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EXISTENCE OF POSITIVE SOLUTIONS TO A DISCRETE FRACTIONAL BOUNDARY VALUE PROBLEM AND CORRESPONDING LYAPUNOV-TYPE INEQUALITIES

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Abstract. We prove existence of positive solutions to a boundary value problem depending on discrete fractional operators. Then, corresponding discrete fractional Lyapunov-type inequalities are obtained.

Keywords: fractional difference equations, Lyapunov-type inequalities, fractional boundary value problems, positive solutions.

Mathematics Subject Classification: 26A33, 26D15, 39A12.

1. INTRODUCTION

Recently, a large debate appeared regarding Lyapunov-type inequalities – see, e.g., [5,8,13,16] and references therein. In 1907, Lyapunov proved in [15] that if $q:[a,b]\to\mathbb{R}$ is a continuous function, then a necessary condition for the boundary value problem

$$\begin{cases} y'' + qy = 0, \quad a < t < b, \\ y(a) = y(b) = 0 \end{cases} \tag{1.1}$$

to have a nontrivial solution is given by

$$\int_a^b |q(s)|\, ds > \frac{4}{b-a}.$$

Ferreira has succeed to generalize the above classical result to the case when the second-order derivative in (1.1) is substituted by a fractional operator of order α, in Caputo or Riemann–Liouville sense [6, 7]. More recently, the authors obtained in [5] a generalized Lyapunov-type inequality for the following fractional boundary value problem:

$$\begin{cases} {}_aD^{\alpha} y + q(t) f(y) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.2}$$

where ${}_aD^{\alpha}$ is the Riemann–Liouville derivative, $1 < \alpha \leq 2$, and $q : [a, b] \to \mathbb{R}_+$ is a Lebesgue integrable function.

Theorem 1.1 ([5])**.** *Let $q : [a, b] \to \mathbb{R}_+$ be a real Lebesgue integrable function. Assume that $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a concave and nondecreasing function. If the fractional boundary value problem* (1.2) *has a nontrivial solution, then*

$$\int_a^b q(t) dt > \frac{4^{\alpha-1} \Gamma(\alpha) \eta}{(b-a)^{\alpha-1} f(\eta)}, \tag{1.3}$$

where $\eta = \max_{t \in [a,b]} y(t)$.

Here we are concerned with the discrete fractional calculus [2, 10]. It turns out that Lyapunov fractional inequalities can also be obtained by considering a discrete fractional difference in (1.1) instead the Caputo or Riemann–Liouville derivatives [8]. Motivated by the results obtained in [1, 5, 8, 9], we prove here some generalizations of the Lypunov inequality of [8]. The new inequalities are, in some sense, similar to that of (1.3) (compare with (3.6) and (3.7)) but, instead of (1.2), they involve the following discrete fractional boundary value problem:

$$\begin{cases} \triangle^{\alpha} y(t) + q(t+\alpha-1) f(y(t+\alpha-1)) = 0, & 1 < \alpha \leq 2, \\ y(\alpha-2) = y(\alpha+b+1) = 0, \quad b \geq 2, & b \in \mathbb{N}, \end{cases} \tag{1.4}$$

where operator $\triangle^{\alpha}$ is defined in Section 2. Interestingly, we show that the hypothesis found in Theorem 1.1, assuming the nonlinear term f to be concave, can be removed in the discrete setting (see Theorems 3.5 and 3.7).

The paper is organized as follows. In Section 2, we recall some notations, definitions and preliminary facts, which are used throughout the work. Our original results are then given in Section 3: using the Guo–Krasnoselskii fixed point theorem, we establish in Section 3.1 an existence result for the discrete fractional boundary value problem (1.4) (see Theorem 3.3); then, in Section 3.2, assuming that function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is only continuous and nondecreasing, we generalize the Lyapunov inequality given in [8, Theorem 3.1] (see Theorems 3.5 and 3.7). Examples illustrating the new results are given.

2. PRELIMINARIES

In this section, we recall some notations, definitions and preliminary facts, which are used throughout the text. We begin by recalling the well-known definition of power function:

$$x^{[y]} = \frac{\Gamma(x+1)}{\Gamma(x-y+1)}$$

for any x and y for which the right-hand side is defined. We borrow from [3] the following notation:

$$\mathbb{N}_a := \{a, a+1, a+2, \ldots\}, \quad a \in \mathbb{R}.$$

Definition 2.1. For a function $f : \mathbb{N}_a \to \mathbb{R}$, the discrete fractional sum of order $\alpha \geq 0$ is defined by

$$({}_a\triangle_t^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{[\alpha-1]} f(s), \quad t \in \mathbb{N}_{a+\alpha}.$$

Definition 2.2. For a function $f : \mathbb{N}_a \to \mathbb{R}$, the discrete fractional difference of order $\alpha > 0$ ($n-1 \leq \alpha \leq n$, where $n \in \mathbb{N}$) is defined by

$$(\triangle^\alpha f)(t) = (\triangle^n \, {}_a\triangle_t^{-(n-\alpha)} f)(t), \quad t \in \mathbb{N}_{a+n-\alpha},$$

where $\triangle^n$ is the standard forward difference of order n.

The reader interested on more details about the discrete fractional calculus is referred to [1, 8–10].

Definition 2.3. Let X be a real Banach space. A nonempty closed convex set $P \subset X$ is called a cone if it satisfies the following two conditions:

(i) $x \in P$, $\lambda \geq 0$, implies $\lambda x \in P$;
(ii) $x \in P$, $-x \in P$, implies $x = 0$.

Lemma 2.4 (Guo–Krasnoselskii fixed point theorem [12])**.** *Let X be a Banach space and let $K \subset X$ be a cone. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator such that*

(i) $\|Tu\| \geq \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for any* $u \in K \cap \partial\Omega_2$*; or*
(ii) $\|Tu\| \leq \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for any* $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. MAIN RESULTS

Let us consider the nonlinear discrete fractional boundary value problem (1.4). We deal with its sum representation involving a Green function.

Lemma 3.1. *Function y is a solution to the boundary value problem* (1.4) *if and only if y satisfies*

$$y(t)=\sum_{s=0}^{b+1}G(t,s)q(s+\alpha-1)f(y(s+\alpha-1)),$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\frac{t^{[\alpha-1]}(\alpha+b-s)^{[\alpha-1]}}{(\alpha+b+1)^{[\alpha-1]}}-(t-s-1)^{[\alpha-1]}, & s<t-\alpha+1\leq b+1,\\ \frac{t^{[\alpha-1]}(\alpha+b-s)^{[\alpha-1]}}{(\alpha+b+1)^{[\alpha-1]}}, & t-\alpha+1<s\leq b+1\end{cases}\tag{3.1}$$

is the Green function associated to problem (1.4)*.*

Proof. Similar to the one found in [1]. □

Lemma 3.2. *The Green function G given by* (3.1) *satisfies the following properties:*

1. $G(t,s)>0$ *for all* $t\in[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}$ *and* $s\in[1,b+1]_{\mathbb{N}_1}$*;*
2. $\max_{t\in[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}}G(t,s)=G(s+\alpha-1,s),\ s\in[1,b+1]_{\mathbb{N}_1}$*;*
3. $G(s+\alpha-1,s)$ *has a unique maximum given by*

$$\max_{s\in[1,b+1]_{\mathbb{N}_1}}G(s+\alpha-1,s)=\begin{cases}\frac{1}{4}\frac{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)}{\Gamma(\alpha)\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)}, & \textit{if } b \textit{ is even},\\ \frac{1}{\Gamma(\alpha)}\frac{\Gamma(b+3)\Gamma^2(\frac{b+1}{2}+\alpha)}{\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})}, & \textit{if } b \textit{ is odd};\end{cases}$$

4. *there exists a positive constant* $\lambda\in(0,1)$ *such that*

$$\min_{t\in\left[\frac{b+\alpha}{4},\frac{3(b+\alpha)}{4}\right]_{\mathbb{N}_{\alpha-1}}}G(t,s)\geq\lambda\max_{t\in[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}}G(t,s)=\lambda G(s+\alpha-1,s)$$

for $s\in[1,b+1]_{\mathbb{N}_1}$.

Proof. Similar to the one found in [8]. □

3.1. EXISTENCE OF POSITIVE SOLUTIONS

Let us consider the Banach space

$$X:=\left\{y:[\alpha-2,\alpha+b+1]_{\mathbb{N}_{\alpha-2}}\to\mathbb{R},\ y(\alpha-2)=y(\alpha+b+1)=0\right\}$$

with the supremum norm. In agreement with Lemma 2.4, to prove existence of a solution to the discrete fractional boundary value problem (1.4), it suffices to prove that a suitable map T has a fixed point in X. We are interested to prove existence of nontrivial positive solutions to (1.4), which are the ones to have a physical meaning [14]. For that, we consider the following two hypotheses:

(H_1) $f(y)\geq\overset{*}{\gamma}r_1$ for $y\in[0,r_1]$,
(H_2) $f(y)\leq\gamma r_2$ for $y\in[0,r_2]$,

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. In what follows, we take

$$\gamma := \left(\sum_{s=0}^{b+1} G(s+\alpha-1, s) q(s+\alpha-1) \right)^{-1} \tag{3.2}$$

and

$$\overset{*}{\gamma} := \left(\sum_{s=\frac{b+\alpha}{4}}^{\frac{3(b+\alpha)}{4}} \lambda G(s+\alpha-1, s) q(s+\alpha-1) \right)^{-1}. \tag{3.3}$$

Theorem 3.3. *Let $q : [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}_+$ be a nontrivial function. Assume that there exist two positive constants $r_2 > r_1 > 0$ such that the assumptions (H_1) and (H_2) are satisfied. Then the discrete fractional boundary value problem* (1.4) *has at least one nontrivial positive solution y belonging to X such that $r_1 \leq \|y\| \leq r_2$.*

Proof. First of all, we define the operator $T : X \to X$ as follows:

$$Ty(t) = \sum_{s=0}^{b+1} G(t,s) q(s+\alpha-1) f(y(s+\alpha-1)). \tag{3.4}$$

We use Lemma 2.4 with the following cone K:

$$K := \left\{ y \in X : \min_{\left[\frac{b+\alpha}{4}, \frac{3(b+\alpha)}{4}\right]_{\mathbb{N}_{\alpha-1}}} y(t) \geq \lambda \|y\| \right\}.$$

To prove existence of a nontrivial solution to the fractional discrete problem (1.4) amounts to show existence of a fixed point to the operator T in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. From Lemma 3.2, we get that $T(K) \subset K$. Taking into account that T is a summation operator on a discrete finite set, it follows that $T : K \to K$ is a completely continuous operator. Now, it remains to consider the first part (i) of Lemma 2.4 to prove our result. Let $\Omega_i = \{y \in K : \|y\| \leq r_i\}$. From (H_1), we have for $t \in \left[\frac{b+\alpha}{4}, \frac{3(b+\alpha)}{4}\right]_{\mathbb{N}_{\alpha-1}}$ and $y \in K \cap \partial\Omega_1$ that

$$\begin{aligned}(Ty)(t) &\geq \sum_{s=0}^{b+1} \min_{t \in \left[\frac{b+\alpha}{4}, \frac{3(b+\alpha)}{4}\right]_{\mathbb{N}_{\alpha-1}}} G(t,s) q(s+\alpha-1) f(y(s+\alpha-1)) \\ &\geq \overset{*}{\gamma} \left(\sum_{s=0}^{b+1} \lambda G(s+\alpha-1, s) q(s+\alpha-1) \right) r_1 = \|y\|.\end{aligned}$$

Thus, $\|Ty\| \geq \|y\|$ for $y \in K \cap \partial\Omega_1$. Let us now prove that $\|Ty\| \leq \|y\|$ for all $y \in K \cap \partial\Omega_2$. From (H_2), it follows that

$$\begin{aligned}\|Ty\| &= \max_{t\in[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} \sum_{s=0}^{b+1} G(t,s)q(s+\alpha-1)f(y(s+\alpha-1)) \\ &\leq \gamma\left(\sum_{s=0}^{b+1} G(s+\alpha-1,s)q(s+\alpha-1)\right) r_2 = \|y\|\end{aligned}$$

for $y \in K \cap \partial\Omega_2$. Thus, from Lemma 2.4, we conclude that the operator T defined by (3.4) has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. Therefore, the discrete fractional boundary problem (1.4) has at least one positive solution belonging to X such that $r_1 \leq \|y\| \leq r_2$. $\square$

Example 3.4. Consider the following discrete fractional boundary value problem:

$$\begin{cases} \triangle^{\frac{3}{2}} y(t) + \frac{(\frac{2t+1}{2})}{y(\frac{2t+1}{2})+20} = 0, \\ y(-\frac{1}{2}) = y(\frac{11}{2}) = 0. \end{cases} \tag{3.5}$$

Note that this problem is of type (1.4) with $b = 3$. The value (3.2) of γ is given by

$$\gamma = \left(\sum_{s=0}^{4} \frac{1}{\Gamma(\frac{3}{2})} \frac{\Gamma(6)\Gamma^2\left(2+\frac{3}{2}\right)}{\Gamma(5+\frac{3}{2})\Gamma^2(3)} \left(\frac{2s+1}{2}\right)\right)^{-1} \approx 0.0616$$

while, from formula (3.3) of [1], the value (3.3) of $\overset{*}{\gamma}$ becomes

$$\overset{*}{\gamma} = \left(\sum_{s=0}^{4} 0.03779 \frac{1}{\Gamma(\frac{3}{2})} \frac{\Gamma(6)\Gamma^2(2+\frac{3}{2})}{\Gamma(5+\frac{3}{2})\Gamma^2(3)} \left(\frac{2s+1}{2}\right)\right)^{-1} \approx 1.6301.$$

Choose $r_1 = 1/100$ and $r_2 = 1$. Then, one gets

1. $f(y) \geq \overset{*}{\gamma} r_1$ for $y \in [0, 1/100]$;
2. $f(y) \leq \gamma r_2$ for $y \in [0, 1]$.

Therefore, from Theorem 3.3, problem (3.5) has at least one nontrivial solution y in X such that $y \in [1/100, 1]$.

3.2. GENERALIZED DISCRETE FRACTIONAL LYAPUNOV INEQUALITIES

The next result generalizes [8, Theorem 3.1]: in the particular case of $f(y) = y$, inequalities (3.6) reduce to those in [8, Theorem 3.1]. Note that $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a nondecreasing function.

Theorem 3.5. *Let $q : [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}$ be a nontrivial function. Assume that $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a nondecreasing function. If the discrete fractional boundary value problem* (1.4) *has a nontrivial solution y, then*

$$\begin{cases} \sum_{s=0}^{b+1} |q(s+\alpha-1)| > \frac{4\Gamma(\alpha)\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)\eta}{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)f(\eta)}, & \textit{if } b \textit{ is even},\\ \sum_{s=0}^{b+1} |q(s+\alpha-1)| > \frac{\Gamma(\alpha)\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})\eta}{\Gamma(b+3)\Gamma^2(\frac{b+1}{2}+\alpha)f(\eta)}, & \textit{if } b \textit{ is odd}, \end{cases} \tag{3.6}$$

where $\eta = \max_{[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} y(s+\alpha-1)$.

Proof. Since the discrete fractional problem (1.4) has a nontrivial solution, we get via Lemma 3.1 that

$$\begin{aligned} \|y\| &\leq \sum_{s=0}^{b+1} G(s+\alpha-1, s)\,|q(s+\alpha-1)|\,f(y(s+\alpha-1))\\ &\leq \begin{cases} \frac{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)}{4\Gamma(\alpha)\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)} \sum_{s=0}^{b+1} |q(s+\alpha-1)|\,f(y(s+\alpha-1)) & \text{if } b \text{ is even},\\ \frac{1}{\Gamma(\alpha)} \frac{\Gamma(b+3)\Gamma^2(\frac{b+1}{2}+\alpha)}{\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})} \sum_{s=0}^{b+1} |q(s+\alpha-1)|\,f(y(s+\alpha-1)) & \text{if } b \text{ is odd}. \end{cases} \end{aligned}$$

Taking into account that f is a nondecreasing function and

$$\eta = \max_{[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} y(s+\alpha-1),$$

we get that

$$\|y\| < \begin{cases} \frac{1}{4} \frac{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)}{\Gamma(\alpha)\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)} \sum_{s=0}^{b+1} |q(s+\alpha-1)|\,f(\eta) & \text{if } b \text{ is even},\\ \frac{1}{\Gamma(\alpha)} \frac{\Gamma(b+3)\Gamma^2(\frac{b+1}{2}+\alpha)}{\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})} \sum_{s=0}^{b+1} |q(s+\alpha-1)|\,f(\eta) & \text{if } b \text{ is odd}. \end{cases}$$

Hence,

$$\begin{cases} \sum_{s=0}^{b+1} |q(s+\alpha-1)| > \frac{4\Gamma(\alpha)\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)\eta}{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)f(\eta)} & \text{if } b \text{ is even},\\ \sum_{s=0}^{b+1} |q(s+\alpha-1)| > \frac{\Gamma(\alpha)\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})\eta}{\Gamma(b+3)\Gamma^2(\frac{b+1}{2}+\alpha)f(\eta)} & \text{if } b \text{ is odd}. \end{cases}$$

This concludes the proof. $\square$

Remark 3.6. Lyapunov inequalities are usually used to get bounds for the eigenvalues of Sturm–Liouville problems [4, 11]. Therefore, if we consider the discrete Sturm–Liouville problem (1.4) with $f(y) = y$ and $q(t) = \lambda$, then inequalities (3.6) give us an interval for the eigenvalues λ [8]. Here we do a generalization of the results obtained in [5, 8].

Most results about Lyapunov inequalities, including classical forms and fractional continuous and discrete versions, assume, similarly to Theorem 3.5, the existence of a nontrivial solution to the considered problem. In the following theorem, we give other assumptions, instead of assuming existence of a nontrivial solution, to have new Lyapunov inequalities when the nonlinear term satisfies certain conditions.

Theorem 3.7. *Consider the discrete fractional boundary value problem*

$$\begin{cases} \triangle^{\alpha} y(t) + q(t+\alpha-1) f(y(t+\alpha-1)) = 0, \quad 1 < \alpha \leq 2, \\ y(\alpha-2) = y(\alpha+b+1) = 0, \quad \mathbb{N} \ni b \geq 2, \end{cases}$$

where $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing and $q : [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}_+$ is a nontrivial function. If there exist two positive constants $r_2 > r_1 > 0$ such that $f(y) \geq \overset{*}{\gamma} r_1$ for $y \in [0, r_1]$ and $f(y) \leq \gamma r_2$ for $y \in [0, r_2]$, then

$$\begin{cases} \sum\limits_{s=0}^{b+1} |q(s+\alpha-1)| > \frac{r_1}{\gamma r_2} \frac{4\Gamma(\alpha)\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)}{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)} & \text{if } b \text{ is even}, \\ \sum\limits_{s=0}^{b+1} |q(s+\alpha-1)| > \frac{r_1}{\gamma r_2} \frac{\Gamma(\alpha)\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})}{\Gamma(b+3)\Gamma^2(\frac{b+1}{2}+\alpha)} & \text{if } b \text{ is odd}. \end{cases} \tag{3.7}$$

Proof. Follows from Theorems 3.3 and 3.5. □

Example 3.8. Consider the following fractional boundary value problem:

$$\begin{cases} \triangle^{\frac{3}{2}} y(t) + \frac{2t+1}{2\Gamma(6)} \ln(2+y) = 0, \\ y(-\frac{1}{2}) = y(\frac{11}{2}) = 0. \end{cases}$$

We have that

(i) $f(y) = \frac{\ln(2+y)}{\Gamma(6)} : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing;

(ii) $q(t) = t : [\frac{1}{2}, \frac{9}{2}]_{\mathbb{N}_{\frac{1}{2}}} \to \mathbb{R}_+$ with $\sum\limits_{s=0}^{4} \frac{2s+1}{2} = \frac{25}{2} > 0$.

We computed before the values of γ and $\overset{*}{\gamma}$. Choosing $r_1 = 1/10000$ and $r_2 = 1$, we get

1. $f(y) = \frac{\ln(2+y)}{\Gamma(6)} \geq \overset{*}{\gamma} r_1$ for $y \in [0, 1/10000]$;
2. $f(y) = \frac{\ln(2+y)}{\Gamma(6)} \leq \gamma r_2$ for $y \in [0, 1]$.

Therefore, from Theorem 3.7, we get that

$$\sum_{s=0}^{4} \left| q\Big(\frac{2s+1}{2}\Big) \right| > \frac{\Gamma(\frac{3}{2})\Gamma(\frac{13}{2})\Gamma^2(3)\Gamma(6)}{10000\Gamma(6)\Gamma^2(\frac{7}{2})0.0616} \approx 0.15.$$

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