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SEMICIRCULAR ELEMENTS INDUCED BY p-ADIC NUMBER FIELDS

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Abstract. In this paper, we study semicircular-like elements, and semicircular elements induced by p-adic analysis, for each prime p. Starting from a p-adic number field $\mathbb{Q}_p$, we construct a Banach $*$-algebra $\mathfrak{L}\mathfrak{S}_p$, for a fixed prime p, and show the generating elements $Q_{p,j}$ of $\mathfrak{L}\mathfrak{S}_p$ form weighted-semicircular elements, and the corresponding scalar-multiples $\Theta_{p,j}$ of $Q_{p,j}$ become semicircular elements, for all $j \in \mathbb{Z}$. The main result of this paper is the very construction of suitable linear functionals $\tau_{p,j}^0$ on $\mathfrak{L}\mathfrak{S}_p$, making $Q_{p,j}$ be weighted-semicircular, for all $j \in \mathbb{Z}$.

Keywords: free probability, primes, p-adic number fields $\mathbb{Q}_p$, Hilbert-space representations, C^*-algebras, wighted-semicircular elements, semicircular elements.

Mathematics Subject Classification: 05E15, 11R47, 11R56, 46L10, 46L40, 47L15, 47L30, 47L55.

1. INTRODUCTION

The main purpose of this paper is to construct *weighted-semicircular*, and *semicircular* elements for a fixed prime p. Starting from a prime p, we consider *p-adic analysis* on the *p-adic number field* $\mathbb{Q}_p$, and a certain *$*$-algebra $\mathcal{M}_p$* of all *measurable functions on* $\mathbb{Q}_p$. By establishing suitable C^*-probabilistic structures on the *C^*-algebra M_p*, generated by $\mathcal{M}_p$, we focus on a *semigroup S_p* in M_p, generating *C^*-subalgebra $\mathfrak{S}_p$ of M_p*. By filterizing, or sectionizing $\mathfrak{S}_p$ from a system of *linear functionals*, we construct-and-study Banach $*$-probabilistic structures, and our associated weighted-semicircular, and semicircular elements. In classical *statistics*, and in applications of it, one consider *Gaussian elements*, or *Gaussian processes* by taking suitable *measures* (or suitable *probability density functions*) (e.g., [1–3] and [20]). By analogy, we construct our semicircular-like, and semicircular elements by taking (a) suitable (system of) linear functionals on a *Banach $*$-algebra*.

Let $\mathbb{Q}_p$ be the *p-adic number fields for* $p \in \mathcal{P}$, where $\mathcal{P}$ is the set of all *primes* in the *natural numbers* (or the positive integers) $\mathbb{N}$. Then one can naturally understand $\mathbb{Q}_p$ as a *measure space* $(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p)$, where μ_p is a both-left-and-right additive invariant *Haar measure* on the *σ-algebra* $\sigma(\mathbb{Q}_p)$, containing the *basis elements of the topology* for $\mathbb{Q}_p$, formed by transforming the *unit disk* $\mathbb{Z}_p$ *of* $\mathbb{Q}_p$, satisfying

$$\mu_p(\mathbb{Z}_p) = 1 = \mu\left(x + \mathbb{Z}_p\right),$$

for all $x \in \mathbb{Q}_p$.

The *$*$-algebra* $\mathcal{M}_p$, consisting of all μ_p-measurable functions on $\mathbb{Q}_p$, is well-determined for $p \in \mathcal{P}$, and we cannot help emphasizing the importance of such algebraic structures not only in various mathematical fields (modern number theory, geometry with "very small" distance, and operator theory, etc, e.g., [15, 16, 18, 19] and [30]), but also in other scientific fields (quantum physics, quantum arithmetic chaos theory, etc., e.g., [3, 6, 8, 9, 13, 14] and [29]).

1.1. BACKGROUND AND MOTIVATION

The relations between primes and operator algebras have been studied in various different approaches (e.g., [3–5, 11, 13, 14, 23, 29] and [32]). For instance, we studied how primes act "on" certain von Neumann algebras generated by p -adic and Adelic measure spaces (e.g., [9]). Independently, in [7] and [8], we have studied primes as *linear functionals* acting on *arithmetic functions.* i.e., each prime p induces a free-probabilistic structure $(\mathcal{A}, g_p)$ on the algebra $\mathcal{A}$ of all arithmetic functions. In such a case, one can understand arithmetic functions as *Krein-space operators*, under certain representations (See [11]). And, free-probabilistic research on *classical Hecke algebras* induced by primes is considered (e.g., [10]).

Motivated by the main results of [9], we realized that our free-probabilistic settings can be applicable, or used for the *applied operator theory* based on *number-theoretic information.* In particular, one may construct *semicircular law*, or *semicircular-like law* from a fixed prime.

1.2. MAIN IDEAS

In this paper, we study certain operators of the C^*-algebras M_p induced by the $*$-algebra $\mathcal{M}_p$ of μ_p-measurable functions over a fixed p-adic number field $\mathbb{Q}_p$. In particular, we are interested in mutually-orthogonal *projections* $\{P_j\}_{j \in \mathbb{Z}}$ of M_p induced by *generating elements of* $\mathcal{M}_p$. We show that such projections generate a well-defined embedded sub-semigroup S_p of M_p. From such a semigroup, the corresponding *semigroup C^*-algebra* $\mathfrak{S}_p$ is constructed and studied.

From the isomorphism theorem of $\mathfrak{S}_p$, we define *Banach-space operators* c_p and a_p acting "on $\mathfrak{S}_p$," and study fundamental properties of these operators. Then we define a new Banach-space operator l_p "on $\mathfrak{S}_p$" by

$$l_p = c_p + a_p,$$

which gives a filterization, or filterings on $\mathfrak{S}_p$.

By fixing a projection P_j generating $\mathfrak{S}_p$ in M_p, construct a system of operators $\{l_p^n \otimes P_j\}_{n=1}^{\infty}$, and study free-distributional data of the elements in the family. We show the family induces (*free-*)*semicircular law*, under additional processes.

By the semicircularity (e.g., [3, 31] and [34]), our semicircular elements have the same free-distributional data with any other semicircular elements in free probability theory under identically free-distributedness. So, more interesting results are from our so-called *weighted-semicircular elements*. We will see that the free-probabilistic information of such semicircular-like elements are determined by the number-theoretic data from $\mathcal{M}_p$.

Constructions of weighted-semicircular elements and semicircular elements, themselves, are the very main results of this paper. It shows that from a p-adic analytic data, one can obtain semicircular-like property, and semicircularity.

1.3. OVERVIEW

In Section 2, we briefly introduce basic concepts for our proceeding works.

In Sections 3, free-probabilistic models on $\mathcal{M}_p$ is considered in terms of the basis elements of the topology for $\mathbb{Q}_p$. In particular, our free-probabilistic structures imply p-adic-analytic information under p-adic integration. See Theorems 3.7 and 3.8.

In Sections 4, the Hilbert-space representations of the free-probabilistic models of $\mathcal{M}_p$ are established, and the corresponding C^*-algebras M_p generated by $\mathcal{M}_p$ are constructed. Our Hilbert space where $\mathcal{M}_p$ act are naturally constructed by defining inner product determined by p-adic integration of Section 3. Then every element of $\mathcal{M}_p$ is acting on it as a *multiplication operator*.

In Section 5, we build suitable free-probabilistic models of M_p, and study fundamental free-distributional data on M_p. See Theorems 5.3 and 5.3, and Corollary 5.4.

In Sections 6, we fix certain projections $\{P_j\}_{j \in \mathbb{Z}}$ in M_p, and establish the corresponding semigroups S_p generated by the projections, and semigroup C^*-algebras $\mathfrak{S}_p$ of S_p in M_p. The C^*-subalgebras $\mathfrak{S}_p$ give certain filterizations on M_p. See Theorems 6.2 and 6.3.

In Section 7, based on the constructions of $\mathfrak{S}_p$, we establish weighted-semicircular elements in a certain Banach $*$-probability space $\mathfrak{L}_p \otimes_{\mathbb{C}} \mathfrak{S}_p$. And then, corresponding semicircular elements are obtained from our weighted-semicircular elements. Of course, one can check our semicircular elements are following the semicircular law, meanwhile, our weighted-semicircular elements followed semicircular-like law determined by a fixed prime p. See Theorems 7.5, 7.11, 7.12 and 7.14.

2. PRELIMINARIES

In this section, we briefly mention about backgrounds of our works.

2.1. FUNDAMENTALS

Readers can check fundamental analytic-and-combinatorial free probability theory from e.g., [25–27, 31, 33] and [34]. *Free probability* is understood as the noncommutative (and hence, covering commutative) operator-algebraic version of classical probability theory. The classical *independence* is replaced by the *freeness*. It has various applications not only in pure mathematics (e.g., [22, 24] and [23]), but also in related mathematical-and-scientific topics (e.g., [5, 6, 8, 9, 11, 12, 17] and [32]). In particular, we will use combinatorial free probabilistic approach of *Speicher* (e.g., [25–27] and [28]). *Free moments* and *free cumulants* of operators will be computed without introducing in detail.

2.2. p-ADIC NUMBER FIELDS $\mathbb{Q}_p$

Let p be a fixed prime in $\mathcal{P}$, and $\mathbb{Q}_p$, the corresponding p-adic number field. Then this set $\mathbb{Q}_p$ is a well-defined *ring*, which is regarded as a *Banach space* equipped with the *p-norm* $|\cdot|_p$, defined by

$$|x|_p = \left|p^k r\right|_p = \frac{1}{p^k},$$

whenever $x = p^k r$ in $\mathbb{Q}$, for some $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For instance,

$$\left|\frac{4}{3}\right|_2 = \left|2^2 \cdot 3^{-1}\right|_2 = \frac{1}{2^2} = \frac{1}{4},$$

and

$$\left|\frac{4}{3}\right|_3 = \left|4 \cdot 3^{-1}\right|_3 = \frac{1}{3^{-1}} = 3,$$

and

$$\left|\frac{4}{3}\right|_q = 0, \text{ whenever } q \in \mathcal{P} \setminus \{2, 3\}.$$

As a topological space, $\mathbb{Q}_p$ has its basis elements transforming the unit disk

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p = 1\} \text{ of } \mathbb{Q}_p,$$

consisting of all *p-adic integers*, i.e.,

$$\mathbb{Q}_p = \bigcup_{k \in \mathbb{Z}} p^k \mathbb{Z}_p, \tag{2.1}$$

where

$$p^k \mathbb{Z}_p = \{p^k y : y \in \mathbb{Z}_p\} \subset \mathbb{Q}_p.$$

Throughout this paper, we write

$$U_k = p^k \mathbb{Z}_p \text{ in } \mathbb{Q}_p, \text{ for all } k \in \mathbb{Z},$$

with $U_0 = \mathbb{Z}_p$, for convenience.

Also, the p-adic number field $\mathbb{Q}_p$ is a measure space,

$$(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p),$$

equipped with the additive left-and-right invariant *Haar measure* μ_p on the σ*-algebra* $\sigma(\mathbb{Q}_p)$.

Note that

$$\mu_p(U_k) = \frac{1}{p^k} = \mu_p(x + U_k), \quad \text{for all} \quad k \in \mathbb{Z}, \tag{2.2}$$

for all $x \in \mathbb{Q}_p$, satisfying,

$$\mu_p(U_0) = \mu_p(\mathbb{Z}_p) = 1.$$

Remark that, by the very definition, one has the following chain relation,

$$\cdots \subset U_2 \subset U_1 \subset U_0 \subset U_{-1} \subset U_{-2} \subset \cdots, \tag{2.3}$$

in $\mathbb{Q}_p$.

In conclusion, a p-adic number $\mathbb{Q}_p$ is a Banach (topological, measure-theoretic) ring, satisfying (2.1), (2.2) and (2.3). For more details, see [29].

Whenever we fix an integer $k \in \mathbb{Z}$, one can determine so-called the *k-th boundary* ∂_k *of* U_k in $\mathbb{Q}_p$;

$$\partial_k = U_k \setminus U_{k+1}, \tag{2.4}$$

by (2.3), where $A \setminus B = A \cap B^c$, for all sets A and B, where B^c is the complement of B (in a universal set containing A and B). Remark that, by (2.2) and (2.4), one can get that

$$\begin{aligned} \mu_p(\partial_k) &= \mu_p(U_k) - \mu_p(U_{k+1}) \\ &= \frac{1}{p^k} - \frac{1}{p^{k+1}} = \mu_p(x + \partial_k), \end{aligned} \tag{2.5}$$

for all $k \in \mathbb{Z}$, for all $x \in \mathbb{Q}_p$. Also, remark that, by (2.4), one obtains the partition of $\mathbb{Q}_p$,

$$\mathbb{Q}_p = \bigsqcup_{k \in \mathbb{Z}_p} U_k, \tag{2.6}$$

where $\bigsqcup$ means the *disjoint union.*

By understanding $\mathbb{Q}_p$ as a measure space, we have the (pure-algebraic) $*$-algebra $\mathcal{M}_p$ consisting of all μ_p-measurable functions over the complex numbers $\mathbb{C}$, i.e.,

$$\mathcal{M}_p = \{f : \mathbb{Q}_p \to \mathbb{C} : f \text{ is } \mu_p\text{-measurable}\}, \tag{2.7}$$

equipped with the usual functional addition, and the usual functional multiplications.

By definition, if $f \in \mathcal{M}_p$, it is expressed by

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S, \quad \text{with} \quad t_S \in \mathbb{C},$$

where χ_S are the usual *characteristic functions for* $S \in \sigma(\mathbb{Q}_p)$, having its *adjoint*,

$$f^* = \sum_{S \in \sigma(\mathbb{Q}_p)} \overline{t_S} \chi_S,$$

where $\overline{z}$ are the *conjugates of* z, for all $z \in \mathbb{C}$, and $\sum$ is a finite sum.

Indeed, the vector space $\mathcal{M}_p$ of (2.7) forms a well-defined $*$-algebra over $\mathbb{C}$.

For all $f \in \mathcal{M}_p$, one can have the *p-adic integral of* f by

$$\int_{\mathbb{Q}_p} f d\mu_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \mu_p(S).$$

Note that, by (2.6), if $S \in \sigma(\mathbb{Q}_p)$, then there exists a subset Λ_S of $\mathbb{Z}$, such that

$$\Lambda_S = \{j \in \mathbb{Z} : S \cap \partial_j \neq \varnothing\}, \tag{2.8}$$

satisfying

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \int_{\mathbb{Q}_p} \sum_{j \in \Lambda_S} \chi_{S \cap \partial_j} d\mu_p = \sum_{j \in \Lambda_S} \mu_p\left(S \cap \partial_j\right) \leq \sum_{j \in \Lambda_S} \mu_p\left(\partial_j\right) = \sum_{j \in \Lambda_S} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),$$

by (2.4), (2.6) and (2.5), i.e.,

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p \leq \sum_{j \in \Lambda_S} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{2.9}$$

for all $S \in \sigma(\mathbb{Q}_p)$, where Λ_S is subset (2.8) of $\mathbb{Z}$. More precisely, one can get the following proposition.

Proposition 2.1. *Let $S \in \sigma(\mathbb{Q}_p)$, and let $\chi_S \in \mathcal{M}_p$. Then there exist $r_j \in \mathbb{R}$, such that*

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \text{ for all } j \in \Lambda_S, \tag{2.10}$$

and

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right).$$

Proof. By (2.9), whenever $S \in \sigma\left(\mathbb{Q}_p\right)$, there exists a subset Λ_S of $\mathbb{Z}$, in the sense of (2.8), such that

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p \leq \sum_{j \in \Lambda_S} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),$$

because

$$\mu_p\left(S \cap \partial_j\right) \leq \mu_p\left(\partial_j\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}},$$

for all $j \in \Lambda_S$.

So, for each $j \in \Lambda_S$, there exists a unique $r_j \in \mathbb{R}$, such that

$$0 \leq r_j \leq 1,$$

and

$$\mu_p (S \cap \partial_j) = r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right).$$

Therefore, one can get that

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right). \qquad \square$$

By (2.10), one obtains that if

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \in \mathcal{M}_p,$$

then

$$\int_{\mathbb{Q}_p} f d\mu_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \left(\sum_{j \in \Lambda_S} r_j^S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \right), \tag{2.11}$$

where r_j^S are in the sense of (2.10), for all $j \in \Lambda_S$, for all $S \in \sigma(\mathbb{Q}_p)$.

The formula (2.11), obtained from (2.10), provides a universal technique to establish *p-adic calculus.*

3. FREE PROBABILITY ON $\mathcal{M}_p$

Throughout this section, fix a prime $p \in \mathcal{P}$, and $\mathbb{Q}_p$, the corresponding p-adic number field, and let $\mathcal{M}_p$ be the $*$-algebra consisting of all μ_p-measurable functions on $\mathbb{Q}_p$. In this section, let's establish a suitable free-probabilistic model on the $*$-algebra $\mathcal{M}_p$. Remark that free probability provides a universal tool to study free distributions on "noncommutative" algebras, and hence, it covers the cases where given algebras are commutative.

As in Section 2.2, let U_k be the basis elements of of the topology for $\mathbb{Q}_p$,

$$U_k = p^k \mathbb{Z}_p, \text{ for all } k \in \mathbb{Z}, \tag{3.1}$$

with their boundaries $\partial_k = U_k \setminus U_{k+1}$.

Define a linear functional $\varphi_p : \mathcal{M}_p \to \mathbb{C}$ by

$$\varphi_p (f) = \int_{\mathbb{Q}_p} f d\mu_p, \quad \text{for all} \quad f \in \mathcal{M}_p. \tag{3.2}$$

Then, by (3.2), one naturally obtains that

$$\varphi_p\left(\chi_{U_j}\right) = \frac{1}{p^j}, \text{ and } \varphi_p\left(\chi_{\partial_j}\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}},$$

for all $j \in \mathbb{Z}$.

Moreover, by the commutativity on $\mathcal{M}_p$,

$$\varphi_p\left(f_1 f_2\right) = \varphi_p\left(f_2 f_1\right), \text{ for all } f_1, f_2 \in \mathcal{M}_p,$$

and hence, this linear functional φ_p of (3.2) is a *trace on* $\mathcal{M}_p$.

Definition 3.1. The free probability space $(\mathcal{M}_p, \varphi_p)$ is called the p-adic free probability space, for $p \in \mathcal{P}$, where φ_p is the linear functional (3.2) on $\mathcal{M}_p$.

Let U_k be in the sense of (3.1) in $\mathbb{Q}_p$, and $\chi_{U_k} \in \mathcal{M}_p$, for all $k \in \mathbb{Z}$. Then

$$\chi_{U_{k_1}} \chi_{U_{k_2}} = \chi_{U_{k_1} \cap U_{k_2}} = \chi_{U_{\max\{k_1, k_2\}}},$$

by (2.3), where $\max\{k_1, k_2\}$ means the *maximum in* $\{k_1, k_2\}$.

Say $k_1 \leq k_2$ in $\mathbb{Z}$. Then $U_{k_1} \supseteq U_{k_2}$ in $\mathbb{Q}_p$, by (2.3). Therefore, $U_{k_1} \cap U_{k_2} = U_{k_2}$ in $\mathbb{Q}_p$. So, if $k_1 \leq k_2$ in $\mathbb{Z}$, then

$$\chi_{U_{k_1}} \chi_{U_{k_2}} = \chi_{U_{k_1} \cap U_{k_2}} = \chi_{U_{k_2}} \text{ in } \mathcal{M}_p.$$

Lemma 3.2. *Let U_k be in the sense of* (3.1) *in $\mathbb{Q}_p$. Then*

$$\chi_{U_{k_1}} \chi_{U_{k_2}} = \chi_{U_{\max\{k_1, k_2\}}} \quad \text{in } \mathcal{M}_p, \tag{3.3}$$

and hence,

$$\varphi_p\left(\chi_{U_{k_1}} \chi_{U_{k_2}}\right) = \frac{1}{p^{\max\{k_1, k_2\}}}.$$

Proof. By the discussion in the very above paragraph,

$$U_{k_1} \cap U_{k_2} = U_{\max\{k_1, k_2\}} \text{ in } \mathbb{Q}_p,$$

by (2.3), for $k_1, k_2 \in \mathbb{Z}$. So,

$$\chi_{U_{k_1}} \chi_{U_{k_2}} = \chi_{U_{\max\{k_1, k_2\}}},$$

and hence,

$$\varphi_p\left(\chi_{U_{k_1}} \chi_{U_{k_2}}\right) = \mu_p\left(U_{\max\{k_1, k_2\}}\right) = \frac{1}{p^{\max\{k_1, k_2\}}}.$$

□

Inductive to (3.3), we obtain the following result.

Proposition 3.3. *Let $(j_1, \ldots, j_N) \in \mathbb{Z}^N$ for $N \in \mathbb{N}$. Then*

$$\prod_{l=1}^{N} \chi_{U_{j_l}} = \chi_{U_{\max\{j_1, \ldots, j_N\}}} \text{ in } \mathcal{M}_p, \tag{3.4}$$

and hence,

$$\varphi_p\left(\prod_{l=1}^{N} \chi_{U_{j_l}}\right) = \frac{1}{p^{\max\{j_1, \ldots, j_N\}}}.$$

Proof. The proof of (3.4) is done by induction on (3.3). □

Now, let ∂_k be the k-th boundary $U_k \setminus U_{k+1}$ of U_k in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. Then, for $k_1, k_2 \in \mathbb{Z}$, one obtains that

$$\chi_{\partial_{k_1}} \chi_{\partial_{k_2}} = \chi_{\partial_{k_1} \cap \partial_{k_2}} = \delta_{k_1,k_2} \chi_{\partial_{k_1}}, \tag{3.5}$$

where δ means the *Kronecker delta*, and hence,

$$\varphi_p \left(\chi_{\partial_{k_1}} \chi_{\partial_{k_2}} \right) = \delta_{k_1,k_2} \varphi_p \left(\chi_{\partial_{k_1}} \right) = \delta_{k_1,k_2} \left(\frac{1}{p^{k_1}} - \frac{1}{p^{k_1+1}} \right).$$

So, we obtain the following computations.

Proposition 3.4. *Let* $(j_1, \ldots, j_N) \in \mathbb{Z}^N$*, for* $N \in \mathbb{N}$*. Then*

$$\prod_{l=1}^{N} \chi_{\partial_{j_l}} = \delta_{(j_1,\ldots,j_N)} \chi_{\partial_{j_1}} \text{ in } \mathcal{M}_p, \tag{3.6}$$

and hence,

$$\varphi_p \left(\prod_{l=1}^{N} \chi_{\partial_{j_l}} \right) = \delta_{(j_1,\ldots,j_N)} \left(\frac{1}{p^{j_1}} - \frac{1}{p^{j_1+1}} \right),$$

where

$$\delta_{(j_1,\ldots,j_N)} = \left(\prod_{l=1}^{N-1} \delta_{j_l, j_{l+1}} \right) \left(\delta_{j_N, j_1} \right).$$

Proof. The proof of (3.6) is done by (3.5). □

Thus, one can get that, for any $S \in \sigma(\mathbb{Q}_p)$,

$$\varphi_p (\chi_S) = \varphi_p \left(\sum_{j \in \Lambda_S} \chi_{S \cap \partial_j} \right)$$

where Λ_S is in the sense of (2.8)

$$\begin{aligned} &= \sum_{j \in \Lambda_S} \varphi_p \left(\chi_{S \cap \partial_j} \right) = \sum_{j \in \Lambda_S} \mu_p (S \cap \partial_j) \\ &= \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \end{aligned} \tag{3.7}$$

by (2.10), where $0 \leq r_j \leq 1$ are in the sense of (2.10), for all $j \in \Lambda_S$.

Also, if $S_1, S_2 \in \sigma(\mathbb{Q}_p)$, then

$$\begin{aligned}\chi_{S_1}\chi_{S_2} &= \left(\sum_{k\in\Lambda_{S_1}} \chi_{S_1\cap\partial_k}\right)\left(\sum_{j\in\Lambda_{S_2}} \chi_{S_2\cap\partial_j}\right)\\ &= \sum_{(k,j)\in\Lambda_{S_1}\times\Lambda_{S_2}} \left(\chi_{S_1\cap\partial_k}\chi_{S_2\cap\partial_j}\right)\\ &= \sum_{(k,j)\in\Lambda_{S_1}\times\Lambda_{S_2}} \delta_{k,j}\chi_{(S_1\cap S_2)\cap\partial_j}\\ &= \sum_{j\in\Lambda_{S_1,S_2}} \chi_{(S_1\cap S_2)\cap\partial_j},\end{aligned} \tag{3.8}$$

where

$$\Lambda_{S_1,S_2} = \Lambda_{S_1} \cap \Lambda_{S_2},$$

because $\partial_k \cap \partial_j = \delta_{k,j}\partial_j$, for all $k, j \in \mathbb{Z}$.

Thus, there exist $w_j \in \mathbb{R}$, such that

$$0 \leq w_j \leq 1, \quad \text{for all} \quad j \in \Lambda_{S_1,S_2}, \tag{3.9}$$

where Λ_{S_1,S_2} is in the sense of (3.8), and

$$\varphi_p(\chi_{S_1}\chi_{S_2}) = \sum_{j\in\Lambda_{S_1,S_2}} w_j\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),$$

by (3.8) and (2.10), for all $S_1, S_2 \in \sigma(\mathbb{Q}_p)$.

Lemma 3.5. *Let $S_l \in \sigma(\mathbb{Q}_p)$, and $\chi_{S_l} \in (\mathcal{M}_p, \varphi_p)$, for $l = 1, 2$, and let*

$$\Lambda_{S_1,S_2} = \Lambda_{S_1} \cap \Lambda_{S_2},$$

where Λ_{S_l} are in the sense of (2.8), for $l = 1$, 2. Then there exist $r_j \in \mathbb{R}$, such that

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \text{ for all } j \in \Lambda_{S_1,S_2}, \tag{3.10}$$

and

$$\varphi_p(\chi_{S_1}\chi_{S_2}) = \sum_{j\in\Lambda_{S_1,S_2}} r_j\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right).$$

Proof. The proof of (3.10) is done by (3.8) and (3.9). □

Remark 3.6. In fact, the above lemma can be re-formulated as follows. If S_1 and S_2 are given as above, then

$$\varphi_p(\chi_{S_1}\chi_{S_2}) = \begin{cases} \sum\limits_{j\in\Lambda_{S_1,S_2}} r_j\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) & \text{if } \Lambda_{S_1,S_2} \neq \varnothing,\\ \mu_p(\varnothing) = 0 & \text{if } \Lambda_{S_1,S_2} = \varnothing.\end{cases} \tag{3.11}$$

In the following text, if we mention (3.10), then it means (3.11), precisely.

By the above lemma, we obtain the following general result under induction.

Theorem 3.7. *Let $S_l \in \sigma(\mathbb{Q}_p)$, and let $\chi_{S_l} \in (\mathcal{M}_p, \varphi_p)$, for $l = 1, \ldots, N$, for $N \in \mathbb{N}$. Let*

$$\Lambda_{S_1,\ldots,S_N} = \bigcap_{l=1}^{N} \Lambda_{S_l} \text{ in } \mathbb{Z},$$

where Λ_{S_l} are in the sense of (2.8), for $l = 1, \ldots, N$. Then there exist $r_j \in \mathbb{R}$, such that

$$0 \le r_j \le 1 \text{ in } \mathbb{R}, \text{ for all } j \in \Lambda_{S_1,\ldots,S_N}, \tag{3.12}$$

and

$$\varphi_p\left(\prod_{l=1}^{N} \chi_{S_l}\right) = \sum_{j \in \Lambda_{S_1,\ldots,S_N}} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right).$$

Proof. The proof of (3.12) is done by induction on (3.10) (or (3.11)). □

Similar to (3.10) and (3.11), the above formula (3.12) is refined by

$$\varphi_p\left(\prod_{l=1}^{N} \chi_{S_l}\right) = \begin{cases} \sum\limits_{j \in \Lambda_{S_1,\ldots,S_N}} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) & \text{if } \Lambda_{S_1,\ldots,S_N} \neq \varnothing, \\ 0 & \text{if } \Lambda_{S_1,\ldots,S_N} = \varnothing. \end{cases} \tag{3.13}$$

By (3.12) (or (3.13)), we obtain that if

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \in (\mathcal{M}_p, \varphi_p), \text{ with } t_S \in \mathbb{C},$$

then

$$\varphi_p(f) = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \varphi_p(\chi_S) = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \left(\sum_{j \in \Lambda_S} r_j^S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)\right),$$

where r_j^S are in the sense of (3.12), for all $j \in \Lambda_S$.

Therefore, one can get the following result.

Theorem 3.8. *Let $f_l = \sum\limits_{S_l \in \sigma(\mathbb{Q}_p)} t_{S_l} \chi_{S_l}$ be elements of our p-adic free probability space $(\mathcal{M}_p, \varphi_p)$, with $t_{S_l} \in \mathbb{C}$, for $l = 1, \ldots, N$, for $N \in \mathbb{N}$. Then*

$$\varphi_p\left(\prod_{l=1}^{N} f_l\right) = \sum_{(S_1,\ldots,S_N) \in \sigma(\mathbb{Q}_p)^N} \left(\prod_{l=1}^{N} t_{S_l}\right) \left(\sum_{j \in \Lambda_{S_1,\ldots,S_N}} r_j^{(S_1,\ldots,S_N)} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)\right), \tag{3.14}$$

where $r_j^{(S_1,\ldots,S_N)}$ are in the sense of (3.12), for all $j \in \Lambda_{S_1,\ldots,S_N}$ (whenever it is nonempty).

Proof. Suppose $f_1, \ldots, f_N$ be given as above in $(\mathcal{M}_p, \varphi_p)$. Then

$$T = \prod_{l=1}^{N} f_l = \sum_{(S_1,\ldots,S_N)\in\sigma(\mathbb{Q}_p)^N} \left(\prod_{l=1}^{N} t_{S_l}\right)\left(\prod_{l=1}^{N} \chi_{S_l}\right)$$

in $\mathcal{M}_p$. Observe that

$$\begin{aligned}\varphi_p(T) &= \sum_{(S_1,\ldots,S_N)\in\sigma(\mathbb{Q}_p)^N} \left(\prod_{l=1}^{N} t_{S_l}\right)\varphi_p\left(\prod_{l=1}^{N} \chi_{S_l}\right)\\ &= \sum_{(S_1,\ldots,S_N)\in\sigma(\mathbb{Q}_p)^N} \left(\prod_{l=1}^{N} t_{S_l}\right)\left(\sum_{j\in\Lambda_{S_1,\ldots,S_N}} r_j^{(S_1,\ldots,S_N)}\left(\frac{1}{p^j}-\frac{1}{p^{j+1}}\right)\right),\end{aligned}$$

by (3.12) (or (3.13)), where $r_j^{(S_1,\ldots,S_N)}$ are in the sense of (3.12). □

The above joint free-moment formula (3.14) provides a universal tool to compute the free distributions of free random variables in our p-adic free probability space $(\mathcal{M}_p, \varphi_p)$.

4. REPRESENTATIONS OF $(\mathcal{M}_p, \varphi_p)$

Fix a prime $p \in \mathcal{P}$. Let $(\mathcal{M}_p, \varphi_p)$ be the p-adic free probability space. Now, we construct a representation of the $*$-algebra $\mathcal{M}_p$. By understanding $\mathbb{Q}_p$ as a measure space, construct the L^2-*space*,

$$H_p \overset{def}{=} L^2(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p) = L^2(\mathbb{Q}_p), \tag{4.1}$$

over $\mathbb{C}$, consisting of all *square-integrable* μ_p-measurable functions on $\mathbb{Q}_p$. Then this L^2-space is a well-defined *Hilbert space* equipped with its *inner product* $\langle\cdot,\cdot\rangle_2$,

$$\langle f_1, f_2\rangle_2 \overset{def}{=} \int_{\mathbb{Q}_p} f_1 f_2^* d\mu_p, \text{ for all } f_1, f_2 \in H_p. \tag{4.2}$$

Naturally, H_p is the $\|\cdot\|_2$-*norm* completion, where

$$\|f\|_2 \overset{def}{=} \sqrt{\langle f, f\rangle_2}, \text{ for all } f \in H_p,$$

where $\langle\cdot,\cdot\rangle_2$ is the inner product (4.2) on H_p.

Definition 4.1. We call the Hilbert space $H_p = L^2(\mathbb{Q}_p)$ of (4.1), the p-adic Hilbert space.

By the very construction (4.1) of the p-adic Hilbert space H_p, our $*$-algebra $\mathcal{M}_p$ acts on H_p, via an *algebra-action* α,

$$\alpha(f)(h) = fh, \text{ for all } h \in H_p, \tag{4.3}$$

for all $f \in \mathcal{M}_p$. i.e., the morphism α of (4.3) is an action of $\mathcal{M}_p$ acting on the Hilbert space H_p. i.e., for any $f \in \mathcal{M}_p$, the image $\alpha(f)$ is an operator on H_p contained in the *operator algebra* $B(H_p)$ of all (bounded linear) operators on H_p.

Denote $\alpha(f)$ by α_f, for all $f \in \mathcal{M}_p$, where α is in the sense of (4.3). Also, for convenience, denote α_{χ_S} simply by α_S, for all $S \in \sigma(\mathbb{Q}_p)$. For instance,

$$\alpha_{U_k} = \alpha_{\chi_{U_k}} = \alpha(\chi_{U_k}),$$

and

$$\alpha_{\partial_k} = \alpha_{\chi_{\partial_k}} = \alpha(\chi_{\partial_k}),$$

for all $k \in \mathbb{Z}$, where U_k are in the sense of (3.1), and ∂_k are the corresponding boundaries of U_k in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$.

By (4.3), the linear morphism α is a well-determined $*$-algebra action of $\mathcal{M}_p$ acting on H_p. Indeed,

$$\alpha_{t_1 f_1 + t_2 f_2}(h) = (t_1 f_1 + t_2 f_2)\, h = t_1 f_1 h + t_2 f_2 h = t_1 \alpha_{f_1}(h) + t_2 \alpha_{f_2}(h),$$

for all $h \in H_p$, for all $f_1, f_2 \in \mathcal{M}_p$, and $t_1, t_2 \in \mathbb{C}$;

$$\alpha_{f_1 f_2}(h) = f_1 f_2 h = f_1 (f_2 h) = f_1 (\alpha_{f_2}(h)) = \alpha_{f_1} \alpha_{f_2}(h),$$

for all $h \in H_p$, for all $f_1, f_2 \in \mathcal{M}_p$; and

$$\begin{aligned}
\langle \alpha_f(h_1), h_2 \rangle_2 = \langle f h_1, h_2 \rangle_2 &= \int_{\mathbb{Q}_p} f h_1 h_2^* d\mu_p \\
&= \int_{\mathbb{Q}_p} h_1 f h_2^* d\mu_p = \int_{\mathbb{Q}_p} h_1 (h_2 f^*)^* \, d\mu_p \\
&= \int_{\mathbb{Q}_p} h_1 (f^* h_2)^* \, d\mu_p = \langle h_1, \alpha_{f^*}(h_2) \rangle_2 \,,
\end{aligned}$$

for all $f \in \mathcal{M}_p$, and for all $h_1, h_2 \in H_p$, which implies that

$$\alpha_f^* = \alpha_{f^*}, \text{ for all } f.$$

Proposition 4.2. *The linear morphism α of* (4.3) *is a well-defined $*$-algebra action of $\mathcal{M}_p$ acting on H_p. Equivalently, the pair (H_p, α) is a well-determined Hilbert-space representation of $\mathcal{M}_p$.*

Proof. By the discussions in the very above paragraphs, the linear morphism α satisfies that

$$\alpha_{f_1 f_2} = \alpha_{f_1}\alpha_{f_2} \text{ on } H_p,$$

and

$$\alpha_{f_1}^* = \alpha_{f_1^*} \text{ on } H_p,$$

for all $f_1, f_2 \in \mathcal{M}_p$. i.e., α is a $*$-homomorphism from $\mathcal{M}_p$ into $B(H_p)$. Therefore, the pair (H_p, α) is a Hilbert-space representation of $\mathcal{M}_p$. □

In the above proposition, we showed that the pair (H_p, α) of the p-adic Hilbert space H_p and the action α of (4.3) is a Hilbert-space representation of $\mathcal{M}_p$.

Definition 4.3. The Hilbert-space representation (H_p, α) is said to be the p-adic (Hilbert-space) representation of $\mathcal{M}_p$.

By the definition (4.3) of the action α of $\mathcal{M}_p$, it generates *multiplication operators* α_f on the p-adic Hilbert space H_p of (4.1) with their symbols f, for all $f \in (\mathcal{M}_p, \varphi_p)$.

Definition 4.4. Let M_p be the operator-norm closure of $\mathcal{M}_p$ in the operator algebra $B(H_p)$, i.e.,

$$M_p \overset{def}{=} \overline{\alpha\left(\mathcal{M}_p\right)} = \overline{\mathbb{C}\left[\alpha_f : f \in \mathcal{M}_p\right]} \text{ in } B(H_p), \tag{4.4}$$

where $\overline{X}$ mean the operator-norm closures of subsets X of $B(H_p)$. Then this C^*-algebra M_p is called the p-adic C^*-algebra of $(\mathcal{M}_p, \varphi_p)$.

5. FREE PROBABILITY ON M_p

Throughout this section, we fix a prime $p \in \mathcal{P}$. Let $(\mathcal{M}_p, \varphi_p)$ be the corresponding p-adic free probability space, and (H_p, α), the p-adic representation of $\mathcal{M}_p$, and let M_p be the corresponding p-adic C^*-algebra of $(\mathcal{M}_p, \varphi_p)$. In this section, we consider suitable free-probabilistic models on M_p. In particular, we are interested in a system $\{\varphi_j^p\}_{j\in\mathbb{Z}}$ of linear functionals on M_p, determined by the j-th boundaries $\{\partial_j\}_{j\in\mathbb{Z}}$ of $\mathbb{Q}_p$.

Define a linear functional $\varphi_j^p : M_p \to \mathbb{C}$ by a linear morphism,

$$\varphi_j^p(a) \overset{def}{=} \left\langle \alpha_a(\chi_{\partial_j}), \chi_{\partial_j}\right\rangle_2, \quad \text{for all} \quad a \in M_p, \tag{5.1}$$

for all $j \in \mathbb{Z}$, where $\langle \cdot, \cdot \rangle_2$ is the inner product (4.2) on the p-adic Hilbert space H_p of (4.1).

First, remark that, if $a \in M_p$, then

$$a = \sum_{S\in\sigma(\mathbb{Q}_p)} t_S \chi_S \text{ in } M_p, \text{ with } t_S \in \mathbb{C},$$

where $\sum$ is a finite or an infinite (limit of finite) sum(s), under C^*-topology of M_p. Thus, the above definition (5.1) is well-defined, and every linear functional φ_j^p are bounded on M_p, for all $j \in \mathbb{Z}$.

Definition 5.1. Let $j \in \mathbb{Z}$, and let φ_j^p be the linear functional (5.1) on the p-adic C^*-algebra M_p. Then the C^*-probability space $\left(M_p, \varphi_j^p\right)$ is said to be the j-th (p-adic) C^*-probability space.

So, one can get the system

$$\{(M_p, \varphi_j^p) : j \in \mathbb{Z}\}$$

of C^*-probability spaces for a fixed C^*-algebra M_p.

Now, fix $j \in \mathbb{Z}$, and take the corresponding j-th C^*-probability space $\left(M_p, \varphi_j^p\right)$ for $S \in \sigma(\mathbb{Q}_p)$, and an element $\chi_S \in M_p$, one has that

$$\begin{aligned}\varphi_j^p(\chi_S) &= \left\langle \alpha_S(\chi_{\partial_j}), \chi_{\partial_j} \right\rangle_2 = \left\langle \chi_{S \cap \partial_j}, \chi_{\partial_j} \right\rangle_2 \\ &= \int_{\mathbb{Q}_p} \chi_{S\cap\partial_j} \chi_{\partial_j}^* d\mu_p = \int_{\mathbb{Q}_p} \chi_{S\cap\partial_j} \chi_{\partial_j} d\mu_p \\ &= \int_{\mathbb{Q}_p} \chi_{S\cap\partial_j} d\mu_p = \mu_p(S \cap \partial_j) = r_S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),\end{aligned}$$

for some $0 \le r_S \le 1$ in $\mathbb{R}$, i.e., there exists $0 \le r_S \le 1$, such that

$$\varphi_j^p(\chi_S) = \mu_p(S \cap \partial_j) = r_S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{5.2}$$

for any $S \in \sigma(\mathbb{Q}_p)$.

Proposition 5.2. *Let $S \in \sigma(\mathbb{Q}_p)$, and $\alpha_S = \alpha_{\chi_S} \in \left(M_p, \varphi_j^p\right)$, for a fixed $j \in \mathbb{Z}$. Then there exists $r_S \in \mathbb{R}$, such that*

$$0 \le r_S \le 1 \quad \textit{in}\ \mathbb{R}, \tag{5.3}$$

and

$$\varphi_j(\alpha_S^n) = r_S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \ \textit{for all}\ n \in \mathbb{N}.$$

Proof. Remark that the element α_S is a projection in M_p in the sense that

$$\alpha_S^* = \alpha_S = \alpha_S^2 \text{ in } M_p,$$

since

$$\alpha_S^* = \alpha(\chi_S)^* = \alpha(\chi_S^*) = \alpha(\chi_S) = \alpha_S,$$

and

$$\alpha_S^2 = \alpha(\chi_S^2) = \alpha(\chi_S) = \alpha_S \text{ in } M_p.$$

So,

$$\alpha_S^n = \alpha_S, \text{ for all } n \in \mathbb{N}.$$

Thus, for any $n \in \mathbb{N}$, we have

$$\varphi_j^p(\alpha_S^n) = \varphi_j^p(\alpha_S) = r_S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),$$

for some $0 \le r_S \le 1$ in $\mathbb{R}$, by (5.2). □

The free-moment formula (5.3) characterizes the free distributions of χ_S in the j-th C^*-probability space $\left(M_p, \varphi_j^p\right)$, for all $S \in \sigma(\mathbb{Q}_p)$, for all $j \in \mathbb{Z}$.

More precisely, we obtain the following theorem.

Theorem 5.3. *Let $S_l \in \sigma(\mathbb{Q}_p)$, and $\alpha_{S_l} = \alpha(\chi_{S_l}) \in \left(M_p, \varphi_j^p\right)$, for a fixed $j \in \mathbb{Z}$, for $l = 1, \ldots, N$, for $N \in \mathbb{N}$. Then there exists $r_{(S_1,\ldots,S_N)} \in \mathbb{R}$, such that*

$$0 \leq r_{(S_1,\ldots,S_N)} \leq 1 \text{ in } \mathbb{R}, \tag{5.4}$$

and

$$\varphi_j\left(\left(\prod_{l=1}^{N}\alpha_{S_l}\right)^n\right) = r_{(S_1,\ldots,S_N)}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),$$

for all $n \in \mathbb{N}$.

Proof. Let $S_1, \ldots, S_N$ be μ_p-measurable subsets of $\mathbb{Q}_p$, for $N \in \mathbb{N}$, and let

$$S = \bigcap_{l=1}^{N} S_l \in \sigma(\mathbb{Q}_p).$$

Then, one has that

$$\alpha_S = \prod_{l=1}^{N}\alpha_{S_l} \text{ in } M_p,$$

satisfying

$$\alpha_S^* = \alpha_S = \alpha_S^2 \text{ in } M_p.$$

Indeed, if $S \neq \varnothing$, then the above projection-property holds in M_p, and if $S = \varnothing$, then $\chi_S = 0_{M_p}$, the zero element of M_p, which is a projection, too. So,

$$\alpha_S^n = \alpha_S, \text{ for all } n \in \mathbb{N}.$$

Therefore,

$$\varphi_j^p\left(\alpha_S^n\right) = \varphi_j^p\left(\alpha_S\right) = r_S\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),$$

for some $0 \leq r_S \leq 1$ in $\mathbb{R}$, by (5.3), for all $n \in \mathbb{N}$. □

The above joint free-moment formula (5.4) characterizes the free-distributions of finitely many projections $\alpha_{S_1}, \ldots, \alpha_{S_N}$ in the j-th C^*-probability space $\left(M_p, \varphi_j^p\right)$, for $j \in \mathbb{Z}$.

As corollaries of (5.4), we obtain the following results.

Corollary 5.4. *Let U_k be in the sense of* (3.1)*, and ∂_k, the k-th boundaries of U_k in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. Then*

$$\varphi_j^p\left(\alpha_{U_k}^n\right) = \begin{cases} \frac{1}{p^j} - \frac{1}{p^{j+1}} & \text{if } k \leq j, \\ 0 & \text{otherwise,} \end{cases} \tag{5.5}$$

and

$$\varphi_j^p\left(\alpha_{\partial_k}^n\right) = \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)$$

for all $n \in \mathbb{N}$, for $k \in \mathbb{Z}$.

Proof. Observe first that, for any $k \in \mathbb{Z}$,

$$\varphi_j^p\left(\alpha_{\partial_k}^n\right) = \varphi_j^p(\alpha_{\partial_k}) = \mu_p\left(\partial_j \cap \partial_k\right) = \delta_{j,k}\mu_p\left(\partial_j\right) = \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),$$

for all $n \in \mathbb{N}$, because

$$\partial_k \cap \partial_j = \delta_{j,k}\partial_j \text{ in } \mathbb{Q}_p.$$

Consider now that, for an arbitrarily given $k \in \mathbb{Z}$, one has

$$U_k \cap \partial_j = \left(\bigsqcup_{l \geq k \text{ in } \mathbb{Z}} \partial_l\right) \cap \partial_j = \begin{cases} \partial_j & \text{if } k \leq j, \\ \varnothing & \text{if } k > j. \end{cases} \tag{5.6}$$

So,

$$\begin{aligned}\varphi_j^p\left(\alpha_{U_k}^n\right) &= \varphi_j^p(\alpha_{U_k}) = \mu_p\left(U_k \cap \partial_j\right) \\ &= \begin{cases} \mu_p\left(\partial_j\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}} & \text{if } k \leq j, \\ \mu_p\left(\varnothing\right) = 0 & \text{otherwise,} \end{cases}\end{aligned}$$

by (5.6), for all $n \in \mathbb{N}$. □

Now, let $S_l \in \sigma\left(\mathbb{Q}_p\right)$, and $a_l = \chi_{S_l} \in \left(M_p, \varphi_j^p\right)$, for $j \in \mathbb{Z}$, for $l = 1, 2$. Then, for

$$(i_1, \ldots, i_n) \in \{1, 2\}^n, \text{ for } n \in \mathbb{N},$$

we have the joint *free cumulant* in terms of φ_j^p,

$$k_n^{p,j}\left(a_{i_1}, \ldots, a_{i_n}\right) = \sum_{\pi \in NC((i_1, \ldots, i_n))} \left(\prod_{V \in \pi} \varphi_j^p\left(\prod_{i_l \in V} a_{i_l}\right)\right)\mu(\pi, 1_n)$$

by the *Möbius inversion of* [26]

$$\begin{aligned} &= \sum_{\pi \in NC((i_1, \ldots, i_n))} \left(\prod_{V \in \pi} \mu_p\left(\left(\bigcap_{i_l \in V} S_{i_l}\right) \cap \partial_j\right)\right)\mu(\pi, 1_n) \\ &= \sum_{\pi \in NC((i_1, \ldots, i_n))} \left(\prod_{V \in \pi}\left(r_{\pi,V}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)\right)\right)\mu(\pi, 1_n), \end{aligned} \tag{5.7}$$

by (5.4), where $0 \leq r_{\pi,V} \leq 1$ are in the sense of (5.4).

Therefore, by the free cumulant formula (5.7), we obtain the following freeness condition on the j-th C^*-probability space $\left(M_p, \varphi_j^p\right)$.

Theorem 5.5. *Let $S_l \in \sigma(\mathbb{Q}_p)$, and $a_l = \alpha_{S_l} \in \left(M_p, \varphi_j^p\right)$, for $j \in \mathbb{Z}$, for $l = 1, 2$. If*

$$j \notin \Lambda_{S_l} \text{ in } \mathbb{Z}, \text{ for all } l = 1, 2,$$

where Λ_{S_l} are in the sense of (2.8), for $l = 1$, 2, then two free random variables a_1 and a_2 are free in $\left(M_p, \varphi_j^p\right)$. i.e.,

$$j \notin \Lambda_{S_l}, l = 1 \text{ or } l = 2 \Rightarrow \alpha_{S_1} \text{ and } \alpha_{S_2} \text{ are free in } \left(M_p, \varphi_j^p\right). \tag{5.8}$$

Proof. Assume that

$$j \notin \Lambda_{S_1}, \text{ and } j \notin \Lambda_{S_2} \text{ in } \mathbb{Z},$$

where $\Lambda_{S_l} = \{k \in \mathbb{Z} : S_l \cap \partial_k \neq \varnothing\}$, for $l = 1, 2$. Then, by (5.3) (or, by (5.4)),

$$\varphi_j^p\left(\alpha_{S_l}^n\right) = 0, \quad \text{for all} \quad l = 1, 2.$$

Moreover, by the above assupmtion, we have

$$j \notin \Lambda_{S_1, S_2} = \Lambda_{S_1} \cap \Lambda_{S_2}.$$

So, if

$$(i_1, \ldots, i_N) \in \{1, 2\}^N$$

is "mixed" in $\{1, 2\}$, for $N \in \mathbb{N} \setminus \{1\}$, then

$$\varphi_j^p\left(\prod_{t=1}^{N} \alpha_{S_{i_t}}\right) = 0,$$

by (5.4).

It shows that, the self-adjoint elements χ_{S_1} and χ_{S_2} have not only vanishing free moments, but also vanishing mixed free moments.

So, by (5.7), we obtain that

$$\begin{aligned} k_n^{p,j}\left(\alpha_{S_{i_1}}, \ldots, \alpha_{S_{i_n}}\right) &= \sum_{\pi \in NC((i_1, \ldots, i_n))} \left(\prod_{V \in \pi}\left(r_{\pi,V}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)\right)\right)\mu(\pi, 1_n) \\ &= \sum_{\pi \in NC((i_1, \ldots, i_n))} \left(\prod_{V \in \pi}(0)\right)\mu(\pi, 1_n) = 0, \end{aligned}$$

for all "mixed" n-tuples $(i_1, \ldots, i_n) \in \{1, 2\}^n$, for all $n \in \mathbb{N} \setminus \{1\}$.

It guarantees that two random variables χ_{S_1} and χ_{S_2} are free in $\left(M_p, \varphi_j^p\right)$. □

6. PROJECTIONS $\{P_j\}_{j \in \mathbb{Z}}$ AND THE SEMIGROUP S_p IN M_p

Let's fix a prime $p \in \mathcal{P}$. In Section 5, we considered the free probability on the j-th C^*-probability space $\left(M_p, \ \varphi_j^p\right)$, where M_p is the p-adic C^*-algebra and φ_j^p is the linear functional (5.1), for $j \in \mathbb{Z}$. In particular, we observed fundamental free distributions of self-adjoint generating elements of M_p.

In this section, we concentrate on a system of *projections*,

$$\{P_j = \alpha_{\partial_j} = \alpha_{\chi_{\partial_j}} : j \in \mathbb{Z}\} \tag{6.1}$$

in M_p. Remark that these projections are mutually orthogonal from each other in the sense that

$$P_{j_1} P_{j_2} = \delta_{j_1,j_2} P_{j_1} \text{ in } M_p, \text{for all} j_1, j_2 \in \mathbb{Z},$$

because

$$\chi_{\partial_{j_1}} \chi_{\partial_{j_2}} = \delta_{j_1,j_2} \chi_{\partial_{j_1}} \text{ in } \mathcal{M}_p, \text{ for all } j_1, j_2 \in \mathbb{Z}.$$

Let $P_k = \alpha_{\partial_k}$ be a projection (6.1) in the p-adic C^*-algebra M_p, for $k \in \mathbb{Z}$. Then, as we have seen in (5.5),

$$\varphi_j^p (P_k) = \delta_{j,k} \mu_p (\partial_j) = \delta_{j,k} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \tag{6.2}$$

for all $j, k \in \mathbb{Z}$.

Now, from the system $\{P_j\}_{j\in\mathbb{Z}}$ of mutually orthogonal projections, let's construct the *multiplicative semigroup* S_p *in* M_p,

$$S_p \overset{def}{=} \left\langle \{P_j\}_{j\in\mathbb{Z}_p} \right\rangle \text{ in } M_p, \tag{6.3}$$

under the inherited operator multiplication on M_p.

Then this algebraic structure S_p of (6.3) is a well-determined commutative semigroup in M_p, which is not a group. Indeed, the inherited operator multiplication is associative on S_p, but it has no identity in S_p. We call S_p, the *projection semigroup* of the system $\{P_j\}_{j\in\mathbb{Z}}$ of (6.1).

Definition 6.1. Define the C^*-subalgebra $\mathfrak{S}_p$ of the p-adic C^*-algebra M_p by the C^*-algebra generated by the projection semigroup S_p of (6.3). We call $\mathfrak{S}_p$ the projection-semigroup C^*-subalgebra of M_p.

Our projection-semigroup C^*-subalgebra $\mathfrak{S}_p$ of M_p has the following structure theorem.

Theorem 6.2. *Let* $\mathfrak{S}_p$ *be the projection-semigroup* C^**-subalgebra of* M_p*. Then*

$$\mathfrak{S}_p \overset{*\text{-}iso}{=} \bigoplus_{k\in\mathbb{Z}} (\mathbb{C} \cdot P_k) \overset{*\text{-}iso}{=} \mathbb{C}^{\oplus|\mathbb{Z}|} \text{ in } M_p, \tag{6.4}$$

where "$\overset{*\text{-}iso}{=}$*" means "being $*$-isomorphic", and where $\oplus$ means topological direct product of* C^**-algebras.*

Proof. Let S_p be the projection semigroup of the system $\{P_j\}_{j\in\mathbb{Z}}$ of mutually orthogonal projections $P_j = \alpha_{\partial_j}$, for all $j \in \mathbb{Z}$, and let $\mathfrak{S}_p = C^* (S_p)$ be the corresponding projection-semigroup C^*-subalgebra of the p-adic C^*-algebra M_p. Then, by the very definition (6.3),

$$\mathfrak{S}_p = C^*(S_p) = C^*(\{P_j\}_{j\in\mathbb{Z}}) = \overline{\mathbb{C}[\{P_j\}_{j\in\mathbb{Z}}]},$$

where $\overline{X}$ are the C^*-norm closures of the subsets X of M_p

$$= \overline{\underline{\bigoplus_{j\in\mathbb{Z}}} (\mathbb{C}\cdot P_j)}$$

by the mutually orthogonality of $\{P_j\}_{j\in\mathbb{Z}}$, and the projection-property of all P_j's, where $\underline{\oplus}$ means pure-algebraic direct sum of algebras

$$= \bigoplus_{j\in\mathbb{Z}} (\mathbb{C}\cdot P_j) = \mathbb{C}^{\oplus|\mathbb{Z}|}. \qquad \square$$

The above structure theorem (6.4) of the projection-semigroup C^*-subalgebra $\mathfrak{S}_p$ shows that the embedded structure $\mathfrak{S}_p$ provides a certain filterization, or diagonalization in the p-adic C^*-algebra M_p.

Let (M_p, φ_j^p) be the j-th C^*-probability spaces of Section 5, for all $j \in \mathbb{Z}$, and let $\mathfrak{S}_p$ be our projection-semigroup C^*-subalgebra of M_p. Then, naturally, one obtains the system of C^*-probability spaces,

$$\left\{\left(\mathfrak{S}_p, \varphi_j^p\right) : j \in \mathbb{Z}\right\}, \tag{6.5}$$

by restricting the linear functionals φ_j^p on M_p to those on $\mathfrak{S}_p$, for all $j \in \mathbb{Z}$.

And free-distributional data on $\mathfrak{S}_p$ is completely determined by the following result.

Theorem 6.3. *Let $(\mathfrak{S}_p, \varphi_j^p)$ be a C^*-probability space in (6.5), for any $j \in \mathbb{Z}$. For an element*

$$T = \sum_{k\in\mathbb{Z}} t_k P_k \in \mathfrak{S}_p, \text{ with } t_k \in \mathbb{C},$$

we have

$$\varphi_j^p(T^n) = t_j^n \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \quad \text{for all} \quad n \in \mathbb{N}, \tag{6.6}$$

for $j \in \mathbb{Z}$.

Proof. Let $T = \sum_{k\in\mathbb{Z}} t_k P_k \in \mathfrak{S}_p$, with $t_k \in \mathbb{C}$. Then, by the structure theorem (6.4), T is equivalent to

$$T \overset{\text{equi}}{=} \bigoplus_{k\in\mathbb{Z}} t_k P_k \text{ in } \mathbb{C}^{\bigoplus|\mathbb{Z}|} = \mathfrak{S}_p,$$

satisfying that

$$T^n \overset{\text{equi}}{=} \left(\bigoplus_{k\in\mathbb{Z}} t_k P_k\right)^n = \bigoplus_{k\in\mathbb{Z}} t_k^n P_k^n \overset{\text{equi}}{=} \sum_{k\in\mathbb{Z}} t_k^n P_k,$$

in $\mathfrak{S}_p$, for all $n \in \mathbb{N}$.

Therefore, for any fixed $j \in \mathbb{Z}$, one can get that

$$\varphi_j^p (T^n) = \varphi_j^p \left(\sum_{k \in \mathbb{Z}} t_k^n P_k \right) = \varphi_j^p \left(t_j^n P_j \right) = t_j^n \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right),$$

by (6.2), for all $n \in \mathbb{N}$. □

7. WEIGHTED-SEMICIRCULARITY INDUCED BY $\mathcal{M}_p$

Let p be a fixed prime in $\mathcal{P}$, and let M_p be the p-adic C^*-algebra induced by the $*$-algebra $\mathcal{M}_p$, under the p-adic representation (H_p, α) of $\mathcal{M}_p$. Let $\mathfrak{S}_p$ be the projection-semigroup C^*-subalgebra of M_p, satisfying the structure theorem (6.4);

$$\mathfrak{S}_p \overset{*\text{-iso}}{=} \bigoplus_{j \in \mathbb{Z}} (\mathbb{C} \cdot P_j),$$

where $P_j = \alpha_{\partial_j}$ are the mutually-orthogonal projections of (6.1), for all $j \in \mathbb{Z}$.

Also, let $\left\{ \left(\mathfrak{S}_p, \varphi_j^p \right) \right\}_{k \in \mathbb{Z}}$ be the system (6.5) of j-th C^*-probability spaces of $\mathfrak{S}_p$. Recall again that

$$\varphi_j^p (\alpha_{\partial_k}) = \delta_{j,k} \left(\frac{1}{p^j} - \frac{1}{p^{k+1}} \right), \text{ for all } j, k \in \mathbb{Z}. \tag{7.1}$$

Recall now that an *arithmetic function* $\phi : \mathbb{N} \to \mathbb{C}$ is a *Euler totient function*, if

$$\phi(n) \overset{def}{=} |\{ k \in \mathbb{N} \, | 1 \leq k \leq n \text{ and } \gcd(k, n) = 1 \}|, \tag{7.2}$$

for all $n \in \mathbb{N}$, where gcd means the *greatest common divisor*, and where $|x|$ mean the *cardinalities of sets* X.

It is a well-determined *multiplicative arithmetic function* in the sense that

$$\phi(n_1 n_2) = \phi(n_1) \phi(n_2),$$

whenever $\gcd(n_1, n_2) = 1$, because

$$\phi(n) = n \left(\prod_{p \in \mathcal{P}, p | n} \left(1 - \frac{1}{p} \right) \right), \text{ for all } n \in \mathbb{N}. \tag{7.3}$$

By (7.2) and (7.3), the Euler totient function ϕ satisfies

$$\phi(p) = p - 1 = p \left(1 - \frac{1}{p} \right), \text{ for all } p \in \mathcal{P}.$$

So, our free-moment computation (7.1) can be re-stated as follows:

$$\varphi_j^p (\alpha_{\partial_k}) = \delta_{j,k} \frac{1}{p^j} \left(1 - \frac{1}{p} \right) = \delta_{j,k} \frac{1}{p^{j+1}} \phi(p), \tag{7.4}$$

for all $j, k \in \mathbb{Z}$.

Define now a morphism τ_j^p on $\mathfrak{S}_p$ by a linear functional satisfying

$$\tau_j^p \overset{def}{=} \frac{1}{\phi(p)} \varphi_j^p, \text{ for all } j \in \mathbb{Z}. \tag{7.5}$$

Then the pairs $\left(\mathfrak{S}_p, \tau_j^p\right)$ are well-determined C^*-probability spaces, satisfying

$$\begin{aligned} \tau_j^p\left(P_k\right) &= \frac{\delta_{j,k}}{\phi(p)} \varphi_j^p\left(P_j\right) = \frac{\delta_{j,k}}{\phi(p)} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) \\ &= \delta_{j,k} \left(\frac{1}{p\left(1 - \frac{1}{p}\right)}\right) \left(\frac{1}{p^j}\left(1 - \frac{1}{p}\right)\right) = \frac{\delta_{j,k}}{p^{j+1}}, \end{aligned} \tag{7.6}$$

for all $j, k \in \mathbb{Z}$.

Definition 7.1. We call the C^*-probability spaces $\left(\mathfrak{S}_p, \tau_j^p\right)$, the j-th projection (-semigroup C^*-)probability spaces, for all $j \in \mathbb{Z}$.

Free distributional data on the j-th projection probability spaces $\left(\mathfrak{S}_p, \tau_j^p\right)$ is characterized as follows.

Proposition 7.2. *Let $\left(\mathfrak{S}_p, \tau_j^p\right)$ be the j-th projection probability space for $j \in \mathbb{Z}$, where τ_j^p is the linear functional* (7.5)*, for a fixed $j \in \mathbb{Z}$. Then*

$$\tau_j^p\left(P_k^n\right) = \frac{\delta_{j,k}}{p^{j+1}}, \quad \textit{for all} \quad j, k \in \mathbb{Z}, \tag{7.7}$$

for all $n \in \mathbb{N}$.

Proof. The free distribution (7.7) of a projection P_k is obtained by (7.6), for all $k, j \in \mathbb{Z}$. □

Now, we have all ingrediants to construct semicircular-like property, and semicircularity induced by $\mathcal{M}_p$.

7.1. WEIGHTED-SEMICIRCULARITY AND SEMICIRCULARITY

Let (A, φ) be an arbitrary (topological, or pure-algebraic) $*$-probability space of a $*$-algebra A, and a linear functional φ on A. Remember that $*$-algebra is an algebra equipped with the *adjoint* $(*)$ *on* A. An element a of a $*$-algebra A is said to be *self-adjoint*, if $a^* = a$ in A, where $a^*$is the adjoint of a.

Definition 7.3. Let a be a free random variable in a $*$-probability space (A, φ), and let $k_n(\ldots)$ be the free cumulant on A in terms of φ (e.g., see [26]). The given free random variable a is said to be semicircular in (A, φ), if (i) a is self-adjoint, and (ii) a satisfies

$$k_n\Big(\underbrace{a, a, \ldots, a}_{n\text{-times}}\Big) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{7.8}$$

for all $n \in \mathbb{N}$.

Formore about free moments and free cumulants, see [26] and cited papers therein. By the *Möbius inversion of* [26], one can get the equivalent definition of the semicircularity (7.8) as follows: A free random variable a is semicircular in (A, φ), if and only if (i) a is self-adjoint, and (ii) all odd free-moments of a vanish, equivalently,

$$\varphi\left(a^{2n-1}\right) = 0, \text{ for all } n \in \mathbb{N}, \tag{7.9}$$

and (iii) all even free-moments of a satisfy

$$\varphi(a^{2n}) = c_n, \text{ for all } n \in \mathbb{N}, \tag{7.10}$$

where c_n are the *n-th Catalan number*,

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!},$$

for all $n \in \mathbb{N}$ (see [26]).

So, the free-moment formulas (7.9) and (7.10) characterize the semicircularity (7.8) under self-adjointness.

Motivated by the definition (7.8) of semicircularity, we define the following semicircular-like property, called the *weighted-semicircularity* as follows.

Definition 7.4. A free random variable $a \in (A, \varphi)$ is said to be weighted-semicircular in (A, φ) with weight $t_0 \in \mathbb{C} \setminus \{0\}$ (in short, t_0-semicircular), if a is self-adjoint in A, and

$$k_n(a, \ldots, a) = \begin{cases} k_2(a, a) = t_0 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{7.11}$$

for all $n \in \mathbb{N}$.

Of course, if $t_0 = 1$, then 1-semicircularity of (7.11) is the same as the semicircularity (7.8).

By the Möbius inversion of [26], if a free random variable a is t_0-semicircular in (A, φ), then

$$\varphi(a^{2m-1}) = 0,$$

and

$$\begin{aligned}\varphi(a^{2m}) &= \sum_{\pi \in NC(2m)} \Big(\prod_{V \in \pi} k_{|V|}\big(\underbrace{a, \ldots, a}_{|V|\text{-times}}\big)\Big) \\ &= \sum_{\pi \in NC_2(2m)} \Big(\prod_{V \in \pi} k_{|V|}\big(\underbrace{a, \ldots, a}_{|V|\text{-times}}\big)\Big)\end{aligned}$$

where

$$NC_2(2m) = \{\theta \in NC(2m) : \forall V \in \pi, |V| = 2\}$$

by (7.11)

$$= \sum_{\pi \in NC_2(2m)} \left(\prod_{V \in \pi} k_2(a, a) \right) = \sum_{\pi \in NC_2(2m)} \left(\prod_{V \in \pi} t_0 \right)$$
$$= \sum_{\pi \in NC_2(2m)} \left(t_0^{|\pi|} \right)$$

where $|\pi|$ means the number of blocks of the partition π

$$= \sum_{\pi \in NC_2(2m)} t_0^m$$

since all noncrossing partitions π in $NC_2(2m)$ has $\frac{2m}{2}$-many blocks

$$= t_0^m \left(\sum_{\pi \in NC_2(2m)} 1 \right) = t_0^m c_m,$$

where c_m means the m-th Catalan number, for all $m \in \mathbb{N}$.

So, by the definition (7.11), one obtains that: if a ist_0-semicircular in (A, φ), then it is self-adjoint, and there exists $t_0 \in \mathbb{C}$, such that

$$\varphi(a^n) = \begin{cases} t_0^{\frac{n}{2}} c_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{7.12}$$

for all $n \in \mathbb{N}$.

Theorem 7.5. *Let $a \in (A, \varphi)$ be a self-adjoint non-zero free random variable. Then a is t_0-semicircular in (A, φ) for some $t_0 \in \mathbb{C}$, if and only if*

$$\varphi(a^n) = \begin{cases} t_0^{\frac{n}{2}} c_{\frac{n}{2}} & \textit{if } n \textit{ is even,} \\ 0 & \textit{if } n \textit{ is odd,} \end{cases} \tag{7.13}$$

for all $n \in \mathbb{N}$.

Proof. The proof of the free-moment characterization (7.13) of the t_0-semicircularity is done by (7.11) and (7.12), via the Möbius inversion of [26].

($\Rightarrow$) If a is t_0-semicircular in (A, φ), then the free-moment formula (7.13) holds by (7.12).

($\Leftarrow$) If a self-adjoint free random variable a satisfies the free-moment computation (7.13) in (A, φ), then

$$k_n(a, \dots, a) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \varphi\left(a^{|V|}\right) \right) \mu(\pi, 1_n), \tag{7.14}$$

by the Möbius inversion, for all $n \in \mathbb{N}$. Since all odd free-moments of a vanish by (7.13), whenever a block V of any noncrossing partition contains odd-many elements, then $\varphi\left(a^{|V|}\right) = 0$, and hence, if a partition π contains a block V_0 with odd-many elements, then

$$\prod_{V \in \pi} \varphi\left(a^{|V|}\right) = \left(\varphi\left(a^{|V_0|}\right)\right) \left(\prod_{V \in \pi, V \neq V_0} \varphi\left(a^{|V|}\right) \right) = 0.$$

Notice now that all noncrossing partitions π of $NC(n)$ contains at least one odd block in π, whenever n is odd in $\mathbb{N}$. So, by (7.14), one obtains that

$$k_n(a, \dots, a) = 0, \text{ whenever } n \text{ is odd in } \mathbb{N}. \tag{7.15}$$

Now, let $k \in \mathbb{N}$, and observe

$$\begin{aligned} k_{2k}(a, \dots, a) &= \sum_{\pi \in NC(2k)} \left(\prod_{V \in \pi} \varphi\left(a^{|V|}\right) \right) \mu(\pi, 1_{2k}) \\ &= \sum_{\pi \in NC_e(2k)} \left(\prod_{V \in \pi} \varphi\left(a^{|V|}\right) \right) \mu(\pi, 1_{2k}), \end{aligned} \tag{7.16}$$

where

$$NC_e(2k) = \{\theta \in NC(2k) : \forall B \in \theta \Rightarrow |B| \text{ is even}\}.$$

It is not difficult to check that the sub-lattice $NC_e(2k)$ of the lattice $NC(2k)$ is equivalent to $NC(k)$, for all $k \in \mathbb{N}$. Thus, the formula (7.16) goes to

$$\begin{aligned} k_{2k}(a, \dots, a) &= \sum_{\theta \in NC(k)} \left(\prod_{B \in \theta} \varphi\left(a^{2|B|}\right) \right) \mu(\theta, 1_k) \\ &= \sum_{\theta \in NC(k)} \left(\prod_{B \in \theta} t_0^{|B|} c_{|B|} \right) \mu(\theta, 1_k) \\ &= \sum_{\theta \in NC(k)} t_0^k \left(\prod_{B \in \theta} c_{|B|} \right) \mu(\theta, 1_k) \\ &= t_0^k \left(\sum_{\theta \in NC(k)} \left(\prod_{B \in \theta} c_{|B|} \right) \mu(\theta, 1_k) \right) \\ &= \begin{cases} t_0 & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{7.17}$$

by (7.9) and (7.10) (which are equivalent to the semicircularity (7.8)). Indeed,

$$\sum_{\theta \in NC(k)} \left(\prod_{B \in \theta} c_{|B|} \right) \mu(\theta, 1_k) = 0,$$

whenever $k \neq 1$, because of the semicircularity.

Therefore, if the free-moment computation (7.13) holds, then a is t_0-semicircular in (A, φ), by (7.15) and (7.17).

Thus, by $(\Rightarrow)$ and $(\Leftarrow)$, a self-adjoint elemnet a is t_0-semicircular in (A, φ), if and only if it satisfies

$$\varphi(a^n) = \begin{cases} t_0^{\frac{n}{2}} c_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$. □

So, by the above free-momental characterization (7.13), our t_0-semicircularity (7.11) ce be re-stated.

7.2. TENSOR PRODUCT BANACH $*$-ALGEBRA $\mathfrak{L}\mathfrak{S}_p$

Let M_p be the p-adic C^*-algebra containing its p-adic projection-semigroup C^*-subalgebra $\mathfrak{S}_p$, and let τ_k^p be linear functionals (7.5) on $\mathfrak{S}_p$, for all $k \in \mathbb{Z}$. Throughout this section, we fix k in $\mathbb{Z}$, and the corresponding k-th C^*-probability space $(\mathfrak{S}_p, \tau_k^p)$. The formula (7.6) says that

$$\tau_k^p (P_j) = \frac{\delta_{k,j}}{p^{k+1}}, \quad \text{for all} \quad j \in \mathbb{Z}. \tag{7.18}$$

Recall that

$$\mathfrak{S}_p \overset{*\text{-iso}}{=} \underset{j \in \mathbb{Z}}{\oplus} (\mathbb{C} \cdot P_j) \overset{*\text{-iso}}{=} \mathbb{C}^{\oplus |\mathbb{Z}|} \text{ in } M_p, \tag{7.19}$$

by the structure theorem (6.4).

By (7.19), one can define a *Banach-space operators* c_p and a_p "acting on $\mathfrak{S}_p$" by linear transformations satisfying

$$c_p (P_j) = P_{j+1}, \text{ and } a_p (P_j) = P_{j-1}, \tag{7.20}$$

acting on $\mathfrak{S}$, for all $j \in \mathbb{Z}$.

By the very definition (7.20), these linear operators c_p and a_p are bounded (or continuous) under the operator-norm of $\mathfrak{S}_p$, inherited from the C^*-norm on the p-adic C^*-algebra M_p. So, they are well-defined Banach-space operators acting "on $\mathfrak{S}_p$."

Definition 7.6. The Banach-space operators c_p and a_p on $\mathfrak{S}_p$ in the sense of (7.20) are called the (p-)creation, respectively, the (p-)annihilation on $\mathfrak{S}_p$. Define a new Banach-space operator l_p on $\mathfrak{S}_p$ by

$$l_p = c_p + a_p \text{ on } \mathfrak{S}_p. \tag{7.21}$$

We call this operator l_p of (7.21), the (p-)radial operator on $\mathfrak{S}_p$.

Let l_p be the radial operator $c_p + a_p$ of (7.21) on $\mathfrak{S}_p$, where c_p and a_p are the creation, respectively, the annihilation of (7.20). Construct a *Banach algebra* $\mathfrak{L}_p$ by

$$\mathfrak{L}_p = \overline{\mathbb{C}[l_p]} \text{ in } B(\mathfrak{S}_p), \tag{7.22}$$

where $B(\mathfrak{S}_p)$ means the (topological) *operator space*, consisting of all bounded (equivalently, continuous) linear transformations on $\mathfrak{S}_p$, equipped with its *operator-norm* $\|\cdot\|$, defined by

$$\|T\| = \sup\{\|Tx\|_{\mathfrak{S}_p} : x \in \mathfrak{S}_p,\ \|x\|_{\mathfrak{S}_p} = 1\},$$

where

$$\|x\|_{\mathfrak{S}_p} = \sup\{\|x(h)\|_p : h \in H_p,\ \|h\|_p = 1\},$$

giving the C^*-norm topology on M_p (and hence, on $\mathfrak{S}_p$), where $\|\cdot\|_p$ means the Hilbert-space norm on the p-adic Hilbert space H_p.

By the definition (7.22) of the Banach algebra $\mathfrak{L}_p$, every element x of $\mathfrak{L}_p$ is expressed by

$$x = \sum_{k=0}^{\infty} t_k l_p^k, \text{ with } t_k \in \mathbb{C},$$

in $\mathfrak{L}_p$, with identity: $l_p^0 = 1_{\mathfrak{L}_p}$, the identity operator on $\mathfrak{L}_p$, satisfying that:

$$1_{\mathfrak{L}_p}(P_j) = P_j, \text{ for all } j \in \mathbb{Z}.$$

Now, define the adjoint on $\mathfrak{L}_p$ by

$$x^* = \left(\sum_{k=0}^{\infty} t_k l_p^k\right)^* \overset{def}{=} \sum_{k=0}^{\infty} \overline{t_k} l_p^k.$$

Then the Banach algebra $\mathfrak{L}_p$ forms a *Banach $*$-algebra*.

Definition 7.7. Let $\mathfrak{L}_p$ be the Banach $*$-algebra (7.22) in the operator space $B(\mathfrak{S}_p)$. We call it the (p-adic) radial (Banach-$*$-)algebra on $\mathfrak{S}_p$.

Let $\mathfrak{L}_p$ be the radial algebra on $\mathfrak{S}_p$. Construct now the *tensor product Banach $*$-algebra* $\mathfrak{L}\mathfrak{S}_p$ by

$$\mathfrak{L}\mathfrak{S}_p = \mathfrak{L}_p \otimes_{\mathbb{C}} \mathfrak{S}_p, \tag{7.23}$$

where $\otimes_{\mathbb{C}}$ means the tensor product of Banach $*$-algebras.

Consider elements $l_p^k \otimes P_j$ of the tensor product Banach $*$-algebra $\mathfrak{L}\mathfrak{S}_p$ of (7.23), for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $j \in \mathbb{Z}$. By the very definition (7.23), such elements $l_p^k \otimes P_j$ generate $\mathfrak{L}\mathfrak{S}_p$. We concentrate on such generating operators $l_p^k \otimes P_j$ of $\mathfrak{L}\mathfrak{S}_p$, later.

Define a morphism

$$E_p : \mathfrak{L}\mathfrak{S}_p \to \mathfrak{S}_p$$

by a linear transformation satisfying that:

$$E_p\left(l_p^k \otimes P_j\right) = \frac{\left(p^{j+1}\right)^{k+1}}{\left[\frac{k}{2}\right] + 1} l_p^k(P_j), \tag{7.24}$$

for all $k \in \mathbb{N}_0, j \in \mathbb{Z}$, where $[\frac{k}{2}]$ means the *minimal integer* greater than or equal to $\frac{k}{2}$, for instance,

$$\left[\frac{3}{2}\right] = 2 = \left[\frac{4}{2}\right].$$

By (7.19), if

$$T = \sum_{n=1}^{N} \left(t_n l_p^{n_k} \otimes s_n P_j\right) \in \mathfrak{L}\mathfrak{S}_p \text{ with } t_n, s_n \in \mathbb{C},$$

for $N \in \mathbb{N}$, then

$$T = \sum_{n=1}^{N} (t_n s_n) \left(l_p^{n_k} \otimes P_j\right),$$

and hence,

$$E_p(T) = \sum_{n=1}^{N} (t_n s_n) E_p \left(l_p^{n_k} \otimes P_j\right) = \sum_{n=1}^{N} (t_n s_n) \frac{\left(p^{j+1}\right)^{n_k+1}}{[\frac{n_k}{2}]+1} l_p^{n_k} (P_j), \tag{7.25}$$

in $\mathfrak{S}$, by (7.24).

Note that, if $Q_l = l_p^{k_l} \otimes P_{j_l} \in \mathfrak{L}\mathfrak{S}_p$, for $4k_l \in \mathbb{N}_0, j_l \in \mathbb{Z}$, for $l = 1, 2$, then

$$\begin{aligned} E_p (Q_1 Q_2) &= E_p \left(l_p^{k_1} l_p^{k_2} \otimes P_{j_1} P_{j_2}\right) \\ &= E_p \left(l_p^{k_1+k_2} \otimes (\delta_{j_1,j_2} P_{j_1}\right) \\ &= \delta_{j_1,j_2} E_p \left(l_p^{k_1+k_2} \otimes P_{j_1}\right) \\ &= \delta_{j_1,j_2} \frac{\left(p^{j+1}\right)^{k_1+k_2+1}}{\left[\frac{k_1+k_2}{2}\right]+1} l_p^{k_1+k_2} (P_{j_1}). \end{aligned} \tag{7.26}$$

Proposition 7.8. *Let $Q_l = l_p^{k_l} \otimes P_{j_l} \in \mathfrak{L}\mathfrak{S}_p$, for $k_l \in \mathbb{N}_0, j_l \in \mathbb{Z}$, for $l = 1, 2$. If E_p is the morphism in the sense of* (7.24)*, then*

$$E_p (Q_1 Q_2) = \delta_{j_1,j_2} \frac{\left(p^{j+1}\right)^{k_1+k_2+1}}{\left[\frac{k_1+k_2}{2}\right]+1} l_p^{k_1+k_2} (P_{j_1}). \tag{7.27}$$

Proof. The proof of (7.27) is directly done by (7.26). □

Now, consider how our radial operator $l_p = c_p + a_p$ acts on $\mathfrak{S}_p$. First, observe that if c_p and a_p are the creation, respectively, the annihilation on $\mathfrak{S}_p$, then

$$c_p a_p (P_j) = c_p (a_p(P_j)) = c_p (P_{j-1}) = P_j, \tag{7.28}$$

and

$$a_p c_p (P_j) = a_p (c_p(P_j)) = a_p (P_{j+1}) = P_j,$$

for all $j \in \mathbb{Z}$.

Lemma 7.9. *Let c_p, a_p be the creation, respectively, the annihilation on $\mathfrak{S}_p$. Then*

$$c_p a_p = 1_{\mathfrak{S}_p} = a_p c_p, \tag{7.29}$$

where $1_{\mathfrak{S}_p}$ is the identity operator on $\mathfrak{S}_p$.

Proof. Since the C^*-algebra $\mathfrak{S}_p$ is $*$-isomorphic to $\underset{j\in\mathbb{Z}}{\oplus}(\mathbb{C}\cdot P_j)$ (by (7.19)), the formula (7.29) holds by (7.28), under the linearity of c_p and a_p on $\mathfrak{S}$. □

The formula (7.29) shows that the Banach-space operators c_p and a_p are commutative on $\mathfrak{S}_p$. Therefore, one can get that

$$l_p^n = (c_p + a_p)^n = \sum_{k=0}^{n} \binom{n}{k} c_p^k a_p^{n-k}, \tag{7.30}$$

with identities:

$$c_p^0 = 1_{\mathfrak{S}_p} = a_p^0,$$

for all $n \in \mathbb{N}$, where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \text{for all} \quad n \in \mathbb{N}, k \in \mathbb{N}_0.$$

Consider now the formulas (7.29) and (7.30) together. Assume first that $n = 2m-1$ is odd, for $m \in \mathbb{N}$. Then

$$l_p^n = l_p^{2m-1} = \sum_{k=0}^{2m-1} \binom{2m-1}{k} c_p^k a_p^{2m-1-k},$$

by (7.30). Thus, we can realize that l_p^{2m-1} has vanishing $1_{\mathfrak{S}_p}$-terms by (7.29), for all $m \in \mathbb{N}$. i.e.,

$$l_p^{2m-1} \text{ does not contain } 1_{\mathfrak{S}_p}\text{-terms, for } m \in \mathbb{N}. \tag{7.31}$$

Now, suppose that $n = 2m$ is even, for $m \in \mathbb{N}$. Then

$$l_p^{2m} = \sum_{k=0}^{2m} \binom{2m}{k} c_p^k a_p^{2m-k}$$

by (7.30)

$$= \left(\binom{2m}{m} c_p^m a_p^m\right) + \sum_{k\neq m\in\{0,1,\ldots,2m\}} \binom{2m}{k} c_p^k a_p^{2m-k}$$

$$= \binom{2m}{m} (1_{\mathfrak{S}_p})^m + \sum_{k\neq m\in\{0,1,\ldots,2m\}} \binom{2m}{k} c_p^k a_p^{2m-k}$$

by (7.29)

$$= \binom{2m}{m} \cdot 1_{\mathfrak{S}_p} + [\mathit{non}\text{-}1_{\mathfrak{S}_p}\text{-terms}],$$

i.e.,

$$l_p^{2m} \text{ contains the term } \binom{2m}{m} \cdot 1_{\mathfrak{S}_p}, \text{ for } m \in \mathbb{N}. \tag{7.32}$$

Proposition 7.10. *Let $l_p \in \mathfrak{L}_p$ be the radial operator on $\mathfrak{S}_p$. Then*

$$l_p^{2m-1} \textit{ does not contain a } 1_{\mathfrak{S}_p}\textit{-term, and} \tag{7.33}$$

$$l_p^{2m} \textit{ contains its } 1_{\mathfrak{S}_p}\textit{-term, } \binom{2m}{m} \cdot 1_{\mathfrak{S}_p}, \tag{7.34}$$

for all $m \in \mathbb{N}$.

Proof. The proofs of (7.33) and (7.34) are done by (7.31), respectively (7.32), with help of (7.29) and (7.30). □

Now, we have all ingredients to construct weighted-semicircular elements in $\mathfrak{L}\mathfrak{S}_p$.

7.3. WEIGHTED-SEMICIRCULAR ELEMENTS Q_j INDUCED BY $\mathbb{H}(G_p)$

Let $\mathfrak{L}\mathfrak{S}_p$ be the tensor product Banach $*$-algebra $\mathfrak{L}_p \otimes_{\mathbb{C}} \mathfrak{S}_p$ in the sense of (7.23), and let $E_p : \mathfrak{L}\mathfrak{S}_p \to \mathfrak{S}_p$ be the linear transformation (7.24). Throughout this section, Fix an element

$$Q_j = l_p \otimes P_j \in \mathfrak{L}\mathfrak{S}_p, \tag{7.35}$$

for some $j \in \mathbb{Z}$.

Observe that

$$Q_j^n = (l_p \otimes P_j)^n = \left(l_p^n \otimes P_j^n\right) = \left(l_p^n \otimes P_j\right), \tag{7.36}$$

for all $n \in \mathbb{N}$, because P_j are projections in $\mathfrak{S}_p$, for all $j \in \mathbb{Z}$.

Consider that, if $Q_j \in \mathfrak{L}\mathfrak{S}_p$ is in the sense of (7.35), for $j \in \mathbb{Z}$, then

$$E_p\left(Q_j^n\right) = E_p\left(l_p^n \otimes P_j\right) = \frac{\left(p^{j+1}\right)^{n+1}}{\left[\frac{n}{2}\right]+1} l_p^n\left(P_j\right) \tag{7.37}$$

in $\mathfrak{S}_p$, by (7.36), for all $n \in \mathbb{N}$.

Now, for each fixed $j \in \mathbb{Z}$, define a linear functional $\tau_{p:j}^0$ on $\mathfrak{L}\mathfrak{S}_p$ by

$$\tau_{p:j}^0 = \tau_j^p \circ E_p \text{ on } \mathfrak{L}\mathfrak{S}_p, \tag{7.38}$$

where τ_j^p is in the sense of (7.5) on $\mathfrak{S}_p$.

By the linearity of both τ_j^p and E_p, the morphism $\tau_{p:j}^0$ of (7.38) is a linear functional on $\mathfrak{L}\mathfrak{S}_p$. i.e., $\left(\mathfrak{L}\mathfrak{S}_p, \tau_{p:j}^0\right)$ forms a *Banach $*$-probability space* in the sense of [26] and [31].

By (7.37) and (7.38), one has that: if Q_j is in the sense of (7.35), then

$$\tau_{p:j}^0\left(Q_j^n\right) = \tau_j^p\left(E_p(Q_j^n)\right) = \frac{\left(p^{j+1}\right)^{n+1}}{\left[\frac{n}{2}\right]+1} \tau_j^p\left(l_p^n(P_j)\right), \tag{7.39}$$

for all $n \in \mathbb{N}$.

Theorem 7.11. *Let* $Q_j = l_p \otimes P_j \in \left(\mathfrak{LS}_p, \tau^0_{p:j}\right)$, *for a fixed* $j \in \mathbb{Z}$. *Then* Q_j *is* $p^{2(j+1)}$*-semicircular in* $\left(\mathfrak{LS}_p, \tau^0_j\right)$. *Moreover,*

$$\tau^0_j \left(Q^n_j\right) = \begin{cases} \left(p^{j+1}\right)^n c_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{7.40}$$

for all $n \in \mathbb{N}$.

Proof. Let us fix $j \in \mathbb{Z}$, and the corresponding Banach $*$-probability space $\left(\mathfrak{LS}_p, \tau^0_{p:j}\right)$, and let Q_j be a generating operator $l_p \otimes P_j$ of $\mathfrak{LS}_p$. Then it is trivial that Q_j is self-adjoint in $\mathfrak{LS}_p$. Indeed, one has

$$Q^*_j = (l_p \otimes P_j)^* = \left(l^*_p \otimes P^*_j\right) = (l_p \otimes P_j) = Q_j.$$

Observe now that

$$\tau^0_{p:j} \left(Q^{2m-1}_j\right) = \frac{\left(p^{j+1}\right)^{(2m-1)+1}}{\left[\frac{2m-1}{2}\right] + 1} \tau^p_j \left(l^{2m-1}_p (P_j)\right)$$

by (7.39)

$$\begin{aligned} &= \frac{\left(p^{j+1}\right)^{2m}}{\left[\frac{2m-1}{2}\right] + 1} \tau^p_j \left(\left(\sum_{k=0}^{2m-1} \binom{2m-1}{k} c^k_p a^{2m-1-k}_p \right) (P_j) \right) \\ &= \frac{\left(p^{j+1}\right)^{2m}}{\left[\frac{2m-1}{2}\right] + 1} \tau^p_j \left(\sum_{k=0}^{2m-1} \binom{2m-1}{k} \left(c^k_p a^{2m-1-k}_p (P_j)\right) \right), \end{aligned} \tag{7.41}$$

for all $m \in \mathbb{N}$.

Remark that the embedded parts

$$c^k_p a^{2m-1-k}_p (P_j) = P_{j-2m+1+2k}$$

of the summands in (7.41) cannot be P_j-terms by (7.33), for all $k = 0, 1, \ldots, 2m-1$.

Therefore, the formula (7.41) vanishes, for all $m \in \mathbb{N}$.

Now, consider that

$$\tau_{p:j}^{0}\left(Q_j^{2m}\right) = \frac{\left(p^{j+1}\right)^{2m+1}}{\left[\frac{2m}{2}\right]+1}\tau_j^p\left(l_p^{2m}(P_j)\right)$$

by (7.39)

$$= \frac{\left(p^{j+1}\right)^{2m+1}}{m+1}\tau_j^p\left(\left(\sum_{k=0}^{2m}\binom{2m}{k}c_p^k a_p^{2m-k}\right)(P_j)\right)$$

$$= \frac{\left(p^{j+1}\right)^{2m+1}}{m+1}\tau_j^p\left(\binom{2m}{m}\cdot P_j + \sum_{k\neq m\in\{0,1,\dots,2m\}}\binom{2m}{k}c_p^k a_p^{2m-k}(P_j)\right)$$

by (7.34)

$$= \frac{\left(p^{j+1}\right)^{2m+1}}{m+1}\tau_j^p\left(\binom{2m}{m}\cdot P_j + [\text{non-}P_j\text{-terms}]\right)$$

$$= \frac{\left(p^{j+1}\right)^{2m+1}}{m+1}\tau_j^p\left(\binom{2m}{m}\cdot P_j\right) = \frac{\left(p^{j+1}\right)^{2m+1}}{m+1}\binom{2m}{m}\tau_j^p\left(P_j\right)$$

$$= \frac{\left(p^{j+1}\right)^{2m+1}}{m+1}\binom{2m}{m}\left(\frac{1}{p^{j+1}}\right) = \frac{1}{m+1}\binom{2m}{m}\left(p^{j+1}\right)^{2m}$$

$$= c_m\left(p^{j+1}\right)^{2m} = \left((p^{j+1})^2\right)^m c_m,$$

where c_m means the m-th Catalan number, i.e.,

$$\tau_{p:j}^{0}\left(Q_j^{2m}\right) = c_m\left(p^{j+1}\right)^{2m} = c_m\left(p^{2(j+1)}\right)^m, \tag{7.42}$$

for all $m \in \mathbb{N}$.

So, by the free-moment computations (7.41) and (7.42), the self-adjoint free random variable Q_j of our Banach $*$-probability space $\left(\mathfrak{LS}_p, \tau_{p:j}^0\right)$ is a weighted-semicircular element with its weight $p^{2(j+1)}$. i.e., there exists

$$\left(p^{j+1}\right)^2 = p^{2(j+1)} \in \mathbb{C},$$

such that

$$\tau_{p:j}^{0}\left(Q_j^{2m}\right) = c_m\left(p^{2(j+1)}\right)^m,$$

and

$$\tau_{p:j}^{0}\left(Q_j^{2m-1}\right) = 0,$$

for all $m \in \mathbb{N}$.

Therefore, the operator Q_j is $p^{2(j+1)}$-semicircular in $\left(\mathfrak{LS}_p, \tau_{p:j}^0\right)$, by (7.11) and (7.13). $\square$

One can construct the system

$$\mathbb{LS}_p = \left\{ \left(\mathfrak{LS}_p, \tau^0_{p:j} \right) \right\}_{j \in \mathbb{Z}} \tag{7.43}$$

of Banach $*$-probability spaces. Then, every Banach $*$-probability space $\left(\mathfrak{LS}_p, \tau^0_{p:j} \right)$ in the family $\mathbb{LS}_p$ of (7.43) has its $p^{2(j+1)}$-semicircular element $Q_j = l_p \otimes P_j$, for all $j \in \mathbb{Z}$. i.e., we have family

$$\mathcal{WS}_p = \left\{ Q_j = l_p \otimes P_j \in \left(\mathfrak{LS}_p, \tau^0_{p,j} \right) \right\}_{j \in \mathbb{Z}} \tag{7.44}$$

of weighted-semicircular elements in the family $\mathbb{LS}_p$ of (7.43).

So, if $k \in \mathbb{Z}$, then one obtains a corresponding $p^{2(k+1)}$-semicircular element $Q_k \in \mathcal{WS}_p$ in a Banach $*$-probability space $\left(\mathfrak{LS}_p, \tau^0_{p:k} \right) \in \mathbb{LS}_p$, for all $p \in \mathcal{P}$, satisfying

$$\tau^0_{p:k}\left(Q^n_k\right) = \begin{cases} c_{\frac{n}{2}} \left(p^{2(j+1)} \right)^{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

equivalently,

$$k^{p.j,0}_n \Big(\underbrace{Q_j, Q_j, \ldots, Q_j}_{n\text{-times}} \Big) = \begin{cases} p^{2(j+1)} & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$, where $k^{p,j,0}_n(\ldots)$ is the free cumulant with respect to the linear functional $\tau^0_{p:j}$ on $\mathfrak{LS}_p$ (in the sense of [26]).

The following theorem re-prove an equivalent free-distributional data of (7.40), in terms of free cumulant. In fact, the following theorem must hold true by (7.40), and by the Möbius inversion of [26]. However, we provide an independent proof of the theorem below.

Theorem 7.12. *Let* $Q_j = l_p \otimes P_j \in \left(\mathfrak{LS}_p, \tau^0_{p:j} \right)$ *be given as in* (7.35), *for* $j \in \mathbb{Z}$. *Then*

$$k^{p.j,0}_n \Big(\underbrace{Q_j, Q_j, \ldots, Q_j}_{n\text{-times}} \Big) = \begin{cases} p^{2(j+1)} & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{7.45}$$

for all $n \in \mathbb{N}$.

Therefore, Q_j *is a* $p^{2(j+1)}$*-semicircular in* $\left(\mathfrak{LS}_p, \tau^0_{p,j} \right)$, *for* $j \in \mathbb{Z}$.

Proof. Let Q_j be a self-adjoint free random variable (7.35) of the Banach $*$-probability space $\left(\mathfrak{LS}_p, \tau^0_{p,j} \right)$, for a fixed $j \in \mathbb{Z}$. Then

$$\tau^0_{p:j}\left(Q^{2m}_j\right) = c_m \left(p^{j+1} \right)^{2m}, \quad \text{and} \quad \tau^0_j\left(Q^{2m-1}_j\right) = 0, \tag{7.46}$$

for all $m \in \mathbb{N}$, by (7.40).

By the Möbius inversion of [26], one has

$$k^{p,j,0}_n \left(Q_j, Q_j, \ldots, Q_j \right) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \tau^0_{p:j} \left(Q^{|V|}_j \right) \right) \mu(\pi, 1_n) \tag{7.47}$$

for all $n \in \mathbb{N}$.

Suppose now that n in (7.47) is odd. Then every partition π in the lattice $NC(n)$ consisting of all noncrossing partitions over $\{1, \ldots, n\}$ contains at least one odd block V_0 in π, i.e., there always exists at least one block V_0 of π has odd-many elements. Then

$$\tau_{p:j}^{0}\left(Q_j^{|V_0|}\right) = 0,$$

and hence, for a partition π,

$$\prod_{V\in\pi}\tau_{p:j}^{0}\left(Q_j^{|V|}\right) = \left(\tau_{p:j}^{0}\left(Q_j^{|V_0|}\right)\right)\Big(\prod_{V\in\pi, V\neq V_0}\tau_j^{0}\left(Q_j^{|V|}\right)\Big) = 0,$$

whenever n is odd in $\mathbb{N}$. Since π is arbitrary in $NC(n)$, the formula (7.47) vanishes, whenever n is odd, i.e.,

$$k_n^{p,j,0}\left(Q_j, \ldots, Q_j\right) = 0, \text{ if } n \text{ is odd.} \tag{7.48}$$

Consider now that, if $n = 2$, then

$$\begin{aligned}
k_2^{p,j,0}\left(Q_j, Q_j\right) &= \tau_{p:j}^{0}\left(Q_j^2\right) - \tau_{p:j}^{0}\left(Q_j\right)\tau_{p:j}^{0}\left(Q_j\right)\\
&= \tau_{p:j}^{0}\left(l_p^2 \otimes P_j\right) - \tau_{p:j}^{0}\left(Q_j\right)^2 = \tau_j^{0}\left(l_p^2 \otimes P_j\right)\\
&= \tau_j^{p}\left(E_p(l_p^2 \otimes P_j)\right) = \left(p^{j+1}\right)^{2+1}\tau_j^{p}\left(P_j\right)\\
&= \left(p^{j+1}\right)^2 = p^{2(j+1)},
\end{aligned}$$

by (7.47), i.e., one obtains that

$$k_2^{p,j,0}\left(Q_j, Q_j\right) = p^{2(j+1)}. \tag{7.49}$$

Let $m > 1$ in $\mathbb{N}$. Then

$$\begin{aligned}
k_{2m}^{p,j,0}\left(Q_j, \ldots, Q_j\right) &= \sum_{\pi\in NC_e(2m)}\left(\prod_{V\in\pi}\tau_{p:j}^{0}\left(Q_j^{|V|}\right)\right)\mu(\pi, 1_{2m})\\
&= \sum_{\pi\in NC_e(2m)}\left(\prod_{V\in\pi}\left(c_{\frac{|V|}{2}}\left(p^{j+1}\right)^{|V|}\right)\right)\mu(\pi, 1_{2m})\\
&= \sum_{\pi\in NC_e(2m)}\left(\left(\prod_{V\in\pi}c_{\frac{|V|}{2}}\right)\left(p^{j+1}\right)^{2m}\right)\mu(\pi, 1_{2m})\\
&= \left(p^{j+1}\right)^{2m}\left(\sum_{\pi\in NC_e(2m)}\left(\prod_{V\in\pi}c_{\frac{|V|}{2}}\right)\mu(\pi, 1_{2m})\right),
\end{aligned} \tag{7.50}$$

by (7.47) and (7.49), where

$$NC_e(2m) = \{\theta \in NC(2m) : B \in \theta \Longleftrightarrow |B| \text{ is even}\}.$$

By the semicircularity (7.9) and (7.10), we know that

$$\sum_{\pi \in NC_e(2m)} \left(\prod_{V \in \pi} c_{\frac{|V|}{2}} \right) \mu(\pi, 1_{2m}) = 0, \tag{7.51}$$

whenever $m > 1$ in $\mathbb{N}$. Thus, the formula (7.50) vanishes whenever $m > 1$.

Thus, we have

$$\text{if } m > 1 \text{ in } \mathbb{N}, \text{ then } k_{2m}^{j,0}(Q_j, \ldots, Q_j) = 0, \tag{7.52}$$

by (7.50) and (7.51).

Therefore, by (7.48), (7.49) and (7.52), we obtain

$$k_n^{p,j,0}(Q_j, \ldots, Q_j) = \begin{cases} p^{2(j+1)} & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$.

It guarantees that the element Q_j is $p^{2(j+1)}$-semicircular in $\left(\mathfrak{LS}_p, \tau_j^0\right)$, for all $j \in \mathbb{Z}$, by (7.11) and (7.13). □

7.4. SEMICIRCULAR ELEMENTS INDUCED BY $\mathcal{M}_p$

In this section, we consider semicircular elements in the Banach $*$-probability spaces $(\mathfrak{LS}_p, \tau_{p:j}^0)$, for $j \in \mathbb{Z}$. We will use the same notations used in Section 7.3. As we have seen in (7.40) and (7.45), the generating operators

$$Q_j = l_p \otimes P_j \text{ of } \mathfrak{LS}_p$$

are $p^{2(j+1)}$-semicircular in the Banach $*$-probability space $\left(\mathfrak{LS}_p, \tau_{p:j}^0\right)$, for each $j \in \mathbb{Z}$.

Throughout this section, let's fix $j \in \mathbb{Z}$, and the corresponding Banach $*$-probability space $\left(\mathfrak{LS}_p, \tau_{p:j}^0\right)$. Define now an operator Θ_j of $\mathfrak{LS}_p$ by a free random variable,

$$\Theta_j = \frac{1}{p^{j+1}} Q_j \in \left(\mathfrak{LS}_p, \tau_{p:j}^0\right). \tag{7.53}$$

Since $p^{j+1} \in \mathbb{Q}$, the quantity $\frac{1}{p^{j+1}} \in \mathbb{Q}$, too, in $\mathbb{R}$, and hence, the operator Θ_j is self-adjoint in $\mathfrak{LS}_p$, by the self-adjointness of Q_j.

Observe now that, if Θ_j is a self-adjoint operator (7.53) in $\mathfrak{LS}_p$, then

$$k_n^{p,j,0}\Big(\underbrace{\Theta_j, \Theta_j, \ldots, \Theta_j}_{n\text{-times}}\Big) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \tau_{p:j}^0 \left(\Theta_j^{|V|} \right) \right) \mu(\pi, 1_n)$$
$$= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \left(\frac{1}{p^{j+1}} \right)^{|V|} \tau_{p:j}^0 \left(Q_j^{|V|} \right) \right) \mu(\pi, 1_n)$$
$$= \sum_{\pi \in NC(n)} \left(\frac{1}{p^{j+1}} \right)^n \left(\prod_{V \in \pi} \tau_{p:j}^0 \left(Q_j^{|V|} \right) \right) \mu(\pi, 1_n) \tag{7.54}$$
$$= \left(\frac{1}{p^{j+1}} \right)^n \left(\sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \tau_{p:j}^0 \left(Q_j^{|V|} \right) \right) \mu(\pi, 1_n) \right)$$
$$= \left(\frac{1}{p^{j+1}} \right)^n k_n^{p,j,0} \Big(\underbrace{Q_j, Q_j, \ldots, Q_j}_{n\text{-times}} \Big),$$

for all $n \in \mathbb{N}$.

Remark that the above formula (7.54) can be directly obtained by the *bimodule-map property of free cumulants.* i.e.,

$$k_n^{p,j,0}\left(X_j, \ldots, X_j\right) = k_n^{p,j,0}\left(\frac{1}{p^{j+1}} Q_j, \ldots, \frac{1}{p^{j+1}} Q_j \right)$$
$$= \left(\frac{1}{p^{j+1}} \right)^n k_n^{p,j,0}\left(Q_j,\ \ldots,\ Q_j\right),$$

for all $n \in \mathbb{N}$ (e.g., see [26]).

Lemma 7.13. *Let* $\Theta_j = \frac{1}{p^{j+1}} Q_j = \frac{1}{p^{j+1}} \left(l_p \otimes P_j\right)$ *be in the sense of* (7.53) *in our Banach* $*$*-probability space* $\left(\mathfrak{LS}_p, \tau_{p:j}^0\right)$, *for a fixed* $j \in \mathbb{Z}$. *Then*

$$k_n^{p,j,0}\left(\Theta_j, \ldots, \Theta_j\right) = \left(\frac{1}{p^{j+1}} \right)^n k_n^{p,j,0}(Q_j, \ldots, Q_j), \tag{7.55}$$

for all $n \in \mathbb{N}$.

Proof. The proof of the free-cumulant formula (7.55) is done by (7.54). □

The above free-cumulant formula (7.55) shows that the free-distributional data of Θ_j are determined by those of Q_j.

Theorem 7.14. *Let* Θ_j *be in the sense of* (7.53) *in* $\mathfrak{LS}_p$, *for* $j \in \mathbb{Z}$. *Then it is semicircular in* $\left(\mathfrak{LS}_p, \tau_{p:j}^0\right)$, *for* $j \in \mathbb{Z}$.

Proof. Observe that

$$k_n^{p,j,0}\left(\Theta_j, \ldots, \Theta_j\right) = \left(\frac{1}{p^{j+1}} \right)^n k_n^{p,j,0}\left(Q_j, \ldots, Q_j\right)$$

by (7.55)

$$= \begin{cases} \left(\frac{1}{p^{j+1}}\right)^2 k_2^{p,j,0}(Q_j, Q_j) & \text{if } n = 2, \\ \left(\frac{1}{p^{j+1}}\right)^n \cdot 0 = 0 & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$, by the $p^{2(j+1)}$-semicircularity (7.45), or (7.40) of Q_j, for $j \in \mathbb{Z}$.

So, we obtain that

$$k_n^{p,j,0}(\Theta_j, \ldots, \Theta_j) = \begin{cases} \left(\frac{1}{p^{j+1}}\right)^2 p^{2(j+1)} = 1 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{7.56}$$

for all $n \in \mathbb{N}$.

Therefore, by (7.8) and (7.56), the self-adjoint operators Θ_j are semicircular in $\left(\mathfrak{LS}_p,\ \tau_{p:j}^0\right)$, for all $j \in \mathbb{Z}$. □

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