^5 + 2783y_2^6 + 92y_3^2),
\end{aligned}$$

$$B_\infty^{(3)} = \mathrm{diag}\left[-r-\frac{7}{18}, -r-\frac{5}{36}, -r+\frac{19}{36}\right].$$

Introduce an algebraic function q_2 of y_1, y_2 by the relation $y_2 = y_1^4 - 3q_2^2$. Then $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned}
p_1 &= \frac{9}{2}(7y_1^{12} - 33y_1^8q_2^2 + 33y_1^4q_2^4 + 33q_2^6),\\
p_2 &= \frac{9}{2}(7y_1^{12} - 33y_1^8q_2^2 + 33y_1^4q_2^4 + 33q_2^6) - 99(y_1^2 + 2q_2)(y_1^2 - q_2)^4,\\
p_3 &= \frac{9}{2}(7y_1^{12} - 33y_1^8q_2^2 + 33y_1^4q_2^4 + 33q_2^6) - 99(y_1^2 - 2q_2)(y_1^2 + q_2)^4.
\end{aligned}$$

The algebraic solution $(t, w_{3,1})$ has the parameter $\theta = (-\frac{5}{12}, 0, 0, \frac{11}{12})$ and is uniformized by

$$t = \frac{(1+2s)^2(1-s)^4}{(1-2s)^2(1+s)^4}, \quad w_{3,1} = \frac{(1-5s^2)(1+2s)(1-s)}{(1-2s)(1+s)^3},$$

where $s = \frac{y_2}{y_1^2}$. It is straightforward to show that

$$P_{yz}s_x s_y s_z s_\infty s_\delta r_y s_y P_{xy} s_\delta s_x \theta = (1/4, 1/4, 1/3, 1/3).$$

By the Bäcklund transformation corresponding to $P_{yz}s_x s_y s_z s_\infty s_\delta r_y s_y P_{xy} s_\delta s_x$, the solution $(t, w_{3,1})$ turns out to be the solution obtained by A. Kitaev [16, p.191] (cf. [20]) up to a coordinate change.

4.6. SOLUTION 23 OBTAINED BY P. BOALCH [3] (LT6)

In this case,

$$d(y_1) = \frac{1}{30}, \quad d(y_2) = \frac{11}{30}, \quad d(y_3) = 1,$$

$$\begin{aligned}
h_1 &= y_1 y_3 - (49837 y_1^{33} + 846858 y_1^{22} y_2 + 3484524 y_1^{11} y_2^2 + 1093184 y_2^3)/(12276 y_1^2),\\
h_2 &= y_2 y_3 - (193401 y_1^{44} + 2414900 y_1^{33} y_2 - 399504 y_1^{22} y_2^2\\
&\qquad - 40121616 y_1^{11} y_2^3 - 2891648 y_2^4)/(64944 y_1^3),\\
h_3 &= \frac{1}{2} y_3^2 + (88496737 y_1^{60} + 2058469880 y_1^{49} y_2 + 16782651480 y_1^{38} y_2^2\\
&\qquad + 38703915040 y_1^{27} y_2^3 + 69160946960 y_1^{16} y_2^4 - 96297379584 y_1^5 y_2^5)/81774,
\end{aligned}$$

$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{13}{30}, -r - \frac{1}{30}, -r + \frac{8}{5}\right].$$

Introduce an algebraic function z of y_1, y_2 by the relation

$$y_2 - \frac{1}{12} x_1^{11} + \frac{12}{55} x_1^3 z^2 = 0. \tag{4.5}$$

Then $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned}
p_1 &= y_1^6 (3290554368 z^6 + 103686739200 z^4 y_1^8\\
&\qquad - 458556120000 z^2 y_1^{16} + 337703815625 y_1^{24})/889440750,\\
p_2 &= y_1^{10} (3290554368 z^5 - 3142022400 z^4 y_1^4\\
&\qquad + 6947820000 z^2 y_1^{12} - 3522990625 y_1^{20})/71874000,\\
p_3 &= y_1^{10} (- 3290554368 z^5 - 3142022400 z^4 y_1^4\\
&\qquad + 6947820000 z^2 y_1^{12} - 3522990625 y_1^{20})/71874000.
\end{aligned}$$

The algebraic solution $(t, w_{3,1})$ has the parameter $\theta = (-\frac{1}{6}, \frac{7}{30}, \frac{7}{30}, \frac{31}{30})$ and is uniformized by

$$t = \frac{(4+s)^2(1-s)^3(5-s)}{(4-s)^2(1+s)^3(5+s)}, \quad w_{3,1} = -\frac{(4+s)(1-s)(5-s)}{(1+s)^2(4-s)},$$

where $s = -\frac{55y_1^4}{12z}$. It is straightforward to show that

$$P_{xy}s_\delta(s_xs_ys_zs_\delta)^4r_x\theta = (2/5, 1/5, 2/5, 2/3).$$

By direct computation, we find that $P_{xy}s_\delta(s_xs_ys_zs_\delta)^4r_x$ transforms the solution $(t, w_{3,1})$ to solution 23 obtained by P. Boalch [3].

Remark 4.2. In spite that in this case, since h_1, h_2 are rational functions of y_1, (h_1, h_2, h_3) is not a polynomial potential vector field, these two are polynomials of y_1, z, y_3, where z is the function defined by (4.5). In this sense, (h_1, h_2, h_3) is regarded as an algebraic potential vector field.

4.7. SOLUTION 22 OBTAINED BY P. BOALCH [3] (LT7)

In this case, z is an algebraic function of y_1, y_2 defined by the relation

$$y_1^5 + y_2z^3 + z^5 = 0.$$

$$d(y_1) = \frac{1}{6},\ d(y_2) = \frac{1}{3},\ d(y_3) = 1,\ d(z) = \frac{1}{6},$$

$$\begin{aligned}
h_1 &= y_1(26y_1^{15} - 57y_1^{10}z^5 + 6y_3z^9 - 192y_1^5z^{10} + 16z^{15})/(6z^9),\\
h_2 &= (-7y_1^{20} + 1322y_1^{15}z^5 - 42y_1^{10}z^{10} + 14y_2y_3z^{12} + 672y_1^5z^{15} + 168z^{20})/(14z^{12}),\\
h_3 &= (91y_1^{30} - 19704y_1^{25}z^5 + 112740y_1^{20}z^{10} + 252320y_1^{15}z^{15} + 15y_3^2z^{18} + 360240y_1^{10}z^{20}\\
&\quad + 7296y_1^5z^{25} + 1216z^{30})/(30z^{18}).
\end{aligned}$$

$$B_\infty^{(3)} = \operatorname{diag}\left[-r - \frac{1}{3}, -r - \frac{1}{6}, -r + \frac{1}{2}\right].$$

Introduce an algebraic function q_2 of y_2, z defined by the relation $3y_2 - 2q_2^2 + 5z^2 = 0$. Then $3y_1^5 + 2q_2^2z^3 - 2z^5 = 0$ is the relation among y_1, z, q_2. As a consequence, we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned}
p_1 &= -2(28q_2^6 + 1005q_2^4z^2 - 5550q_2^2z^4 + 4625z^6)/81,\\
p_2 &= -2(28q_2^6 + 1005q_2^4z^2 - 5550q_2^2z^4 + 4625z^6)/81\\
&\quad + \frac{10}{81}(q_2 + z)(q_2 + 5z)^3(-4q_2 + 5z)^2,\\
p_3 &= -2(28q_2^6 + 1005q_2^4z^2 - 5550q_2^2z^4 + 4625z^6)/81\\
&\quad + \frac{10}{81}(-q_2 + z)(-q_2 + 5z)^3(4q_2 + 5z)^2.
\end{aligned}$$

The algebraic solution $(t, w_{3,1})$ has the parameter $\theta = (\frac{13}{30}, \frac{1}{30}, \frac{1}{30}, \frac{5}{6})$ and the curve $\{(t, w_{3,1})\}$ is uniformized by

$$t = \frac{(-5+s)(-1+s)^3(4+s)^2}{(-4+s)^2(1+s)^3(5+s)}, \quad w_{3,1} = \frac{(-1+s)(4+s)(-13+s^2)}{(-4+s)(1+s)^2(5+s)},$$

where $z = sq_2/5$. In this case,

$$P_{xy}s_x s_y s_z t_\infty s_\delta s_\infty t_\infty s_\delta t_\infty^{-1} s_\delta r_x \theta = (1/5, 2/5, 1/5, 1/3).$$

On the other hand, the algebraic solution to Painlevé VI equation

$$\begin{aligned}(t_a, w_a) &= \left(\frac{p_2 - p_1}{p_3 - p_1}, \frac{p_{3,2} - p_1}{p_3 - p_1}\right) \\ &= \left(\frac{(-4+s)^2(1+s)^3(5+s)}{(-5+s)(-1+s)^3(4+s)^2}, \frac{(-4+s)(1+s)(5+s)(2+s^2)}{(-1+s)^2(4+s)(-10+s^2)}\right).\end{aligned}$$

has the parameter $\theta' = (3/5, 1/5, 1/5, 2/3)$ and

$$P_{xy}t_x r_x s_\delta s_y s_z s_\infty r_x s_x s_\delta r_x \theta' = (1/5, 2/5, 1/5, 1/3).$$

By direct computation, we find that $P_{xy}t_x r_x s_\delta s_y s_z s_\infty r_x s_x s_\delta r_x$ transforms (t_a, w_a) to solution 22 obtained by P. Boalch [3] up to a coordinate change.

4.8. KLEIN SOLUTION OF P. BOALCH [2] (LT8)

In this case,

$$\begin{aligned} d(y_1) &= \frac{2}{7},\ d(y_2) = \frac{3}{7},\ d(y_3) = 1, \\ h_1 &= (-2y_1^3 y_2 + y_2^3 + 12 y_1 y_3)/12, \\ h_2 &= (2y_1^5 + 5y_1^2 y_2^2 + 10 y_2 y_3)/10, \\ h_3 &= (-8y_1^7 + 21 y_1^4 y_2^2 + 7 y_1 y_2^4 + 28 y_3^2)/56. \\ B_\infty^{(3)} &= \mathrm{diag}\left[-r - \frac{2}{7}, -r - \frac{1}{7}, -r + \frac{3}{7}\right] \end{aligned}$$

Putting

$$\begin{aligned} y_1 &= -\frac{2^{3/7}}{4}(3u_1^2 + 4u_2^2), \\ y_2 &= \frac{1}{2^{6/7}} u_1(u_1 - 2u_2)(u_1 + 2u_2), \end{aligned}$$

we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned} p_1 &= (1/224)u_1(267u_1^6 + 700u_1^4u_2^2 + 784u_1^2u_2^4 + 1344u_2^6), \\ p_2 &= (1/224)(-165u_1^7 - 308u_1^5u_2^2 - 1568u_1^4u_2^3 - 112u_1^3u_2^4 \\ &\qquad - 1792u_1^2u_2^5 - 448u_1u_2^6 - 512u_2^7), \\ p_3 &= (1/224)(-165u_1^7 - 308u_1^5u_2^2 + 1568u_1^4u_2^3 - 112u_1^3u_2^4 \\ &\qquad + 1792u_1^2u_2^5 - 448u_1u_2^6 + 512u_2^7). \end{aligned}$$

In this case we take $(t_a, w_a) = \left(\frac{p_2 - p_3}{p_1 - p_3}, \frac{p_{3,2} - p_3}{p_1 - p_3}\right)$ instead of the algebraic solution $(t, w_{3,1})$. Then by the substitution $u_1 = \frac{2}{3}u_2(1 - 2s)$, we find that (t_a, w_a) coincides with Klein solution of [2].

Remark 4.3. The polynomial h defined above is regarded as the discriminant of the complex reflection group ST24 in the sense of Shephard-Todd [30]. We already constructed another potential vector field which also implies Klein solution. For the details, see [12, §7.3].

4.9. SOLUTION OBTAINED BY KITAEV [16, P.192] (LT9)

In this case, z is an algebraic function of y_1, y_2 defined by the relation

$$y_1^2 + y_2 - z^2 = 0$$

$$d(y_1) = \frac{1}{4}, \quad d(y_2) = \frac{1}{2}, \quad d(y_3) = 1, \quad d(z) = \frac{1}{4},$$

$$\begin{aligned}
h_1 &= \frac{1}{16}(-93y_1^5 + 16y_1y_3 + 70y_1^3z^2 - 15y_1z^4 + 16z^5),\\
h_2 &= \frac{1}{4}(25y_1^6 + 4y_2y_3 + 55y_1^4z^2 - 30y_1^2z^4 + 40y_1z^5 - 10z^6),\\
h_3 &= \frac{1}{1792}(502125y_1^8 + 896y_3^2 - 318500y_1^6z^2 + 57750y_1^4z^4\\
&\qquad + 56000y_1^3z^5 + 65100y_1^2z^6 - 33600y_1z^7 + 7525z^8),
\end{aligned}$$

$$B_\infty^{(3)} = \operatorname{diag}\left[-r - \frac{1}{4}, -r - \frac{1}{12}, -r + \frac{1}{12}\right].$$

Putting $z = \dfrac{2(-2+s+s^2)y_1}{-2-2s+s^2}$, we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned}
p_1 &= \frac{135y_1^4(-304 - 64s + 608s^2 - 32s^3 - 680s^4 + 16s^5 + 248s^6 + 8s^7 + 5s^8)}{16(-2-2s+s^2)^4},\\
p_2 &= \frac{135y_1^4(80 - 64s - 544s^2 - 32s^3 + 472s^4 + 16s^5 - 136s^6 + 8s^7 + 5s^8)}{16(-2-2s+s^2)^4},\\
p_3 &= -\frac{135y_1^4(-80 + 64s - 992s^2 + 32s^3 + 680s^4 - 16s^5 - 152s^6 - 8s^7 + 19s^8)}{16(-2-2s+s^2)^4}.
\end{aligned}$$

The algebraic solution $(t, w_{3,2})$ has the parameter $\theta = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2})$ and is uniformized by

$$t = \frac{(-2+s^2)(2+s^2)^3}{16(-1+s)^3(1+s)^3}, \quad w_{3,2} = \frac{(2+s^2)^2(2+2s+s^2)}{4(-1+s)(1+s)^2(-2+4s+s^2)}.$$

This is the solution obtained by A.V. Kitaev [16, p. 192, lines 19–20] by $s \to -s$.

4.10. OCTAHEDRAL SOLUTION 9 BY P. BOALCH [4] (LT10)

In this case,

$$d(y_1) = \frac{1}{24}, \quad d(y_2) = \frac{1}{4}, \quad d(y_3) = 1,$$

$$\begin{aligned}
h_1 &= -\frac{1}{552}y_1^{25} + \frac{1}{19}y_1^{19}y_2 + \frac{1}{2}y_1^{13}y_2^2 - \frac{2}{3}y_1^7y_2^3 - \frac{7}{4}y_1y_2^4 + y_1y_3,\\
h_2 &= -\frac{2}{145}y_1^{30} - \frac{1}{23}y_1^{24}y_2 - 4y_1^{12}y_2^3 + 8y_1^6y_2^4 + \frac{1}{5}y_2^5 + y_2y_3,\\
h_3 &= -\frac{531}{99452}y_1^{48} - \frac{6}{23}y_1^{42}y_2 + \frac{12}{23}y_1^{36}y_2^2 + \frac{122}{23}y_1^{30}y_2^3 + \frac{395}{23}y_1^{24}y_2^4\\
&\quad - 24y_1^{18}y_2^5 + 164y_1^{12}y_2^6 - 88y_1^6y_2^7 + \frac{11}{8}y_2^8 + \frac{1}{2}y_3^2,\\
&B_\infty^{(3)} = \operatorname{diag}\left[-r-\frac{7}{18}, -r-\frac{13}{72}, -r+\frac{41}{72}\right].
\end{aligned}$$

Introduce an algebraic function q_2 of y_1, y_2 by the relation $8y_2 + y_1^6 + 15q_2^2 = 0$. Then we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned}
p_1 &= \frac{1}{276}(-34y_1^{24} + 552y_1^{18}y_2 + 1104y_1^{12}y_2^2 + 4048y_1^6y_2^3 - 759y_2^4),\\
p_2 &= \frac{1}{368}\{39y_1^{24} - 552y_1^{18}y_2 + 276y_1^{12}y_2^2\\
&\qquad - 7728y_1^6y_2^3 + 644y_2^4 - 345q_2y_1^3(y_1^6 - 2y_2)^2(y_1^6 + 8y_2)\},\\
p_3 &= \frac{1}{368}\{39y_1^{24} - 552y_1^{18}y_2 + 276y_1^{12}y_2^2\\
&\qquad - 7728y_1^6y_2^3 + 644y_2^4 + 345q_2y_1^3(y_1^6 - 2y_2)^2(y_1^6 + 8y_2)\}.
\end{aligned}$$

The algebraic solution $(t, w_{3,2})$ is uniformized by

$$t = \frac{(1+2s+9s^2)^2(1-s)^4}{(1-2s+9s^2)^2(1+s)^4}, \quad w_{3,2} = \frac{(1-s)(1+2s+9s^2)(1-18s^2-15s^4)}{(1+s)^3(1-2s+9s^2)(1+15s^2)}.$$

Then the algebraic solution to Painlevé VI equation $\left(t, \dfrac{t(w_{3,2}-1)}{w_{3,2}-t}\right)$ coincides with octahedral solution 9 obtained by P. Boalch [4] by the substitution $s = \frac{1-r}{1+r}$.

4.11. SOLUTION 24 BY P. BOALCH [3] (LT11)

This case is treated in §2. We use the notation in §2 without any comment.

Let z be an algebraic function of y_1, y_2 defined by the equation

$$3y_1^2 - y_2 - z^2 = 0$$

which is same as (2.21). The potential vector field obtained in §2 is (h_1, h_2, h_3), where

$$\begin{aligned}
h_1 &= y_1y_3 - \frac{27y_1^5}{4000} - \frac{y_1^3z^2}{1200} + \frac{y_1z^4}{800} + \frac{z^5}{3750},\\
h_2 &= y_2y_3 - \frac{37y_1^6}{1000} + \frac{9y_1^4z^2}{200} - \frac{3y_1^2z^4}{200} - \frac{4y_1z^5}{625} - \frac{z^6}{1000},\\
h_3 &= \frac{y_3^2}{2} + \frac{897y_1^8}{4480000} - \frac{13y_1^6z^2}{160000} - \frac{3y_1^4z^4}{320000} + \frac{y_1^3z^5}{187500} + \frac{7y_1^2z^6}{800000} + \frac{y_1z^7}{437500} + \frac{13z^8}{48000000}.
\end{aligned}$$

We recall the algebraic solution $(t, w_{3,1})$ defined in (2.24). Then $\theta = (\frac{7}{20}, \frac{1}{20}, \frac{1}{20}, \frac{3}{4})$ is the parameter corresponding to $(t, w_{3,1})$. It is easy to check that

$$P_{xy} r_z s_z s_\infty s_\delta (s_x s_y s_z s_\delta)^4 \theta = \Big(\frac{1}{2}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5}\Big).$$

By direct computation, we find that $P_{xy} r_z s_z s_\infty s_\delta (s_x s_y s_z s_\delta)^4$ transforms the solution $(t, w_{3,1})$ to solution 24 in [3] (cf. (2.1)).

4.12. SOLUTION 25 BY P. BOALCH [3] (LT12)

In this case,

$$d(y_1) = \frac{1}{20}, \quad d(y_2) = \frac{1}{4}, \quad d(y_3) = 1,$$

$$\begin{aligned} h_1 &= \frac{1}{114} y_1^{21} - y_1^{11} y_2^2 - \frac{2}{3} y_1^6 y_2^3 + \frac{1}{2} y_1 y_2^4 + y_1 y_3, \\ h_2 &= \frac{2}{5} y_1^{25} + \frac{51}{38} y_1^{20} y_2 - 2 y_1^{15} y_2^2 + 7 y_1^{10} y_2^3 - \frac{3}{10} y_2^5 + y_2 y_3, \\ h_3 &= \frac{366}{361} y_1^{40} - \frac{96}{7} y_1^{35} y_2 - \frac{440}{19} y_1^{30} y_2^2 - \frac{192}{19} y_1^{25} y_2^3 \\ &\quad + \frac{1740}{19} y_1^{20} y_2^4 - 32 y_1^{15} y_2^5 + 56 y_1^{10} y_2^6 + \frac{2}{7} y_2^8 + \frac{1}{2} y_3^2, \end{aligned}$$

$$B_\infty^{(3)} = \operatorname{diag}\left[-r - \frac{23}{60}, -r - \frac{11}{60}, -r + \frac{17}{30}\right].$$

To solve the cubic equation $h = 0$ with respect to y_3, we define $q_1 = y_1^5 + 5y_2$ and introduce an algebraic function q_2 of y_1, y_2 by the relation $5(y_1^5 + 5y_2)(3y_1^5 - y_2) + q_2^2 = 0$. Then we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned} p_1 &= (-184375 q_1^8 + 167500 q_1^6 q_2^2 - 4250 q_1^4 q_2^4 + 300 q_1^2 q_2^6 + 1369 q_2^8)/(24320000 q_1^4), \\ p_2 &= (28125 q_1^8 - 17500 q_1^6 q_2^2 - 2250 q_1^4 q_2^4 - 12160 q_1^3 q_2^5 - 780 q_1^2 q_2^6 - 299 q_2^8)/(6080000 q_1^4), \\ p_3 &= (28125 q_1^8 - 17500 q_1^6 q_2^2 - 2250 q_1^4 q_2^4 + 12160 q_1^3 q_2^5 - 780 q_1^2 q_2^6 - 299 q_2^8)/(6080000 q_1^4). \end{aligned}$$

The algebraic solution $(t, w_{3,1})$ has the parameter $\theta = (-\frac{1}{4}, -\frac{3}{20}, -\frac{3}{20}, \frac{19}{20})$ and is uniformized by

$$t = \frac{(5 + 2s + s^2)^2 (3 - s)^3 (5 + s)}{(5 - 2s + s^2)^2 (3 + s)^3 (5 - s)}, \quad w_{3,1} = \frac{(5 + 2s + s^2)(15 + s^2)(3 - s)}{(5 - 2s + s^2)(3 + s)^2 (5 - s)},$$

where $s = \frac{5q_1}{q_2}$. Note that $(2/5, 1/2, 2/5, 4/5) = P_{xy} s_x r_x s_\infty t_\infty s_x s_\delta s_\infty t_\infty s_y s_z r_x \theta$ is the parameter of solution 25 obtained by P. Boalch [3]. By direct computation, we find that $P_{xy} s_x r_x s_\infty t_\infty s_x s_\delta s_\infty t_\infty s_y s_z r_x$ transforms $(t, w_{3,1})$ to the solution 25 up to a coordinate change.

4.13. SOLUTION 27 BY P. BOALCH [3] (LT13)

In this case,

$$\begin{aligned}
&d(y_1) = \frac{1}{15}, \quad d(y_2) = \frac{1}{3}, \quad d(y_3) = 1,\\
&h_1 = -\frac{1}{11}y_1^{11}y_2 - \frac{1}{3}y_1y_2^3 + y_1y_3,\\
&h_2 = -\frac{5}{76}y_1^{20} + \frac{3}{2}y_1^{10}y_2^2 + \frac{1}{4}y_2^4 + y_2y_3,\\
&h_3 = \frac{10}{87}y_1^{30} + 2y_1^{20}y_2^2 - 6y_1^{10}y_2^4 + \frac{2}{15}y_2^6 + \frac{1}{2}y_3^2,\\
&B_\infty^{(3)} = \operatorname{diag}\left[-r - \frac{2}{5}, -r - \frac{2}{15}, -r + \frac{8}{15}\right].
\end{aligned}$$

We introduce new variables (q_1, q_2, q_3) by the relations

$$\begin{aligned}
y_1 &= 3^{7/30}(-q_2)^{1/10}/(2^{7/15}5^{1/30}),\\
y_2 &= -(5/2)^{1/3}3^{2/3}q_1/4,\\
y_3 &= (5q_1^3 - 9q_1q_2 + 64q_3)/64
\end{aligned}$$

and regard h as a polynomial of q_3. Moreover we define $m = q_2/q_1^2$ and $m = \frac{(8s-3)(s-1)^2}{s^2}$. Then we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned}
p_1 &= q_1^3(81s^2 - 432s^3 + 936s^4 - 1280s^5 + 720s^6 + 3s^2(-3+2s)^2(-1+6s)u)/(256s^5),\\
p_2 &= q_1^3(81s^2 - 432s^3 + 936s^4 - 1280s^5 + 720s^6 - 3s^2(-3+2s)^2(-1+6s)u)/(256s^5),\\
p_3 &= q_1^3(-81 + 432s - 936s^2 + 1280s^3 - 720s^4)/(128s^3),
\end{aligned}$$

p_1, p_2, p_3 are defined by and u by the equation $u^2 = (3-8s)(1-6s)(3-2s)$. In this case $p_{3,1} = \frac{q_1(25q_1^2 - 9q_2)}{16}$. The algebraic solution $(t, w_{3,1})$ has the parameter $\theta = (-\frac{2}{15}, -\frac{2}{15}, -\frac{2}{15}, \frac{16}{15})$ and is uniformized by

$$t = \frac{1}{2} + \frac{81 - 432s + 936s^2 - 1280s^3 + 720s^4}{2u(1-6s)(3-2s)^2}, \quad w_{3,1} = \frac{1}{2} + \frac{9 - 36s + 52s^2}{2u(3-2s)}.$$

In this case, it is easy to see that

$$s_xs_ys_zs_\delta s_\infty\theta = (2/5, 2/5, 2/5, 2/3).$$

We now recall solution 27 obtained by P. Boalch [3]. We put

$$t_A = \frac{1}{2} + \frac{(25s^4 + 170s^3 + 42s^2 + 8s - 2)u_A}{54s^3(5s+4)^2}, \quad w_A = \frac{1}{2} + \frac{350s^3 + 63s^2 - 6s - 2}{30s(2s+1)u_A},$$

where $u_A^2 = s(8s+1)(5s+4)$. Then (t_A, w_A) is solution 27. By the change of parameter $s \to \frac{3(2s_1+1)}{2(8s_1+1)}$ and $u \to -\frac{18v}{(8s_1+1)^2}$, where $v^2 = s_1(8s_1+1)(5s_1+4)$, $(t, w_{3,1})$ is transformed to (t_A, w_A).

Remark 4.4. The polynomial h in this case is related with the polynomial $F_{B,6}$ introduced in [28]. See also [12, §8.1].

4.14. SOLUTION OBTAINED BY A. KITAEV [16, P.179] (LT14)

In this case, z is an algebraic function of y_1, y_2 defined by

$$y_1^2 + y_2 z^6 + z^{16} = 0.$$

$$d(y_1) = \frac{8}{15},\ d(y_2) = \frac{2}{3},\ d(y_3) = 1,\ d(z) = \frac{1}{15},$$

$$h_1 = -(2093y_1^4 - 897y_1y_3z^9 + 3450y_1^2z^{16} + 525z^{32})/(897z^9),$$
$$h_2 = (-238y_1^5 + 85y_2y_3z^{15} + 1700y_1^3z^{16} - 750y_1z^{32})/(85z^{15}),$$
$$h_3 = (49y_1^6 + 2415y_1^4z^{16} + 3y_3^2z^{18} + 795y_1^2z^{32} - 35z^{48})/(6z^{18}),$$

$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{1}{5}, -r - \frac{1}{15}, -r + \frac{4}{15}\right]$$

We are going to obtain solutions of $h = 0$, regarding h as a polynomial of y_3. For this purpose, we put $m = \frac{5z^{16}}{3y_1^2}$ and introduce s by the relation $m = -\frac{(1-s)^2(3-8s)}{s^2}$. Then we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where p_1, p_2, p_3 are defined by

$$p_1 = -y_1^3(189 - 1188s + 3024s^2 - 4080s^3 + 2160s^4 + 7(-3+2s)^2(-1+6s)u)/(30s^3z^9),$$
$$p_2 = -y_1^3(189 - 1188s + 3024s^2 - 4080s^3 + 2160s^4 - 7(-3+2s)^2(-1+6s)u)/(30s^3z^9),$$
$$p_3 = y_1^3(189 - 918s + 1764s^2 - 2440s^3 + 1440s^4)/(15s^3z^9),$$

and u by the equation $u^2 = (3-8s)(1-6s)(3-2s)$. The algebraic solution $(t, w_{3,1})$ corresponding to $p_{3,1} = (-259y_1^4 - 486y_1^2z^{16} - 35z^{32})/(24y_1z^9)$ has the parameter $\theta = (\frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{7}{15})$ and is uniformized by

$$t = \frac{1}{2} - \frac{81 - 432s + 936s^2 - 1280s^3 + 720s^4}{2(3-2s)^2(1-6s)u}, \quad w_{3,1} = \frac{1}{2} - \frac{-9 + 30s - 20s^2 + 24s^3}{8su}.$$

Note that

$$s_\delta(s_x s_y s_z s_\delta)^4\theta = (1/5, 1/5, 1/5, 1/3).$$

In order to continue the computation, we recall the solution obtained by Kitaev [16, p. 11]. Let t_A and u_A be the rational function and the elliptic function given in §4.13, respectively. Moreover we define

$$w_B = \frac{1}{2} + \frac{(8s+1)(50s^3 + 24s^2 + 9s - 2)}{2(20s^3 + 60s^2 + 1)u_A}.$$

Then by the substitution

$$s = -\frac{4(s_1^2 + 2s_1 + 1)}{5s_1^2 + 118s_1 + 5}, \quad u_A = -\frac{216s_1(s_1^2 - 1)}{(5s_1^2 + 118s_1 + 5)u_B},$$

where $u_B^2 = s_1(5s_1^2 + 118s_1 + 5)$, we find that, as functions of s_1, (t_A, u_B) is nothing but the solution obtained by Kitaev [16, p. 179, lines 15–17] (cf. [20]).

By operating the Bäcklund transformation corresponding to $s_\delta(s_x s_y s_z s_\delta)^4$ on $(t, w_{3,1})$, we obtain the algebraic solution $(\tilde{t}, \tilde{w})$ to Painleé VI defined by

$$\tilde{t} = \frac{1}{2} + \frac{720s^4 - 1280s^3 + 936s^2 - 432s + 81}{2(2s-3)^2(6s-1)u},$$
$$\tilde{w} = \frac{1}{2} + \frac{(8s-3)(72s^3 - 100s^2 + 42s - 9)}{2(24s^3 - 52s^2 + 42s - 9)u},$$

where $u = \pm\sqrt{(3-8s)(1-6s)(3-2s)}$ as before. Then by the substitution $s = \frac{3(2s_1+1)}{2(8s_1+1)}$, $u = -\frac{18s_1(5s_1+4)}{(8s_1+1)u_A}$, we find that $(\tilde{t}, \tilde{w})$ turns out to be $(t_A, w_B)|_{s \to s_1}$. As a consequence, the algebraic solution $(t, w_{3,1})$ is equivalent to the solution obtained by Kitaev [16, p. 179, lines 15–17].

4.15. SOLUTION 28 BY BOALCH [3] (LT15)

In this case, z is an algebraic function of y_1, y_2 defined by the relation

$$-128y_1^3 + 3y_2 - 60y_1^2 z + 125z^3 = 0.$$

$$d(y_1) = \frac{1}{5}, \quad d(y_2) = \frac{3}{5}, \quad d(y_3) = 1, \quad d(z) = \frac{1}{5},$$

$$\begin{aligned}
h_1 = (&-131488y_1^6 + 90y_1y_3 - 74400y_1^5z - 36000y_1^4z^2 \\
&+ 110000y_1^3z^3 + 93750y_1^2z^4 + 168750y_1z^5 + 78125z^6)/90, \\
h_2 = (&-743168y_1^8 + 20y_2y_3 + 1578240y_1^7z + 2496000y_1^6z^2 \\
&+ 472000y_1^5z^3 - 5100000y_1^4z^4 - 13050000y_1^3z^5 \\
&- 11000000y_1^2z^6 - 4687500y_1z^7 - 1171875z^8)/20, \\
h_3 = (&1900883200y_1^{10} + 63y_3^2 + 750528000y_1^9z - 332640000y_1^8z^2 - 2294400000y_1^7z^3 \\
&+ 308700000y_1^6z^4 + 2305800000y_1^5z^5 + 2682750000y_1^4z^6 + 1845000000y_1^3z^7 \\
&+ 1676953125y_1^2z^8 + 464843750y_1z^9 + 105468750z^{10})/126,
\end{aligned}$$

$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{2}{5}, -r, -r + \frac{2}{5}\right].$$

We write y_1, y_2, z by u, q as follows:

$$\begin{aligned}
y_1 &= \frac{1}{6}q(-5 + 2u + u^2), \\
y_2 &= \frac{2}{9}q^3(-125 + 150u + 135u^2 - 4u^3 - 27u^4 + 6u^5 + u^6), \\
z &= \frac{1}{15}q(-5 - 10u + u^2).
\end{aligned}$$

Then $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned} p_1 = &-\frac{2q^5}{15}(- 15625 + 31250u + 625u^2 - 19000u^3 + 750u^4 \\ &\qquad + 6796u^5 - 150u^6 - 760u^7 - 5u^8 + 50u^9 + 5u^{10}), \\ p_2 = &\frac{2q^5}{15}(65625 + 68750u + 9375u^2 - 45000u^3 - 4750u^4 + 14900u^5 \\ &\qquad - 650u^6 - 1800u^7 + 85u^8 + 110u^9 + 11u^{10}), \\ p_3 = &-\frac{2q^5}{15}(34375 - 68750u + 10625u^2 + 45000u^3 - 3250u^4 \\ &\qquad - 14900u^5 - 950u^6 + 1800u^7 + 75u^8 - 110u^9 + 21u^{10}). \end{aligned}$$

The algebraic solution $(t, w_{3,1})$ corresponding to

$$p_{3,1} = -\frac{2q^5\begin{pmatrix} 109375 + 37500u - 98750u^2 - 105500u^3 + 25625u^4 + 41400u^5 \\ -5700u^6 - 8280u^7 + 1025u^8 + 844u^9 - 158u^{10} - 12u^{11} + 7u^{12} \end{pmatrix}}{5(-5 + 2u + u^2)}$$

has the parameter $\theta = (\frac{1}{5}, \frac{1}{10}, \frac{1}{10}, \frac{4}{5})$ and is uniformized by

$$\begin{aligned} t &= -\frac{(-5+u)^3(-1+u)^3(5+4u+u^2)^2}{(1+u)^3(5+u)^3(5-4u+u^2)^2}, \\ w_{3,1} &= -\frac{(-5+u)^2(-1+u)(-5-2u+u^2)(5+4u+u^2)}{(1+u)(5+u)^2(5-4u+u^2)(-5+2u+u^2)}. \end{aligned}$$

Note that

$$t_z s_\infty r_z s_\delta s_\infty t_\infty s_x P_{xy}\theta = \left(\frac{1}{2}, \frac{1}{5}, \frac{1}{2}, \frac{3}{5}\right)$$

and the latter is the parameter for solution 28 of Boalch [3]. Then we find that $t_z s_\infty r_z s_\delta s_\infty t_\infty s_x P_{xy}$ transforms $(t, w_{3,1})$ to an algebraic solution which coincides with solution 28 of Boalch [3] up to a coordinate change.

4.16. SOLUTION 29 BY P. BOALCH [3] (LT18)

In this case,

$$d(y_1) = \frac{1}{10}, \quad d(y_2) = \frac{1}{5}, \quad d(y_3) = 1,$$

$$\begin{aligned} h_1 &= -y_1(5y_1^6y_2^2 - 14y_2^5 - 2y_3)/2, \\ h_2 &= (5y_1^{12} + 275y_1^6y_2^3 - 55y_2^6 + 33y_2y_3)/33, \\ h_3 &= (-100y_1^{18}y_2 + 2550y_1^{12}y_2^4 + 12750y_1^6y_2^7 + 595y_2^{10} + 9y_3^2)/18, \end{aligned}$$

$$B_\infty^{(3)} = \operatorname{diag}\left[-r - \frac{1}{3}, -r - \frac{7}{30}, -r + \frac{17}{30}\right].$$

$$h = \frac{1}{2}(-4y_1^{30} + 1520y_1^{24}y_2^3 - 2780y_1^{18}y_2^6 + 150280y_1^{12}y_2^9 + 103615y_1^6y_2^{12} - 1666y_2^{15}) + 3y_2(10y_1^{18} - 260y_1^{12}y_2^3 - 1325y_1^6y_2^6 - 63y_2^9)y_3 + \frac{3}{2}y_2^2(5y_1^6 - 2y_2^3)y_3^2 + y_3^3.$$

To obtain solutions of $h = 0$ as a cubic equation of y_3, we observe that h only depends on y_1^6. Noting this, we put

$$y_1^6 = \frac{y_2^3}{J(s)} \quad \left(J(s) = \frac{4(1-s+s^2)^3}{27s^2(1-s)^2}\right)$$

and erase y_1 in h. Then we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned} p_1 = -y_2^5(&56 - 1480s + 7080s^2 - 19080s^3 + 32925s^4 - 39138s^5 \\ &+ 32925s^6 - 19080s^7 + 7080s^8 - 1480s^9 + 56s^{10})/8(1-s+s^2)^5, \\ p_2 = -y_2^5(&-136 + 680s - 3315s^2 + 9180s^3 - 15810s^4 + 18156s^5 \\ &- 14955s^6 + 9000s^7 - 3720s^8 + 920s^9 + 56s^{10})(8(1-s+s^2)^5, \\ p_3 = y_2^5(&-56 - 920s + 3720s^2 - 9000s^3 + 14955s^4 - 18156s^5 \\ &+ 15810s^6 - 9180s^7 + 3315s^8 - 680s^9 + 136s^{10})/8(1-s+s^2)^5. \end{aligned}$$

The algebraic solution $(t, w_{3,1})$ corresponding to $p_{3,1} = -34y_1^6y_2^2 + 73y_2^5$ has the parameter $\theta = (\frac{7}{30}, \frac{7}{30}, \frac{7}{30}, \frac{9}{10})$ and it is uniformized by

$$t = -\frac{(5-5s+8s^2)^2(2-s)^5s}{(8-5s+5s^2)^2(1-2s)^5}, \quad w = \frac{(5-5s+8s^2)(4-s+4s^2)(2-s)^2}{3(8-5s+5s^2)(1-2s)^3}.$$

Since $(1/3, 1/3, 1/3, 4/5) = s_\infty t_\infty s_\delta s_\infty t_\infty \theta$ is the parameter of solution 29 in P. Boalch [3], we find that $s_\infty t_\infty s_\delta s_\infty t_\infty$ transforms $(t, w_{3,1})$ to an algebraic solution which coincides with solution 29 of Boalch [3] up to a coordinate change.

Remark 4.5. The polynomial h in this case is related with the polynomial $F_{H,2}$ introduced in [28]. See also [12, §8.2].

4.17. SOLUTION 30 BY P. BOALCH [3] (LT19)

In this case, z is an algebraic function of y_1, y_2 defined by

$$y_1^6 + y_2z^6 + z^9 = 0.$$

$$d(y_1) = \frac{3}{10},\ d(y_2) = \frac{3}{5},\ d(y_3) = 1,\ d(z) = \frac{1}{5}$$

$$\begin{aligned} h_1 &= \frac{y_1}{z}\left(y_3z - \frac{8}{91}y_2^2 + \frac{33}{182}y_2z^3 + \frac{9}{130}z^5\right), \\ h_2 &= y_2y_3 - 3y_2^2z^2 - 9y_2z^5 - \frac{27}{4}z^8, \\ h_3 &= \frac{1}{2}y_3^2 - \frac{8}{17z^{17}}y_1^{18} + 6y_2^2z^4 - \frac{48}{7}y_2z^7 - \frac{127}{10}z^{10}. \end{aligned}$$

$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{1}{3}, -r - \frac{1}{30}, -r + \frac{11}{30}\right].$$

To solve the equation $h = 0$ as a cubic equation of y_3, putting $m = y_2^6/z^9$ and $m = \frac{(1-s)^2(2+s)^2(1+2s)^2}{8(1+s+s^2)^3}$, we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned}
p_1 = & -27z^5(16 + 80s + 180s^2 + 240s^3 + 195s^4 + 162s^5 \\
& + 195s^6 + 240s^7 + 180s^8 + 80s^9 + 16s^{10})/80(1 + s + s^2)^5, \\
p_2 = & -27z^5(- 20 - 100s - 225s^2 - 300s^3 - 210s^4 + 195s^6 \\
& + 240s^7 + 180s^8 + 80s^9 + 16s^{10})/80(1 + s + s^2)^5, \\
p_3 = & \, 27z^5(- 16 - 80s - 180s^2 - 240s^3 - 195s^4 + 210s^6 \\
& + 300s^7 + 225s^8 + 100s^9 + 20s^{10})/80(1 + s + s^2)^5.
\end{aligned}$$

The algebraic solution $(t, w_{3,1})$ corresponding to $p_{3,1} = -\frac{(4y_2^2 - 78y_2z^3 - 45z^6)}{15z}$ has the parameter $\theta = (1/30, 1/30, 1/30, 7/10)$ and is uniformized by

$$t = \frac{s^5(2 + s)(3 + 3s + 2s^2)^2}{(1 + 2s)(2 + 3s + 3s^2)^2}, \quad w = -\frac{s^4(3 + 3s + 2s^2)}{(1 + 2s)(1 + s + s^2)(2 + 3s + 3s^2)}.$$

In this case,

$$s_\infty t_\infty s_\delta s_\infty t_\infty \theta = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{5}\right).$$

On the other hand the algebraic solution to Painlevé VI equation $(t, w_{3,2})$ corresponding to

$$p_{3,2} = -\frac{-39y_1^6y_2 - 234y_1^6z^3 + 5y_2^2z^6 + 3y_2z^9}{10y_2z^4}$$

coincides with the solution 30 obtained by P. Boalch [3].

4.18. OCTAHEDRAL SOLUTION 12 BY P. BOALCH [4] (LT20)

In this case, z is an algebraic function of y_1, y_2 defined by

$$27y_2 + 180y_1^4z - 20y_1^2z^3 + z^5 = 0.$$

$$d(y_1) = \frac{1}{12}, \quad d(y_2) = \frac{5}{12}, \quad d(y_3) = 1, \quad d(z) = \frac{1}{12},$$

$$\begin{aligned}
h_1 &= y_1y_3 + \frac{396}{7}y_1^{12}z - \frac{572}{189}y_1^{10}z^3 - \frac{407}{567}y_1^8z^5 + \frac{110}{729}y_1^6z^7 \\
&\quad - \frac{935}{78732}y_1^4z^9 + \frac{13}{26244}y_1^2z^{11} - \frac{11}{1364688}z^{13}, \\
h_2 &= -\frac{213840}{17}y_1^{17} + y_2y_3 + \frac{52800}{7}y_1^{15}z^2 - \frac{981200}{567}y_1^{13}z^4 \\
&\quad + \frac{1007600}{5103}y_1^{11}z^6 - \frac{194150}{15309}y_1^9z^8 + \frac{211420}{413343}y_1^7z^{10} \\
&\quad - \frac{5275}{531441}y_1^5z^{12} - \frac{25}{531441}y_1^3z^{14} + \frac{25}{2834352}y_1z^{16}, \\
h_3 &= \frac{406467072}{161}y_1^{24} + \frac{1}{2}y_3^2 + \frac{30108672}{49}y_1^{22}z^2 - \frac{8642304}{49}y_1^{20}z^4 \\
&\quad + \frac{13598464}{441}y_1^{18}z^6 - \frac{46723424}{11907}y_1^{16}z^8 + \frac{120976768}{321489}y_1^{14}z^{10} \\
&\quad - \frac{3539360}{137781}y_1^{12}z^{12} + \frac{509344}{413343}y_1^{10}z^{14} - \frac{52426}{1240029}y_1^8z^{16} \\
&\quad + \frac{572}{531441}y_1^6z^{18} - \frac{583}{30292137}y_1^4z^{20} + \frac{17}{90876411}y_1^2z^{22} + \frac{11}{6543101592}z^{24},
\end{aligned}$$

$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{1}{2}, -r + \frac{1}{12}, -r + \frac{5}{12}\right].$$

It is straightforward to show $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where p_1, p_2, p_3 are defined by

$$\begin{aligned}
p_1 &= \frac{1}{1102248}\{77u_A^3(18y_1^2 + z^2)^3 - 8y_1z(62355744y_1^{10} - 13215312y_1^8z^2 \\
&\qquad + 1753488y_1^6z^4 - 116424y_1^4z^6 + 3850y_1^2z^8 - 105z^{10})\}, \\
p_2 &= \frac{1}{1102248}\{-77u_A^3(18y_1^2 + z^2)^3 - 8y_1z(62355744y_1^{10} - 13215312y_1^8z^2 \\
&\qquad + 1753488y_1^6z^4 - 116424y_1^4z^6 + 3850y_1^2z^8 - 105z^{10})\}, \\
p_3 &= \frac{2}{15309}y_1z(20785248y_1^{10} - 3207600y_1^8z^2 + 318384y_1^6z^4 \\
&\qquad - 16632y_1^4z^6 + 462y_1^2z^8 - 7z^{10}),
\end{aligned}$$

and u_A by

$$u_A^2 = 2(18y_1^2 - 8y_1z + z^2)(18y_1^2 + 8y_1z + z^2).$$

Let $(t, w_{3,1})$ be the algebraic solution corresponding to

$$\begin{aligned}
p_{3,1} = \frac{z}{1102248y_1}&(2702082240y_1^{12} - 300231360y_1^{10}z^2 + 12416976y_1^8z^4 \\
&+ 864864y_1^6z^6 - 108108y_1^4z^8 + 5460y_1^2z^{10} - 77z^{12}).
\end{aligned}$$

To obtain the parametric expression of $(t, w_{3,1})$, we introduce m, u_B by $m = \frac{t_1}{z}$, $u_B = \frac{m}{z^2}u_A$. Then

$$t = \frac{1}{2} + t_C, \quad w_{3,1} = \frac{1}{2} + w_C,$$

where

$$t_C = -\frac{4m^4(-1+18m^2)(3-104m^2+3528m^4-33696m^6+314928m^8)}{(1+18m^2)^3 u_B^3},$$
$$w_C = -\frac{(-1+50m^2-684m^4+7128m^6)}{2(1+18m^2)^2 u_B}.$$

Noting that both t_C, w_C are rational with respect to m^2, we put $s = 18m^2$. Moreover, we put $u = 9u_B$. Then we find that the algebraic solution $(t, w_{3,1})$ has the parameter $\theta = (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{11}{12})$ and is uniformized by

$$t = \frac{1}{2} + \frac{2s^2(1-s)(27-52s+98s^2-52s^3+27s^4)}{(1+s)^3 u^3}, \quad w = \frac{1}{2} + \frac{9-25s+19s^2-11s^3}{2(1+s)^2 u},$$

where u is defined by the relation

$$u^2 = s(9s^2-14s+9).$$

Note that $(1/2, 1/2, 1/2, 2/3) = (s_\delta s_x s_y s_z)^4 s_\delta s_\infty \theta$ is the parameter of solution 12 of Boalch [4]. Then we find that by the Bäcklund transformation corresponding to $(s_\delta s_x s_y s_z)^4 s_\delta s_\infty$, (t_A, w_A) is transformed to an algebraic solution (t_a, w_a) identified with octahedral solution 12 in [4] by changing the variable s with $2s_1 + 1$.

4.19. OCTAHEDRAL SOLUTION 11 BY P. BOALCH [4] (LT21)

In this case, z is an algebraic function of y_1, y_2 defined by

$$-y_2 + z(3y_1^2 - 3y_1 z + z^2) = 0,$$
$$d(y_1) = \frac{1}{6}, \quad d(y_2) = \frac{1}{2}, \quad d(y_3) = 1, \quad d(z) = \frac{1}{6},$$
$$h_1 = y_1 y_3 - \frac{4608}{35} y_1^7 - 768 y_1^5 z^2 + \frac{5120}{3} y_1^4 z^3 - \frac{5120}{3} y_1^3 z^4 + \frac{2816}{3} y_1^2 z^5 - \frac{2560}{9} y_1 z^6 + \frac{256}{7} z^7,$$
$$h_2 = y_2 y_3 - \frac{41472}{5} y_1^8 z + \frac{179712}{5} y_1^7 z^2 - \frac{347904}{5} y_1^6 z^3 + 77568 y_1^5 z^4 - 55040 y_1^4 z^5 + 25600 y_1^3 z^6 - \frac{22784}{3} y_1^2 z^7 + \frac{3968}{3} y_1 z^8 - \frac{2816}{27} z^9,$$
$$h_3 = \frac{1}{2} y_3^2 + \frac{382205952}{275} y_1^{12} + \frac{42467328}{5} y_1^{10} z^2 - 28311552 y_1^9 z^3 + 50724864 y_1^8 z^4 - \frac{303562752}{5} y_1^7 z^5 + 51380224 y_1^6 z^6 - \frac{468189184}{15} y_1^5 z^7 + \frac{122945536}{9} y_1^4 z^8 - \frac{38273024}{9} y_1^3 z^9 + \frac{72876032}{81} y_1^2 z^{10} - \frac{103284736}{891} y_1 z^{11} + \frac{1703936}{243} z^{12},$$
$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{7}{18}, -r - \frac{1}{18}, -r + \frac{4}{9}\right].$$

We put

$$y_1 = (-3 + u^2 + \sqrt{6}u)y_0, \quad z = 3\sqrt{6}uy_0.$$

Then $h = 45 \cdot 2025(y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned}
p_1 &= -\frac{9216}{5}\{729 + 9477u^2 - 8505u^4 + 5886u^6 - 945u^8 + 117u^{10} + u^{12} \\
&\quad + 6u(-3 + u^2)(3 + u^2)^2(3 - 3u + u^2)(3 + 3u + u^2)\sqrt{6}\}y_0^6, \\
p_2 &= \frac{13824}{5}\{729 - 7290u + 7047u^2 - 2430u^3 - 6885u^4 + 1620u^5 + 4266u^6 + 540u^7 \\
&\quad - 765u^8 - 90u^9 + 87u^{10} - 30u^{11} + u^{12} \\
&\quad - 4u(-3 + u^2)(3 + u^2)^2(3 - 3u + u^2)(3 + 3u + u^2)\sqrt{6}\}y_0^6, \\
p_3 &= \frac{13824}{5}\{729 + 7290u + 7047u^2 + 2430u^3 - 6885u^4 - 1620u^5 + 4266u^6 - 540u^7 \\
&\quad - 765u^8 + 90u^9 + 87u^{10} + 30u^{11} + u^{12} \\
&\quad - 4u(-3 + u^2)(3 + u^2)^2(3 - 3u + u^2)(3 + 3u + u^2)\sqrt{6}\}y_0^6.
\end{aligned}$$

We consider the algebraic solution $(t, w_{3,2})$. Then

$$\begin{aligned}
&(t, w_{3,2}) \\
&= \Bigg(\frac{(3 - 3u + u^2)^2(3 + 6u + u^2)^4}{(3 + 3u + u^2)^2(3 - 6u + u^2)^4}, \\
&\quad -\frac{(3 - 3u + u^2)(3 + 6u + u^2)\{81 - 54u^2 + 162u^4 - 6u^6 + u^8 - 3\sqrt{6}u(-3 + u^2)^3\}}{(3 + 3u + u^2)(3 - 6u + u^2)^3\{9 + u^4 + \sqrt{6}u(3 - u^2)\}}\Bigg)
\end{aligned}$$

has the parameter $\theta = (\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{3}{2})$. It is easy to show that

$$r_y P_{yz}\theta = \Big(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}\Big).$$

By the corresponding Bäcklund transformation, we have

$$(t, w_{3,2}) \overset{P_{yz}}{\longrightarrow} \left(\frac{1}{t}, \frac{w_{3,2}}{t}\right) \overset{r_y}{\longrightarrow} \left(\frac{1}{t}, \frac{w_{3,2} - 1}{w_{3,2} - t}\right).$$

By the substitution $u = \frac{(2+\sqrt{6})s+\sqrt{6}}{s-(2+\sqrt{6})}$, we conclude that $\left(\frac{1}{t}, \frac{w_{3,2}-1}{w_{3,2}-t}\right)$ coincides with octahedral solution 11 in [4].

4.20. SOLUTION 38 BY P. BOALCH [3] (LT26)

In this case,

$$d(y_1) = \frac{1}{5},\ d(y_2) = \frac{2}{5},\ d(y_3) = 1,$$

$$h_1 = -\frac{y_1^6}{30} - \frac{y_1^4 y_2}{2} + \frac{y_1^2 y_2^2}{2} + \frac{y_2^3}{3} + y_1 y_3,$$

$$h_2 = \frac{5y_1^7}{6} + \frac{y_1^5 y_2}{2} + \frac{5y_1^3 y_2^2}{2} - \frac{5y_1 y_2^3}{6} + y_2 y_3,$$

$$h_3 = -\frac{21y_1^{10}}{8} + 5y_1^8 y_2 + \frac{35y_1^6 y_2^2}{4} + \frac{35y_1^2 y_2^4}{8} - \frac{7y_2^5}{20} + \frac{y_3^2}{2},$$

$$B_\infty^{(3)} = \operatorname{diag}\left[-r - \frac{1}{3}, -r - \frac{2}{15}, -r + \frac{7}{15}\right].$$

We put $y_2 = m y_1^2$, where $m = -\frac{1+3s+6s^2+5s^3}{3s^2}$. Then we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where p_1, p_2, p_3 are defined by

$$\begin{aligned} p_1 = \{&4 + 25s + 135s^2 + 475s^3 + 1150s^4 + 1665s^5 \\ &+ 1375s^6 + 625s^7 - (2+s)^2(1+5s)^2(1+s+4s^2)v\} y_1^5/(90s^5), \\ p_2 = \{&4 + 25s + 135s^2 + 475s^3 + 1150s^4 + 1665s^5 \\ &+ 1375s^6 + 625s^7 + (2+s)^2(1+5s)^2(1+s+4s^2)v\} y_1^5/(90s^5), \\ p_3 = -&(8 + 35s + 180s^2 + 545s^3 + 1340s^4 + 1827s^5 + 1400s^6 + 875s^7) y_1^5/(90s^5), \end{aligned}$$

and v by the equation $v^2 = (1+5s)(1+s+4s^2)$. The algebraic solution $(t, w_{3,1})$ corresponding to $p_{3,1} = (-5y_1^6 - 58y_1^4 y_2 + 41y_1^2 y_2^2 + 16y_2^3)/(10y_1)$ has the parameter $\theta = (-\frac{2}{15}, -\frac{2}{15}, -\frac{2}{15}, \frac{4}{5})$ and is uniformized by

$$\begin{aligned} t &= \frac{1}{2} - \frac{3(4 + 20s + 105s^2 + 340s^3 + 830s^4 + 1164s^5 + 925s^6 + 500s^7)}{2(2+s)^2(1+5s)v^3}, \\ w_{3,1} &= \frac{1}{2} - \frac{(8 + 26s + 71s^2 + 64s^3 + 55s^4 + 100s^5)}{6s(2+s)(1+5s)v}. \end{aligned}$$

Note that $(1/3, 1/3, 1/3, 3/5) = s_x s_y s_z s_\delta \theta$ is the parameter of solution 38 of Boalch [3]. By direct computation, we conclude that the Bäcklund transformation corresponding to $s_x s_y s_z s_\delta$ transforms $(t, w_{3,1})$ to solution 38.

Remark 4.6. The polynomial h defined above is regarded as the discriminant of the complex reflection group ST27 in the sense of Shephard-Todd [30].

4.21. SOLUTION 37 BY P. BOALCH [3] (LT27)

In this case, z is an algebraic function of y_1, y_2 defined by

$$-y_2 - y_1 z + 2z^3 = 0,$$

$$d(y_1) = \frac{2}{5}, \quad d(y_2) = \frac{3}{5}, \quad d(y_3) = 1, \quad d(z) = \frac{1}{5},$$

$$h_1 = (175y_1y_3 - 70y_1^3z + 70y_1^2z^3 + 378y_1z^5 - 540z^7)/175,$$
$$h_2 = (10y_1^4 - 120y_1y_2^2 + 75y_2y_3 + 30y_1^2z^4 - 192y_1z^6 + 324z^8)/75,$$
$$h_3 = (16y_1^5 + 80y_1^2y_2^2 + 25y_3^2 - 80y_1^3z^4 + 540y_1^2z^6 - 1080y_1z^8 + 432z^{10})/50,$$
$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{4}{15}, -r - \frac{1}{15}, -r + \frac{1}{3}\right].$$

We put $y_1 = mz^2$, where $m = \frac{1+3s+9s^2+5s^3}{3s^2}$. Then we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where p_1, p_2, p_3 are defined by

$$p_1 = 2(4 + 10s + 45s^2 + 145s^3 + 415s^4 + 567s^5 + 400s^6 + 250s^7 - (2+s)^2(1+5s)^2(1+s+4s^2)v)z^5/(225s^5),$$
$$p_2 = 2(4 + 10s + 45s^2 + 145s^3 + 415s^4 + 567s^5 + 400s^6 + 250s^7 + (2+s)^2(1+5s)^2(1+s+4s^2)v)z^5/(225s^5),$$
$$p_3 = -2(8 + 50s + 270s^2 + 875s^3 + 2075s^4 + 2925s^5 + 2375s^6 + 1250s^7)z^5/(225s^5),$$

and v by the equation $v^2 = (1+5s)(1+s+4s^2)$. The algebraic solution $(t, w_{3,1})$ corresponding to $p_{3,1} = -2(-38y_1^2y_2 + 8y_1^3z + 27y_2^2z + 45y_1y_2z^2)/(25y_1)$ has the parameter $\theta = (-\frac{1}{15}, -\frac{1}{15}, -\frac{1}{15}, \frac{3}{5})$ and is uniformized by

$$t = \frac{1}{2} - \frac{3(4 + 20s + 105s^2 + 340s^3 + 830s^4 + 1164s^5 + 925s^6 + 500s^7)}{2(2+s)^2(1+5s)v^3},$$
$$w_{3,1} = \frac{1}{2} - \frac{(1+2s)(2 + 7s + 33s^2 + 31s^3 + 35s^4)}{2(2+s)(1+3s+9s^2+5s^3)v}.$$

Note that $(1/3, 1/3, 1/3, 1/5) = s_xs_ys_zs_\delta s_xs_ys_z\theta$ is the parameter of solution 37 of Boalch [3]. Then it is possible to confirm that the Bäcklund transformation corresponding to $s_xs_ys_zs_\delta s_xs_ys_z$ transforms $(t, w_{3,1})$ to solution 37.

4.22. OCTAHEDRAL SOLUTION 13 BY P. BOALCH [4] (LT30)

In this case,

$$d(y_1) = \frac{1}{8}, \quad d(y_2) = \frac{3}{8}, \quad d(y_3) = 1,$$
$$h_1 = \frac{5}{9}y_1^9 - \frac{28}{3}y_1^6y_2 - \frac{70}{3}y_1^3y_2^2 + \frac{140}{9}y_2^3 + y_1y_3,$$
$$h_2 = \frac{140}{11}y_1^{11} - 15y_1^8y_2 + 84y_1^5y_2^2 + 70y_1^2y_2^3 + y_2y_3,$$
$$h_3 = -\frac{3680}{3}y_1^{16} - \frac{216160}{39}y_1^{13}y_2 + \frac{30016}{3}y_1^{10}y_2^2 - \frac{2240}{3}y_1^7y_2^3 + \frac{39200}{3}y_1^4y_2^4 + \frac{7840}{3}y_1y_2^5 + \frac{1}{2}y_3^2,$$
$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{3}{8}, -r - \frac{1}{8}, -r + \frac{1}{2}\right].$$

We change y_2, y_3 with m, q_3 defined by the relations

$$y_2 = (4m+5)y_1^3, \quad y_3 = -\frac{2}{3}(2025 + 3024m + 1120m^2)y_1^8 + q_3.$$

The entries of the matrix T are polynomials of y_1, m, q_3. Then h is also a polynomial of y_1, m, q_3. Putting

$$m = -\frac{3(u^2 - i)(u^2 + 2iu + 1)(u^2 + 2u - 1)}{2(u^2 + (1+i)u - i)^3},$$

we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned}
p_1 &= \frac{420y_1^8(1 + 76u^4 - 282u^8 + 76u^{12} + u^{16})}{(-i + (1+i)u + u^2)^8}, \\
p_2 &= -\frac{210y_1^8(1 - 48u^2 + 76u^4 - 144u^6 - 282u^8 - 144u^{10} + 76u^{12} - 48u^{14} + u^{16})}{(-i + (1+i)u + u^2)^8}, \\
p_3 &= -\frac{210y_1^8(1 + 48u^2 + 76u^4 + 144u^6 - 282u^8 + 144u^{10} + 76u^{12} + 48u^{14} + u^{16})}{(-i + (1+i)u + u^2)^8}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
p_{2,1} &= \frac{70y_1^8(8019 + 23652m + 25968m^2 + 12544m^3 + 2240m^4)}{99 + 180m + 80m^2}, \\
p_{3,1} &= \frac{70}{3}y_1^8(243 + 864m + 896m^2 + 288m^3), \\
p_{3,2} &= \frac{70y_1^8(1215 + 2700m + 1984m^2 + 480m^3)}{3(5 + 4m)}.
\end{aligned}$$

Then by direct computation, we conclude that the algebraic solution $(t, w_{2,1})$ (resp. $(t, w_{3,1})$, $(t, w_{3,2})$) has the parameter $\theta = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ (resp. $(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{7}{8})$, $(\frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{5}{8})$). Among these three solutions, the simplest one is $(t, w_{3,1})$, where

$$\begin{aligned}
t &= \frac{(-1+u)^2(1+u)^2(1 + 6u^2 + u^4)^3}{(1+u^2)^2(-1 - 2u + u^2)^3(-1 + 2u + u^2)^3}, \\
w_{3,1} &= \frac{(-1+u)(1+u)(-i - (1+i)u + u^2)(1 - 2iu + u^2)^2(1 + 2iu + u^2)}{(-i+u)(i+u)(-1 - 2u + u^2)(-i + (1+i)u + u^2)(-1 + 2u + u^2)^2}.
\end{aligned}$$

Noting that$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}) = P_{yz}P_{xy}P_{yz}s_\delta(s_x s_y s_z s_\delta)^4 s_\infty(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{7}{8})$ is the parameter of solution 13 of Boalch [4], we conclude by direct computation that the solution $(t, w_{3,1})$ is equivalent to solution 13.

Remark 4.7. As was pointed out in [12, §5], the polynomial h is related with E_{14}-singularity.

4.23. THE SOLUTION BY P. BOALCH [4, P.110] (LT32)

$$d(y_1) = \frac{1}{42}, \quad d(y_2) = \frac{1}{3}, \quad d(y_3) = 1,$$

$$\begin{aligned} h_1 &= \frac{90}{43}y_1^{43} + \frac{1230}{29}y_1^{29}y_2 - \frac{369}{2}y_1^{15}y_2^2 - \frac{410}{3}y_1y_2^3 + y_1y_3, \\ h_2 &= -\frac{369}{2}y_1^{56} - 324y_1^{42}y_2 - 2214y_1^{28}y_2^2 + 3321y_1^{14}y_2^3 + 123y_2^4 + y_2y_3, \\ h_3 &= -\frac{35459514}{83}y_1^{84} + 3234654y_1^{70}y_2 + 6687756y_1^{56}y_2^2 + 1680426y_1^{42}y_2^3 \\ &\quad + 16702416y_1^{28}y_2^4 - 6263406y_1^{14}y_2^5 + \frac{77326}{3}y_2^6 + \frac{1}{2}y_3^2, \end{aligned}$$

$$B_\infty^{(3)} = \operatorname{diag}\left[-r - \frac{3}{7}, -r - \frac{5}{42}, -r + \frac{23}{42}\right].$$

Putting $y_2 = my_1^{14}$, where $m = \frac{-91-63s-21s^2-5s^3}{(-7+s)^2 s}$, we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where p_1, p_2, p_3 are defined by

$$\begin{aligned} p_1 = 30758t1^{42}(&- 10045 - 271215s - 281547s^2 - 167433s^3 \\ &- 47628s^4 - 8820s^5 - 903s^6 + 189s^7 + 63s^8 \\ &+ 19s^9 - 15498(1+s)^2(7+s+s^2)v)/3(-7+s)^6s^3, \\ p_2 = 30758t1^{42}(&- 10045 - 271215s - 281547s^2 - 167433s^3 \\ &- 47628s^4 - 8820s^5 - 903s^6 + 189s^7 + 63s^8 \\ &+ 19s^9 + 15498(1+s)^2(7+s+s^2)v)/3(-7+s)^6s^3, \\ p_3 = -15379y_1^{42}(&- 184828 - 1247589s - 950544s^2 - 824964s^3 \\ &- 243432s^4 - 73206s^5 - 20496s^6 - 756s^7 \\ &- 252s^8 + 211s^9)/6(-7+s)^6s^3, \end{aligned}$$

and v by the equation $v^2 = s(7+s+s^2)$. The algebraic solution $(t, w_{3,1})$ corresponding to $p_{3,1} = \frac{2}{3}(486y_1^{42} + 7257y_1^{28}y_2 - 20295y_1^{14}y_2^2 - 6683y_2^3)$ has the parameter $\theta = (5/42, 5/42, 5/42, 41/42)$ and is uniformized by

$$\begin{aligned} t &= \frac{1}{2} - \frac{-784 - 8127s - 7236s^2 - 5208s^3 - 1512s^4 - 378s^5 - 84s^6 + s^9}{432(1+s)^2(7+s+s^2)v}, \\ w_{3,1} &= \frac{1}{2} + \frac{(56 + 5s + 26s^2 + 16s^3 + 4s^4 + s^5)}{36(1+s)v}. \end{aligned}$$

Note that $(4/7, 4/7, 4/7, 1/3) = s_x s_y s_z s_\delta s_\infty \theta$ is the parameter of the solution obtained by P. Boalch [4, p.110]. It is possible to confirm by direct computation that the Bäcklund transformation corresponding to $s_x s_y s_z s_\delta s_\infty$ transforms $(t, w_{3,1})$ to solution obtained by P. Boalch.

4.24. SOLUTION OBTAINED BY A. KITAEV [17, P.15] (LT33)

In this case, z is an algebraic function of y_1, y_2 defined by the relation

$$95y_2 + 19y_1z^5 + 5z^{19} = 0,$$

$$d(y_1) = \frac{1}{3}, \quad d(y_2) = \frac{19}{42}, \quad d(y_3) = 1, \quad d(z) = \frac{1}{42},$$

$$\begin{aligned}
h_1 &= (2553y_1^4 + 74y_1y_3 + 45954y_1^3z^{14} + 33966y_1^2z^{28} + 8154y_1z^{42} + 621z^{56})/74,\\
h_2 &= (28670y_2y_3 + 1450702y_1^4z^5 + 725351y_1^3z^{19}\\
&\quad + 335439y_1^2z^{33} + 114741y_1z^{47} + 13959z^{61})/28670,\\
h_3 &= (296769y_1^6 + 5y_3^2 - 7284330y_1^5z^{14} - 4909005y_1^4z^{28} - 633420y_1^3z^{42} + 250695y_1^2z^{56}\\
&\quad + 82134y_1z^{70} + 6885z^{84})/10,
\end{aligned}$$

$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{11}{42}, -r - \frac{1}{7}, -r + \frac{17}{42}\right].$$

Then putting $y_1 = mz^{14}$, where $m = \frac{-14-21s-s^3}{(-7+s)^2s}$, we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where p_1, p_2, p_3 are defined by

$$\begin{aligned}
p_1 = 56(&-12397 - 111573s - 108108s^2 - 72177s^3\\
&- 21609s^4 - 5292s^5 - 1218s^6 - 63s^7 + 13s^9\\
&- 8694(1+s)^2(7+s+s^2)v)z^{42}/(-7+s)^6s^3,\\
p_2 = 56(&-12397 - 111573s - 108108s^2 - 72177s^3\\
&- 21609s^4 - 5292s^5 - 1218s^6 - 63s^7 + 13s^9\\
&+ 8694(1+s)^2(7+s+s^2)v)z^{42}/(-7+s)^6s^3,\\
p_3 = -14(&-76636 - 862155s - 732564s^2 - 549780s^3\\
&- 156996s^4 - 39690s^5 - 8652s^6 + 252s^7\\
&+ 109s^9)z^{42}/(-7+s)^6s^3,
\end{aligned}$$

and v by the equation $v^2 = s(7+s+s^2)$. The algebraic solution $(t, w_{3,1})$ corresponding to $p_{3,1} = (1771y_1^4 + 16146y_1^3z^{14} + 11898y_1^2z^{28} + 2826y_1z^{42} + 207z^{56})/(7y_1)$ has the parameter $\theta = (1/7, 1/7, 1/7, 2/3)$ and is uniformized by

$$\begin{aligned}
t &= \frac{1}{2} - \frac{-784 - 8127s - 7236s^2 - 5208s^3 - 1512s^4 - 378s^5 - 84s^6 + s^9}{432(1+s)^2(7+s+s^2)v},\\
w_{3,1} &= \frac{1}{2} - \frac{s(-623 - 330s - 279s^2 - 62s^3 - 3s^4 + s^6)}{12(1+s)(14+21s+s^3)v}.
\end{aligned}$$

We now recall the algebraic solution (t_{KiA}, w_{KiA}) (cf. [17, p.15]) defined by

$$\begin{aligned} t_{KiA} &= \frac{1}{2} \\ &\quad - \frac{(14s^9 - 105s^8 + 252s^7 - 392s^6 + 420s^5 - 336s^4 + 112s^3 + 72s^2 - 96s + 32)}{16(s+2)^2(s-1)^3(s^2-s+1)v_b}, \\ w_{KiA} &= 1 + \frac{(3s-2)(s^2-2s+4)^2}{4(s+2)(s-1)^2(s^2-s+1)(3s^2-4s+4)} \\ &\qquad \times \frac{(-14s^5 + 25s^4 - 20s^3 - 8s^2 + 16s - 8 - 8(s-1)(s^2-s+1)vb)}{(2s+1)(3s^3-10s^2+6s-2) - 14(s-1)v_b}, \end{aligned}$$

where v_b is defined by the equation $v_b^2 - (2s+1)(1-s)(s^2-s+1) = 0$. Moreover, it is known (cf. [20]) that (t_{KiA}, w_{KiA}) has the parameter $(1/3, 1/7, 1/7, 6/7)$. It is easy to show that

$$s_\infty s_x r_x \theta = (1/3, 1/7, 1/7, 6/7)$$

and that $s_\infty s_x r_x$ transforms the algebraic solution $(t, w_{3,1})$ to $(t, t/w_{3,1})$. By the substitution $s = \frac{(-1-2s_1}{(-1+s_1)}$, $v = \frac{3v_b}{(-1+s1)^2}$, we conclude that $(t, t/w_{3,1})$ turns out to be $(t_{KiA}, w_{KiA})|_{s\to s_1}$. As a consequence, $(t, w_{3,1})$ is equivalent to (t_{KiA}, w_{KiA}).

4.25. THE 2,3,7 SOLUTION BY P. BOALCH [4] (LT34)

In this case, z is an algebraic function of y_1, y_2 defined by the relation

$$-\frac{121}{672}y_1^2 + y_2 z^6 + y_1 z^{17} + z^{34} = 0.$$

$$d(y_1) = \frac{17}{42}, \quad d(y_2) = \frac{2}{3}, \quad d(y_3) = 1, \quad d(z) = \frac{1}{42},$$

$$\begin{aligned} h_1 &= (133891945y_1^4 + 133891945y_1y_3z^9 - 1205027505y_1^3z^{17} \\ &\qquad - 57568896y_1^2z^{34} + 2355091200y_1z^{51} + 1327656960z^{68})/(133891945z^9), \\ h_2 &= (-187785466y_1^5 + 217272440y_2y_3z^{15} - 2735304825y_1^4z^{17} - 16165069536y_1^3z^{34} \\ &\qquad - 34015167648y_1^2z^{51} - 17644290048y_1z^{68} - 3446489088z^{85})/(217272440z^{15}), \\ h_3 &= (115727222325y_1^6 - 65143905512400y_1^5z^{17} + 11869458700y_3^2z^{18} \\ &\qquad + 58725922432320y_1^4z^{34} + 107178140805120y_1^3z^{51} + 87229921185792y_1^2z^{68} \\ &\qquad + 564378845184 00y_1z^{85} + 9062315458560z^{102})/(23738917400z^{18}), \end{aligned}$$

$$B_\infty^{(3)} = \operatorname{diag}\left[-r - \frac{2}{7}, -r - \frac{1}{42}, -r + \frac{13}{42}\right].$$

Then putting $y_1 = mz^{17}$, where $m = -\frac{56(17+15s+3s^2+s^3)}{11s(-7+s)^2}$, we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned} p_1 = &-8804096(-490 - 6615s - 4347s^2 - 4956s^3 - 756s^4 - 189s^5 - 147s^6 \\ &+ 4s^9 - 1890(1+s)^2(7+s+s^2)v)z^{42}/6655(-7+s)^6s^3, \\ p_2 = &-8804096(-490 - 6615s - 4347s^2 - 4956s^3 - 756s^4 - 189s^5 - 147s^6 \\ &+ 4s^9 + 1890(1+s)^2(7+s+s^2)v)z^{42}/6655(-7+s)^6s^3, \\ p_3 = &\ 2201024(-25480 - 257985s - 235872s^2 - 162456s^3 - 49896s^4 - 12474s^5 \\ &- 2352s^6 + 19s^9)z^{42}/6655(-7+s)^6s^3, \end{aligned}$$

where v is defined by the equation $v^2 = s(7+s+s^2)$. The algebraic solution $(t, w_{3,1})$ corresponding to

$$p_{3,1} = \frac{17203175y_1^4 - 198611820y_1^3z^{17} - 162705312y_1^2z^{34} + 9757440y_1z^{51} + 31610880z^{68}}{1244485y_1z^9}$$

has the parameter $\theta = (1/42, 1/42, 1/42, 25/42)$ and is uniformized by $(t, w_{3,1})$, where

$$\begin{aligned} t &= \frac{1}{2} - \frac{-784 - 8127s - 7236s^2 - 5208s^3 - 1512s^4 - 378s^5 - 84s^6 + s^9}{432(1+s)^2(7+s+s^2)v}, \\ w_{3,1} &= \frac{1}{2} + \frac{(28 + 258s + 201s^2 + 124s^3 + 30s^4 + 6s^5 + s^6)}{6(1+s)(17+15s+3s^2+s^3)v)}. \end{aligned}$$

Note that $(2/7, 2/7, 2/7, 1/3) = s_\delta(s_xs_ys_zs_\delta)^4\theta$ is the parameter of the 2,3,7 solution obtained by P. Boalch [4, p.104]. By the Bäcklund transformation corresponding to $s_\delta(s_xs_ys_zs_\delta)^4$, $(t, w_{3,1})$ is transformed to (t_a, w_a), which coincides with the one shown in Boalch [4].

4.26. SOLUTION 46 BY P. BOALCH [3] (LT39)

In this case, z is an algebraic function of y_1, y_2 defined by the relation

$$y_1^2 + 25(-y_2 + z^2) = 0,$$

$$d(y_1) = \frac{1}{4}, \quad d(y_2) = \frac{1}{2}, \quad d(y_3) = 1, \quad d(z) = \frac{1}{4}$$

$$\begin{aligned} h_1 &= t_1t_3 + \frac{183}{2000}t_1^5 + \frac{11}{8}t_1^3z^2 + \frac{75}{16}t_1z^4 + 25z^5, \\ h_2 &= t_2t_3 + \frac{11}{12500}t_1^6 - \frac{139}{100}t_1^4z^2 - \frac{27}{4}t_1^2z^4 + 8t_1z^5 - \frac{25}{4}z^6, \\ h_3 &= \frac{1}{2}t_3^2 + \frac{128991}{160000}t_1^8 + \frac{6479}{1600}t_1^6z^2 + \frac{285}{128}t_1^4z^4 \\ &\quad - 215t_1^3z^5 + \frac{375}{64}t_1^2z^6 - 375t_1z^7 + \frac{59375}{256}z^5, \end{aligned}$$

$$B_\infty^3 = \operatorname{diag}\left[-r-\frac{1}{3}, -r-\frac{1}{12}, -r+\frac{5}{12}\right].$$

Putting $y_1 = mz$, where $m = \frac{(-4+6s-3s^2-2s^3)}{3s^2}$, we find that $h = (y_3-p_1)(y_3-p_2)(y_3-p_3)$, where p_1, p_2, p_3 are defined by

$$\begin{aligned}
p_1 = z^4(&-5504 + 33024s - 90816s^2 + 137600s^3 - 86760s^4 + 141984s^5 \\
&- 263784s^6 + 269496s^7 + 50850s^8 + 40880s^9 + 29184s^{10} + 14136s^{11} \\
&+ 2356s^{12} + 675(2+s)^2(2-2s+3s^2)^2(2-7s+8s^2)v)/(8100s^8), \\
p_2 = z^4(&-5504 + 33024s - 90816s^2 + 137600s^3 - 86760s^4 + 141984s^5 \\
&- 263784s^6 + 269496s^7 + 50850s^8 + 40880s^9 + 29184s^{10} + 14136s^{11} \\
&+ 2356s^{12} - 675(2+s)^2(2-2s+3s^2)^2(2-7s+8s^2)v)/(8100s^8), \\
p_3 = -z^4(&-37696 + 226176s - 621984s^2 + 942400s^3 - 844740s^4 + 781416s^5 \\
&- 983616s^6 + 791604s^7 - 20475s^8 - 54380s^9 + 19416s^{10} + 18264s^{11} \\
&+ 3044s^{12})/(8100s^8),
\end{aligned}$$

and v by the equation $v^2 = (2+s)(2-7s+8s^2)$. The algebraic solution $(t, w_{3,1})$ corresponding to $p_{3,1} = \frac{(589m^5+6450m^3-6500m^2+13125m+37500)z^4}{400m}$ has the parameter $\theta = (1/12, 1/12, 1/12, 3/4)$ and is uniformized by

$$\begin{aligned}
t = \frac{1}{2} &+ (-2+4s+s^2)(32 - 128s + 288s^2 - 288s^3 + 258s^4 \\
&- 312s^5 + 429s^6 - 84s^7 + 24s^8 + 16s^9 \\
&+ 8s^{10})/2(2+s)(2-2s+3s^2)^2v^3, \\
w_{3,1} = \frac{1}{2} &+ (-2+s)(-16 + 48s - 64s^2 + 22s^3 - 34s^4 + 91s^5 + 30s^6 \\
&+ 4s^7)/2(2+s)(2-2s+3s^2)(4-6s+3s^2+2s^3)v.
\end{aligned}$$

It is easy to show that $s_x s_y s_x s_\delta s_x s_y s_z \theta = (1/3, 1/3, 1/3, 1/2)$. The latter parameter corresponds to solution 46 obtained by P. Boalch [3]. Let $(\tilde{t}, \tilde{w})$ be the algebraic solution obtained from $(t, w_{3,1})$ by the Bäcklund transformation corresponding to $s_x s_y s_x s_\delta s_x s_y s_z$. Then

$$\begin{aligned}
\tilde{t} = \frac{1}{2} &+ (-2+4s+s^2)(32 - 128s + 288s^2 - 288s^3 + 258s^4 - 312s^5 \\
&+ 429s^6 - 84s^7 + 24s^8 + 16s^9 \\
&+ 8s^{10})/2(2+s)^2(2-2s+3s^2)^2(2-7s+8s^2)u, \\
\tilde{w} = \frac{1}{2} &+ (-64 + 320s - 736s^2 + 880s^3 + 1324s^4 - 3272s^5 + 2668s^6 \\
&+ 808s^7 + 415s^8 + 458s^9 + 302s^{10} \\
&+ 56s^{11})/2(2+s)(2-2s+3s^2)(8 - 24s + 30s^2 \\
&- 10s^3 + 105s^4 + 6s^5 + 2s^6)u.
\end{aligned}$$

Now we recall solution 46 obtained by Boalch. We denote it by $(t_{\mathrm{Bo46}}, w_{\mathrm{Bo46}})$. Then $t_{\mathrm{Bo46}} = \tilde{t}$ and

$$\begin{aligned} w_{\mathrm{Bo46}} = \frac{1}{2} - (&416 - 2288s + 6216s^2 - 9836s^3 + 9760s^4 - 8312s^5 + 6562s^6 - 3143s^7 \\ &- 242s^8 + 50s^9 + 72s^{10} + 16s^{11}) \\ &/2(2 - 2s + 3s^2)(104 - 312s + 270s^2 + 50s^3 - 75s^4 + 18s^5 + 26s^6)u. \end{aligned}$$

To identify $(\tilde{t}, \tilde{w})$ with $(t_{\mathrm{Bo46}}, w_{\mathrm{Bo46}})$, we define a map $\varphi : (s, u) \to (s_1, v)$ by

$$s = \frac{2(1 - s_1)}{2 + s_1}, \quad u = -\frac{4v}{2 + s_1},$$

where $v = \pm\sqrt{(s_1 + 2)(8s_1^2 - 7s_1 + 2)}$. Then it is straightforward to show that

$$(\tilde{t}, \tilde{w}) \circ \varphi = (t_{\mathrm{Bo46}}, w_{\mathrm{Bo46}})|_{s \to s_1},$$

which implies that $(t, w_{3,1})$ is equivalent to solution 46.

4.27. SOLUTION III

In this case,

$$d(y_1) = \frac{1}{3}, \quad d(y_2) = \frac{2}{3}, \quad d(y_3) = 1$$

$$\begin{aligned} h_1 &= \frac{1}{2}(27\delta(1 + 6\delta)y_1^4 + 36\delta y_1^2 y_2 + y_2^2) + y_1 y_3, \\ h_2 &= -\frac{9}{10}(-1 + 2\delta)y_1(81\delta(3 + 2\delta)y_1^4 + 30y_1^2 y_2 - 5y_2^2) + y_2 y_3, \\ h_3 &= -\frac{3}{10}(1 + 2\delta)(729(-1 + 2\delta)(1 + 4\delta^2)y_1^6 + 1215\delta(-1 + 2\delta)y_1^4 y_2 \\ &\qquad + 135(-1 + 2\delta)y_1^2 y_2^2 + 5y_2^3) + \frac{1}{2}y_3^2. \end{aligned}$$

$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{1}{3}, -r, -r + \frac{1}{3}\right]$$

$$\begin{aligned} h = &(27(1 + 2\delta)^2 y_1^3 + 9(1 + 2\delta)y_1 y_2 + y_3) \\ &\times (729(-1 + 2\delta)^2(-1 + 2\delta + 4\delta^2)y_1^6 + 486(-1 + 2\delta)^2(1 + 2\delta)y_1^4 y_2 \\ &+ 27(-1 + 2\delta)(1 + 6\delta)y_1^2 y_2^2 + 8y_2^3 + 108\delta(-1 + 2\delta)y_1^3 y_3 \\ &+ 18(-1 + 2\delta)y_1 y_2 y_3 + y_3^2). \end{aligned}$$

(δ is a constant.)

Putting $y_2 = -\frac{1}{2}(-9 + 18\delta + m^2)y_1^2$, we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$\begin{aligned} p_1 &= \frac{9}{2}(1 + 2\delta)(-15 + 6\delta + m^2)y_1^3, \\ p_2 &= \frac{1}{2}(81 - 216\delta + 108\delta^2 - 9m^2 + 18\delta m^2 - 2m^3)y_1^3, \\ p_3 &= \frac{1}{2}(81 - 216\delta + 108\delta^2 - 9m^2 + 18\delta m^2 + 2m^3)y_1^3. \end{aligned}$$

The algebraic solution $(t, w_{3,1})$ corresponding to $p_{3,1} = \frac{1}{6}(-27+18\delta+m^2)(-9+18\delta+2m^2)y_1^3$ is uniformized by

$$t = \frac{(-1+s)^2(2+s)}{(1+s)^2(-2+s)}, \quad w_{3,1} = \frac{(-1+s)(2+s)}{s(1+s)},$$

where $m = \frac{6}{s}$. This coincides with Solution III given in [20].

Remark 4.8. We now discuss the case where the potential vector field is integrable. For this purpose we introduce

$$F = \frac{1}{2}(y_1y_3^2 + y_2^2y_3) + cy_1y_2^3 + 6c^2y_1^3y_2^2 + \frac{216c^4}{35}y_1^7,$$

where c is a constant. It is clear from the definition that F is a prepotential. If $c = \frac{1}{6}$, then F coincides with the second prepotential in the introduction under the identification $(y_1, y_2, y_3) = (x_1, x_2, x_3)$. On the other hand, if $c = -\frac{3}{2}$ and $\delta = 0$, then $(\partial_{y_3}F, \partial_{y_2}F, \partial_{y_1}F) = (h_1, h_2, h_3)$. In this case, we find that

$$h = (27y_1^3 + 9y_1y_2 + y_3)(-729y_1^6 + 486y_1^4y_2 - 27y_1^2y_2^2 + 8y_2^3 - 18y_1y_2y_3 + y_3^2)$$

is regarded as the discriminant of the Weyl group of type B_3.

4.28. SOLUTION IV

In this case,

$$d(y_1) = \frac{3}{2q}, \quad d(y_2) = \frac{1}{2}, \quad d(y_3) = 1,$$

$$B_\infty^{(3)} = \mathrm{diag}\left[-r - \frac{q-2}{2q}, -r - \frac{1}{2q}, -r + \frac{q-1}{2q}\right],$$

$$\begin{aligned} h_1 &= y_1y_3 + \frac{16}{q}y_1y_2^2, \\ h_2 &= y_2y_3 + \frac{1}{12q(q-1)}(-64(q-1)(q-2)y_2^3 + 3q^2y_1^q), \\ h_3 &= \frac{1}{2}y_3^2 + \frac{1}{3q^2}(256(q-1)y_2^4 + 24q^2y_1^qy_2), \end{aligned}$$

$$\begin{aligned} h = \frac{1}{4q^3}(&27q^3u_1^2 + 256(8q-9)q^2u_1y_2^3 + 16384(1-q)y_2^6 \\ &- 144q^3u_1y_2y_3 + 1024(3-2q)qy_2^4y_3 + 64(3-q)q^2y_2^2y_3^2 + 4q^3y_3^3). \end{aligned}$$

(We put $u_1 = y_1^q$ for simplicity.)

Introduce a parameter s by

$$u_1 = \frac{128(-1+s)^2(2+s)^2(1+2s)^2y_2^3}{27(1+s+s^2)^3}.$$

Then we find that $h = (y_3 - p_1)(y_3 - p_2)(y_3 - p_3)$, where

$$p_1 = \frac{16y_2^2\{-3(1+s+s^2)^2 + q(-1+s)(1+2s)(5+5s+2s^2)\}}{3q(1+s+s^2)^2},$$
$$p_2 = \frac{16y_2^2\{-3(1+s+s^2)^2 + q(2+s)(1+2s)(2-s+2s^2)\}}{3q(1+s+s^2)^2},$$
$$p_3 = -\frac{16y_2^2\{3(1+s+s^2)^2 + q(-1+s)(2+s)(2+5s+5s^2)\}}{3q(1+s+s^2)^2}.$$

The algebraic solution $(t, w_{3,2})$ corresponding to

$$p_{3,2} = \frac{16y_2^2\{q(-1+s)^2(2+s)^2(1+2s)^2 - 3(1+s+s^2)^3\}}{3q(1+s+s^2)^3}$$

is uniformized by

$$t = -\frac{(-1+s)(1+s)^3}{1+2s}, \quad w_{3,2} = -\frac{(-1+s)(1+s)^2}{1+s+s^2}.$$

Then $(t, w_{3,2})$ coincides with Solution IV given in [20] by the substitution $s \to -s_1 - 1$.

Remark 4.9. As in the case of Solution III, we discuss the case of Solution IV where the potential vector field is integrable. The polynomial

$$F = \frac{1}{2}(y_1y_3^2 + y_2^2y_3) + \frac{c}{2}y_1^2y_2^2 + \frac{c^2}{15}y_2^5,$$

where c is a constant, is a prepotential. If $c = \frac{1}{2}$, then F coincides with the first prepotential in the introduction under the identification $(y_1, y_2, y_3) = (x_1, x_2, x_3)$. On the other hand, if $c = 8$ and $q = 2$ and $y_1 \leftrightarrow y_2$, then $(\partial_{y_3}F, \partial_{y_1}F, \partial_{y_2}F) = (h_2, h_1, h_3)$. This means that changing the roles of $y_1$and y_2 and also the roles of h_1 and h_2, we find that F is a prepotential and (h_2, h_1, h_3) is the associated potential vector field. In this case, we find that

$$h = \frac{27}{4}y_1^4 + 224y_1^2y_2^3 - 512y_2^6 - 36y_1^2y_2y_3 - 64y_2^4y_3 + 8y_2^2y_3^2 + y_3^3$$

is regarded as the discriminant of the Weyl group of type A_3. Indeed, the discriminant of $x^4 - 4y_2x^2 + y_1x - \frac{1}{4}y_3 + 2y_2^2$ as a polynomial x coincides with h.

Remark 4.10. In this case, it follows from the definition that (h_1, h_2, h_3) is a polynomial potential vector field if q is a natural number. As a consequence, there are an infinite number of polynomial potential vector fields. On the other hand, (h_1, h_2, h_3) is an algebraic potential vector field if q is a rational number and is a non-algebraic potential vector field if q is not rational.

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